STUDIA MATHEMATICA 142 (1) (2000)

Long-time asymptotics for the nonlinear heat equation with a fractional Laplacian in a ball

by

VLADIMIR VARLAMOV (Austin, TX)

Abstract. The nonlinear heat equation with a fractional Laplacian

$$u_t + (-\Delta)^{\alpha/2}u = u^2, \quad 0 < \alpha \le 2,$$

is considered in a unit ball B. Homogeneous boundary conditions and small initial conditions are examined. For $3/2 + \varepsilon_1 \le \alpha \le 2$, where $\varepsilon_1 > 0$ is small, the global-in-time mild solution from the space $C^0([0,\infty), H_0^\kappa(B))$ with $\kappa < \alpha - 1/2$ is constructed in the form of an eigenfunction expansion series. The uniqueness is proved for $0 < \kappa < \alpha - 1/2$, and the higher-order long-time asymptotics is calculated.

1. Introduction. The aim of the present paper is to study the nonlinear heat equation with a fractional Laplacian

$$u_t + (-\Delta)^{\alpha/2}u = u^2, \quad 0 < \alpha \le 2. \tag{1.1}$$

The case $\alpha = 2$ corresponds to the standard (Gaussian) diffusion, and $0 < \alpha < 2$ accounts for the anomalous diffusion (see [4]). The nonlinear heat equation with $\alpha = 2$ and the power nonlinearity

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.2}$$

was studied in [5, 6, 8, 10, 12, 13, 15, 16, 20], where the questions of well-posedness and asymptotic behavior of its solutions were investigated. An overview of these results can be found in [3, 31]. We point out that the authors of the above-mentioned papers considered mainly initial-value problems imposing some restrictions on the intial data and discussing the long-time behavior of solutions in terms of the parameters N, p, and the exponents of decay of the initial data. In [9] the long-time asymptotics for the equation of type (1.2) with the nonlinearity $\partial_{x_1}|u|^{q-1} + \partial_{x_2}|u|^{p-1}$ was studied for a certain range of exponents. C. E. Wayne [34] examined the Cauchy problem for (1.2) with a sufficiently smooth nonlinear term $F(u)$

2000 *Mathematics Subject Classification*: 35Q55, 35B40.

Key words and phrases: nonlinear heat equation, fractional Laplacian, first initial-boundary value problem, construction of solutions, higher-order long-time asymptotics.

from the point of view of finite-dimensional invariant manifolds. He constructed these manifolds and showed how they could be used for obtaining long-time asymptotics.

For parabolic partial differential equations on bounded domains the theory of invariant manifolds permits one to establish stability and even calculate the lower-order long-time asymptotics (see [1, 11, 32] and the references therein). Since the linear operator of the equation has a point spectrum, one can separate the phase space of the linear problem into stable, unstable, and central manifolds. In this case the solutions enjoy exponential decay in time, while the power-law decay is typical for solutions of Cauchy problems.

Fractional derivatives dissipative equations were considered in the papers [2, 3, 4, 7, 17, 19]. Nonlocal Burgers-type equations appeared as model equations simplifying the multidimensional Navier–Stokes system with modified dissipativity [2], governing hereditary effects for nonlinear acoustic waves [21], and modeling interfacial growth mechanisms including trapping effects [17]. The linear equation

$$u_t = -\sum_{i=1}^{N} \gamma_i(-\Delta)^{\alpha_i/2}u, \qquad \gamma_i = \text{const} \ge 0,\ 0 < \alpha_i \le 2,$$

appears in the problems of identification of images [7]. The paper [21] contains various examples of fractional differential equations with applications to hydrodynamics, statistical physics, and molecular biology.

We point out that we do not use any of the methods mentioned above in the present note. The basic ideas of our approach were developed in [26–31], and in a certain sense they represent a further development of the methods of [19]. We apply the spectral and perturbation theories in order to construct solutions and then find the long-time asymptotics. We shall consider below the first initial-boundary value problem for (1.1) in a unit ball and construct its small solutions in the form of an eigenfunction expansion series. The well-posedness follows from the construction. The Laplace operator in a ball has a point spectrum, therefore the exponential decay in time is well expected. However, our purpose is not just to establish exponential stabilization. The series representation allows us to calculate the higher-order long-time asymptotics. The function $\sin(\pi r)/(\pi r)$ present in its major term describes the space evolution, and the coefficient in this term is calculated by means of nonlinear iterations. Therefore, the asymptotics is nonlinear. In the case of anomalous diffusion, $3/2 + \varepsilon_1 \le \alpha < \alpha_{\text{cr}}$, where $\alpha_{\text{cr}} \simeq 1.937$, the second-order asymptotics is obtained (the calculation of the coefficient in the second term is also based on nonlinear approximations). The second term contains the spherical harmonic $Y_1(\theta,\varphi)$ and shows the dependence on the angles. We must point out that the success of constructing solutions

depends greatly on the convergence of the spatial eigenfunction series, and this convergence is rather poor. This factor plays a crucial role in determining the eigenfunction expansion coefficients of the nonlinearity. That is why the restriction $3/2 + \varepsilon_1 \le \alpha \le 2$ appears, where $\varepsilon_1 > 0$ is small.

2. Preliminaries. Denote by B a ball of unit radius and introduce the coordinate system with the origin at the center of the ball, so that in spherical coordinates $B = \{(r,\theta,\varphi) : |r| < 1,\ 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi\}$. In our study of the first initial-boundary value problem for the nonlinear heat equation in B we shall use the expansion in the eigenfunctions of the Laplace operator in this ball. We denote by $L_2(B)$ the space of real functions square integrable over B with the norm

$$\|f\|_B^2 = \int_0^1\int_0^{2\pi}\int_0^{\pi} |f(r,\theta,\varphi)|^2 r^2 \sin\theta\, d\theta\, d\varphi\, dr.$$

Then for a function $f(r,\theta,\varphi) \in L_2(B)$ we have the expansion

$$f(r,\theta,\varphi) = \sum_{m\ge 0,\, n\ge 1} \widehat{f}_{mn}\chi_{mn}(r,\theta,\varphi),$$

where $\chi_{mn}(r,\theta,\varphi)$ are the eigenfunctions of the Laplace operator in the ball B, i.e.,

$$\begin{aligned} &\Delta\chi = -\Lambda\chi, \quad (r,\theta,\varphi) \in B,\\ &\chi|_S = 0, \quad |\chi(0,\theta,\varphi)| < \infty, \quad \chi(r,\theta,\varphi+2\pi) = \chi(r,\theta,\varphi), \end{aligned} \tag{2.1}$$

where $\Delta = (1/r^2)\partial_r(r^2\partial_r) + (1/r^2)\Delta_{\theta,\varphi}$ and $\Delta_{\theta,\varphi} = (1/\sin\theta)\partial_\theta(\sin\theta\partial_\theta) + (1/\sin^2\theta)\partial_\varphi^2$.

The angular eigenfunctions $Y(\theta,\varphi)$ are nontrivial solutions of the problem

$$\begin{aligned} &\Delta_{\theta,\varphi}Y + \mu Y = 0, \quad (\theta,\varphi) \in S,\\ &|Y|_{\theta=0,\pi} < \infty, \quad Y(\theta,\varphi+2\pi) = Y(\theta,\varphi). \end{aligned}$$

The eigenvalues of the Laplace operator on the unit sphere are

$$\mu_m = m(m+1), \qquad m = 0,1,2,\ldots,$$

and the corresponding real eigenfunctions are the spherical harmonics of mth order. Further separation of variables leads to

$$Y_m(\theta,\varphi) = \sum_{l=0}^{m}[C_{lm}^{(1)}\cos l\varphi + C_{lm}^{(2)}\sin l\varphi]P_m^l(\cos\theta),$$

where $P_m^l(\cos\theta)$ are the associated Legendre functions [18, 22, 35]. Thus, $Y_m(\theta,\varphi)$ is represented by a linear combination of tesseral harmonics.

For the radial functions we have the problem

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(\Lambda-\frac{m(m+1)}{r^2}\right)R=0,$$
$$R(1)=0,\quad |R(0)|<\infty,$$

whose nontrivial solutions and the corresponding eigenvalues are

$$R_m(r)=j_m(\lambda_{mn}r)=\sqrt{\frac{\pi}{2r}}\,J_{m+1/2}(\lambda_{mn}r),\qquad \Lambda_{mn}=\lambda_{mn}^2,$$

where λ_{mn} are the positive zeros of the Bessel function $J_{m+1/2}(z)$ numbered in increasing order, $m=0,1,2,\ldots$; $n=1,2,\ldots$; n is the number of the zero. Note that $\lambda_{0n}=\pi n$. Our radial functions differ from the spherical Bessel functions $\tilde{j}_m(\lambda_{mn}r)=\sqrt{\frac{\pi}{2\lambda_{mn}r}}J_{m+1/2}(\lambda_{mn}r)$ by the factor $\sqrt{\lambda_{mn}}$ (see [35, p. 357]).

Introducing the real space $L_{2,r}(0,1)$ with the norm $\|f\|_r^2=\int_0^1 rf^2(r)\,dr$ we can write

$$\|J_\nu(\lambda_{\nu n}r)\|_r^2=\int_0^1 J_\nu^2(\lambda_{\nu n}r)r\,dr=\frac{1}{2}[J_\nu'(\lambda_{\nu n}r)]^2,\qquad \nu\ge 0,\ n=1,2,\ldots$$

For sufficiently large $\lambda>0$ the following inequality holds [25, p. 219]:

$$\frac{c_1}{\lambda}\le\|J_\nu(\lambda r)\|_r^2\le\frac{c_2}{\lambda}.$$

We also use the real space $L_{2,r^2}(0,1)$ ($L_2(0,1)$ with weight r^2) with the scalar product $(f,g)=\int_0^1 r^2f(r)g(r)\,dr$ and the norm $\|f\|^2=\int_0^1 r^2f^2(r)\,dr$. Then we can write

$$\|j_m\|_{(n)}^2=\int_0^1 j_m^2(\lambda_{mn}r)r^2\,dr=\frac{\pi}{2}\int_0^1 J_{m+1/2}^2(\lambda_{mn}r)r\,dr$$

and for sufficiently large $\lambda>0$,

$$C_1/\lambda\le\|j_m(\lambda r)\|^2\le C_2/\lambda. \tag{2.2}$$

Note that large positive zeros of $J_m(z)$ with $0\le m\le m_0<\infty$ have the following asymptotics uniform in m (McMahon's expansion, see [14, p. 153]):

$$\lambda_{mn}=\mu_{mn}+O(1/\mu_{mn}),\qquad \mu_{mn}=(m+2n-1/2)\pi/2,\qquad n\to\infty. \tag{2.3}$$

For $0\le n\le n_0<\infty$ and $m\to\infty$ the asymptotic formula is

$$\lambda_{mn}=m+c(n)m^{1/3}+O(m^{-1/3}),$$

where the coefficient $c(n)$ and the constant in the estimate of the remainder depend on n, but are bounded for bounded n.

Having described the radial eigenfunctions, we return to the angular ones $Y_m(\theta,\varphi)$ (sometimes also called surface harmonics [35, p. 298]). Let P and Q be two variable points on the unit sphere S and let $\gamma(P,Q)$ be the angle (between 0 and π) formed by the two vector radii OP and OQ, where O is the center of the unit sphere. Then for P fixed and Q varying over S, $P_m[\cos\gamma(P,Q)]$, where $P_m(x)$ is a Legendre polynomial, is a spherical harmonic of the mth order of the spherical coordinates of Q, and for fixed Q and variable P this function is also a spherical harmonic with respect to P.

Introducing the scalar product in the real space $L_2(S)$ by the formula $(f,g)_S=\int_S fg\,dS$ and denoting by $\|\cdot\|_S$ the corresponding norm we can write (see [22, p. 266])

$$(Y_m,Y_k)_S=\int_S Y_m(Q)P_k[\cos\gamma(P,Q)]\,dS_Q=0,\qquad m\ne k,$$
$$\|Y_m\|_S^2=\frac{4\pi}{2m+1}, \tag{2.4}$$
$$\frac{2m+1}{4\pi}\int_S Y_m(Q)P_m[\cos\gamma(P,Q)]\,dS_Q=Y_m(P).$$

The spherical harmonic expressed as a symmetric function of the two points P and Q is called a Laplace coefficient [22, p. 272], the name coming from the expansion of a function $f(P)$ into the Laplace series

$$f(P)\sim\sum_{m=0}^{\infty}Y_m(P),$$
$$Y_m(P)=\frac{2m+1}{4\pi}\int_S f(Q)P_m[\cos\gamma(P,Q)]\,dS_Q,\qquad m=0,1,\ldots$$

Considered as a function of Q, $P_m[\cos\gamma(P,Q)]$ contains two arbitrary parameters, the coordinates (θ',φ') of the point P, which can be chosen by the choice of the coordinate system. If we direct the z-axis of the coordinate system through P, the spherical harmonics will turn out to be zonal and the constants will be determined. Then the last formula in (2.4) will yield

$$\frac{2m+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}[P_m(\cos\gamma)]^2\sin\gamma\,d\gamma\,d\chi=Y_m(P)=1,$$

where (γ,χ) are the spherical coordinates in the system with the north pole at the point P. Another consequence of (2.4) is the formula

$$\frac{2m+1}{4\pi}\int_S P_m[\cos\gamma(P,Q')]P_m[\cos\gamma(Q,Q')]\,dS_{Q'}=P_m[\cos\gamma(P,Q)].$$

If $P=(\theta',\varphi')$ and $Q=(\theta,\varphi)$, then by the addition theorem for spherical harmonics we can represent them in terms of the tesseral ones:

$$\begin{aligned}P_m[\cos\gamma(P,Q)] &= P_m[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')]\\ &= P_m(\cos\theta)P_m(\cos\theta')\\ &\quad + 2\sum_{l=1}^{m}\frac{(m-l)!}{(m+l)!}P_m^l(\cos\theta)P_m^l(\cos\theta')\cos[l(\varphi-\varphi')].\end{aligned}$$

If we combine the north pole of the coordinate system with the point P, then $\theta'=0$, $Y_m(P)=P_m(1)=1$, and $P_m[\cos\gamma(P,Q)]=P_m(\cos\theta)$.

Next, we shall find the coefficients of the eigenfunction expansion of the function $f(r,Q)\in L_1(B)$, $Q\in S$. Following [35, p. 357] we can write

$$f(r,Q)=\sum_{k\ge 0,\, n\ge 1}\widehat{f}_{kn}j_k(\lambda_{kn}r)Y_k(Q).$$

Taking a fixed point $P\in S$ we multiply both sides of this formula by $P_m[\cos\gamma(P,Q)]$, integrate over S and use (2.4) to get

$$\int_S f(r,Q)P_m[\cos\gamma(P,Q)]\,dS_Q=\frac{4\pi}{2m+1}Y_m(P)\sum_{n=1}^{\infty}\widehat{f}_{mn}j_m(\lambda_{mn}r).$$

Substituting the expansion

$$f(r,Q)=\sum_{n=1}^{\infty}\frac{(f,j_m)_{(n)}(Q)}{\|j_m\|_{(n)}^2}j_m(\lambda_{mn}r)$$

into the last equation and equating the coefficients of $j_m(\lambda_{mn}r)$ we find

$$\begin{aligned}\widehat{f}_{mn}Y_m(P)&=\frac{1}{\|j_m\|_{(n)}^2\|Y_m\|_S^2}\int_0^1 r^2 j_m(\lambda_{mn}r)\,dr\\ &\quad\times\int_S f(r,Q)P_m[\cos\gamma(P,Q)]\,dS_Q.\end{aligned}$$

If we combine the north pole of our coordinate system with the point P, then $Y_m(P)=1$, $P_m[\cos\gamma(P,Q)]=P_m(\cos\theta)$, and we deduce the expression for the eigenfunction expansion coefficients

$$\widehat{f}_{mn}=\frac{((f,j_m)_{(n)}(Q),Y_m(Q))_S}{\|j_m\|_{(n)}^2\|Y_m\|_S^2}.$$

Now we shall give some facts concerning Legendre polynomials $P_m(x)$, $-1<x<1$ (see [22, pp. 176–200]). These functions satisfy the equation

$$\frac{d}{dx}\left[(1-x)^2\frac{d}{dx}P_m(x)\right]+m(m+1)P_m(x)=0,\qquad x\in(-1,1).\tag{2.5}$$

The following properties will be essentially used in our analysis:

$$\begin{gathered}|P_m(x)|<1,\quad x\in(-1;1),\quad P_m(1)=1,\quad P_m(-1)=(-1)^m,\\ \int_{-1}^{1}P_m^2(x)\,dx=\frac{2}{2m+1}.\end{gathered}\tag{2.6}$$

First Theorem of Stieltjes. *For $\theta\in(0,\pi)$, $m=1,2,\ldots,$*

$$|P_m(\cos\theta)|\le\frac{4\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{m}\sqrt{\sin\theta}}.\tag{2.7}$$

Second Theorem of Stieltjes. *For $x\in[-1,1]$, $m=0,1,\ldots,$*

$$|P_{m+2}(x)-P_m(x)|\le\frac{4}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{m+2}}.\tag{2.8}$$

The relation

$$P'_{m+1}(x)-P'_{m-1}(x)=(2m+1)P_m(x),\qquad m\ge 1,\tag{2.9}$$

implies that

$$\int_{-1}^{x}P_m(\xi)\,d\xi=\frac{P_{m+1}(x)-P_{m-1}(x)}{2m+1},\qquad m\ge 1.\tag{2.10}$$

It follows from (2.8) and (2.10) that

$$\left|\int_{-1}^{x}P_m(\xi)\,d\xi\right|\le\frac{4}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{m+1}\,(2m+1)}.\tag{2.11}$$

Finally, since $P_0(x)=1$ and $P_1(x)=x$ the orthogonality relation $\int_{-1}^{1}P_k(x)P_m(x)\,dx=0$, $k\ne m$, implies that

$$\int_{-1}^{1}P_m(\xi)\,d\xi=0,\quad m\ge 1,\quad\text{and}\quad\int_{-1}^{1}\xi P_m(\xi)\,d\xi=0,\quad m\ge 2.\tag{2.12}$$

We introduce the Sobolev space $H^\kappa(B)$ with the equivalent norm

$$\|f\|_\kappa^2=\sum_{m\ge 0,\,n\ge 1}\lambda_{mn}^{2\kappa}|\widehat{f}_{mn}|^2\|\chi_{mn}\|_B^2,$$

the space $H_0^\kappa(B)=H^\kappa(B)\cap\{u|_{\partial B}=0\}$, and the Banach space $C^n([0,\infty),H^\kappa(B))$ equipped with the norm

$$\|u\|_{C^n}=\sum_{j=0}^{n}\sup_{t\in[0,\infty)}\|\partial_t^j u(t)\|_k.$$

3. Main results. We study the first initial-boundary value problem for the nonlinear heat equation with a fractional Laplacian

$$
\begin{aligned}
&u_t + (-\Delta)^{\alpha/2} u = u^2, \quad (r,\theta,\varphi) \in B,\ t>0,\\
&u(r,\theta,\varphi,0) = \varepsilon^2\phi(r,\theta,\varphi), \quad (r,\theta,\varphi)\in B,\\
&u|_{\partial B} = 0, \quad t>0,\\
&|u(0,\theta,\varphi,t)| < \infty,\\
&\text{periodicity conditions in } \varphi \text{ with period } 2\pi,
\end{aligned}
\tag{3.1}
$$

where $0<\alpha\le 2$, $\varepsilon = \text{const} > 0$; $\phi(r,\theta,\varphi)$ is a real-valued function.

We set $A = (-\Delta)^{\alpha/2}$, where Δ is defined on sufficiently smooth functions satisfying the conditions (2.1).

DEFINITION. The function $u(t)$ is called a *mild solution* of the problem (3.1) if it satisfies the integral equation

$$u(t) = \varepsilon^2 \exp(-tA)\phi + \int_0^t \exp(-(t-\tau)A)u^2(\tau)\,d\tau, \quad t>0,$$

in some Banach space $C^0([0,\infty),X)$.

Set $\Pi = \{(\theta,\varphi) : \theta\in[0,\pi],\ \varphi\in[0,2\pi]\}$. We fix some small $\delta>0$ and define $B_\delta^{(1)} = \{(r,\theta,\varphi) : r\in[0,\delta),\ (\theta,\varphi)\in\Pi\}$ and $B_\delta = \overline{B}\setminus B_\delta^{(1)}$, so that B_δ is a closed domain. In what follows we shall use the notation $D_\theta = -(1/\sin\theta)\partial_\theta$ and denote by $V_0^1(f(r,\theta,\varphi))$ the total variation of the function $f(r,\theta,\varphi)$ in $r\in[0,1]$. Next, we formulate some assumptions on a sufficiently smooth function $f(r,Q)$, $r\in(0,1)$, $Q\in S$.

ASSUMPTIONS A.

$$
\begin{gathered}
f(0,Q) = f(1,Q) = D_\theta^2 f(0,Q) = D_\theta^2 f(1,Q) = 0;\\
V_0^1(r\partial_r f(r,Q)) = V_{1,0}(Q) \in L_1(S),\\
\lim_{r\to 0+} r\partial_r f(r,Q) = F_{1,0}(Q)\in L_1(S);\\
V_0^1(r\partial_r D_\theta^2 f(r,Q)) = V_{1,2}(Q)\in L_1(S),\\
\lim_{r\to 0+} r\partial_r D_\theta^2 f(r,Q) = F_{1,2}(Q) \in L_1(S).
\end{gathered}
$$

THEOREM 1. *If $3/2+\varepsilon_1 \le \alpha \le 2$ with some small $\varepsilon_1>0$ and the function $\phi(r,\theta,\varphi)$ satisfies Assumptions* A, *then there is $\varepsilon_0>0$ such that for $\varepsilon\in[0,\varepsilon_0]$ there exists a mild solution of the problem* (2.1) *in the space $C^0([0,\infty), H_0^\kappa(B))$, $\kappa<\alpha-1/2$. It can be represented as*

$$u(r,\theta,\varphi,t) = \sum_{m\ge 0,\,n\ge 1}^{\infty} \widehat{u}_{mn}(t) j_m(\lambda_{mn} r) Y_m(\theta,\varphi), \tag{3.2}$$

where the coefficients $\widehat{u}_{mn}(t)$ are defined below (see (3.5), (3.6)). *If $0<\kappa<\alpha-1/2$, the solution is unique.*

COROLLARY 3.1. *Under the hypotheses of Theorem* 1, *$u(r,\theta,\varphi,t)$ is continuous and bounded in $B_\delta^{(2)}\times[0,\infty)$ and can be represented there as*

$$u(r,\theta,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r,\theta,\varphi,t), \tag{3.3}$$

where the functions $u^{(N)}(r,\theta,\varphi,t)$ are defined in the proof (see (6.1)), *and the series converges absolutely and uniformly with respect to $(r,\theta,\varphi)\in B_\delta$, $t\in[0,\infty)$, and $\varepsilon\in[0,\varepsilon_0]$.*

REMARK 3.1. We give an example of an initial function satisfying Assumptions A. Using separation of variables we set $\phi(r,\theta,\varphi) = R_1(r)\Theta_1(\theta)\Phi_1(\varphi)$ and impose the following restrictions:

$$
\begin{gathered}
R_1(0) = R_1(1) = 0, \quad \lim_{r\to 0+} rR_1'(r) = c_3 < \infty,\\
V_0^1(rR_1'(r)) = c_4 < \infty; \quad \Phi_1(\varphi) \in L_1(0,2\pi);\\
D_\theta^2\Theta_1(\theta) \text{ exists for } \theta\in[0,\pi] \quad\text{and}\quad \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)\Theta_1(\theta)\in L_1(0,\pi).
\end{gathered}
$$

We briefly sketch the proof of Theorem 1. We seek solutions of (3.1) in the form of an expansion in eigenfunctions of the Laplace operator in a disk

$$u(r,\theta,\varphi,t) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \widehat{u}_{mn}(t)\chi_{mn}(r,\theta,\varphi).$$

The coefficients of the corresponding expansion of the nonlinearity $(u^2)^\wedge_{mn}(t)$ are calculated via multiplying two series, namely

$$
\begin{aligned}
(u^2)^\wedge_{mn}(t) &= \frac{1}{\|\chi_{mn}\|_B^2}\Big(\sum_{p\ge 0,\, q\ge 1}\widehat{u}_{pq}(t)\chi_{pq}\cdot \sum_{k\ge 0,\, s\ge 1}\widehat{u}_{ks}(t)\chi_{ks}, \chi_{mn}\Big)_B\\
&= \sum_{p,k\ge 0;\, q,s\ge 1} a(m,n,p,q,k,s)\widehat{u}_{pq}(t)\widehat{u}_{ks}(t),
\end{aligned}
\tag{3.4}
$$

where

$$a(m,n,p,q,k,s) = \frac{(j_p j_k, j_m)_{(q,s,n)}\,(Y_pY_k,Y_m)_S}{\|j_m\|^2_{(n)}\|Y_m\|_S^2}.$$

Substituting these series into (3.1) we obtain the Cauchy problem for $\widehat{u}_{mn}(t)$ with $(u^2)^\wedge_{mn}(t)$ on the right-hand side of the equation. Integrating this problem with respect to t we reduce it to a nonlinear integral equation for $\widehat{u}_{mn}(t)$. For solving it we apply perturbation theory, i.e., we represent

$\widehat{u}_{mn}(t)$ as a formal series in ε,

$$\widehat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{v}_{mn}^{(N)}(t), \tag{3.5}$$

substitute it into the integral equation and obtain the following recurrence formulas for integers $m \ge 0$, $n \ge 1$ (it is convenient to keep ε in the coefficients in order to simplify some estimates):

$$\begin{aligned} &\widehat{v}_{mn}^{(0)}(t) = \widehat{\Phi}_{mn} \exp(-\lambda_{mn}^{\alpha} t), \quad \widehat{\Phi}_{mn} = \varepsilon \widehat{\phi}_{mn}, \\ &\widehat{v}_{mn}^{(N)}(t) = \int_0^t \exp[-\lambda_{mn}^{\alpha}(t-\tau)] \sum_{p,k\ge 0;\, q,s\ge 1} a(m,n,p,q,k,s) \\ &\qquad \times \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(\tau) \widehat{v}_{ks}^{(N-j)}(\tau) d\tau, \quad N \ge 1. \end{aligned} \tag{3.6}$$

Then some time estimates of $\widehat{v}_{mn}^{(N)}(t)$ are deduced which permit us to establish that the formally constructed function (3.2), (3.5), (3.6) really represents the mild solution of (3.1) from the required function space.

THEOREM 2. *Under the hypotheses of Theorem* 1, *there exists a constant C such that the following asymptotics holds for all $t \ge 0$ uniformly in space*:

$$\|u - \widetilde{u}_0\|_\kappa \le C \begin{cases} t \exp(-2\pi^\alpha t), & \alpha = \alpha_{\mathrm{cr}}, \\ \exp(-2\pi^\alpha t), & \alpha_{\mathrm{cr}} < \alpha \le 2; \end{cases} \tag{3.7}$$

$$\|u - \widetilde{u}_0 - \widetilde{u}_1\|_\kappa \le C \exp(-2\pi^\alpha t), \quad 3/2 + \varepsilon_1 \le \alpha < \alpha_{\mathrm{cr}},$$

where

$$\widetilde{u}_0(r,\theta,\varphi,t) = \widetilde{A}_0(\varepsilon) \frac{\sin(\pi r)}{\pi r} \exp(-\pi^\alpha t),$$

$$\widetilde{u}_1(r,\theta,\varphi,t) = \widetilde{A}_1(\varepsilon) \frac{1}{\lambda_{11} r} \left[\frac{\sin(\lambda_{11} r)}{\lambda_{11} r} - \cos(\lambda_{11} r) \right] Y_1(\theta,\varphi) \exp(-\lambda_{11}^\alpha t),$$

$$\alpha_{\mathrm{cr}} = \frac{\ln 2}{\ln(\lambda_{11}/\pi)} \simeq 1.937, \quad \lambda_{11} \simeq 4.493;$$

$$2\pi^\alpha > \lambda_{11}^\alpha \quad \textit{for } 3/2 + \varepsilon_1 \le \alpha < \alpha_{\mathrm{cr}},$$

and the constant coefficients $\widetilde{A}_{0,1}(\varepsilon) \sim c\varepsilon^2$ are defined in the proof (see (7.5)).

4. Auxiliary results. In this section we prove several propositions which will permit us to obtain the estimates of the eigenfunction expansion coefficients of the solution. Let the function $f(r,Q)$ be defined on the unit ball B, Q being a point on the unit sphere S. For integers $m \ge 0$

consider the integral

$$\Im_m(\lambda, M) = \int_0^1 r^{3/2} f(r,M) J_{m+1/2}(\lambda r)\, dr, \quad \lambda > 0, \ M \in \Pi.$$

LEMMA 1. *Let $f(r,M)$ have a partial derivative $\partial_r f(r,M)$ for $r \in (0,1)$, $M \in \Pi$, and $f(0,M) = f(1,M) = 0$ (in case $m = 0$ only $f(1,M) = 0$). Moreover, assume that for each fixed $M \in \Pi$ the function $r\partial_r f(r,M)$ has a bounded total variation in $r \in [0,1]$ which is absolutely integrable over Π, i.e.,*

$$V_0^1(r\partial_r f(r,M)) = V_{f,1}(M) \in L_1(\Pi),$$
$$\lim_{r\to 0^+} r\partial_r f(r,M) = F_{1,0}(M) \in L_1(\Pi).$$

Then there exists $C_M \in L_1(\Pi)$ such that for $m \ge 0$, $\lambda > 0$, $M \in \Pi$,

$$|\Im_m(\lambda, M)| \le \frac{C_M(m+1)}{\lambda^{5/2}}.$$

Proof. See [31, Lemma 2].

Next, consider the integral

$$H(m,n,p,k,\lambda_{mn},\lambda_j) = \int_0^1 J_{m+1/2}(\lambda_{mn} r) J_{p+1/2}(\lambda_1 r) J_{k+1/2}(\lambda_2 r) \sqrt{r}\, dr, \quad j = 1,2,$$

where $m,p,k \ge 0$, $n \ge 1$ are integers, $\lambda_1, \lambda_2 > 0$, and λ_{mn} is one of the positive zeros of the function $J_{m+1/2}(x)$. Our purpose is to obtain an estimate of this integral as $\lambda_1, \lambda_2 \to \infty$ tracing the dependence on λ_{mn} as well. First, we present a few auxiliary results concerning the Fresnel integrals [14, p. 28]

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\, dt, \quad S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\, dt.$$

They have the following properties:

$$C(0) = S(0) = 0, \quad C(\infty) = S(\infty) = 1/2,$$
$$C(x) = \frac{1}{2} + \frac{\sin x}{\sqrt{2\pi x}} + O\left(\frac{1}{x^{3/2}}\right),$$
$$S(x) = \frac{1}{2} - \frac{\cos x}{\sqrt{2\pi x}} + O\left(\frac{1}{x^{3/2}}\right) \quad \text{as } x \to \infty.$$

LEMMA 2. *For any fixed $n \ge 1$, any $m,p,k \ge 0$, and positive $\lambda_1, \lambda_2 \to \infty$ there exists a constant C* independent of m,n,p,k,j *such that the following* *estimates hold*:

$$\text{(4.1)}\qquad |H(m,n,p,k,\lambda_{mn},\lambda_j)| \le \frac{C}{\lambda_{mn}^{1/2}\lambda_1\lambda_2^{1/2}},\qquad \lambda_1>\lambda_2;$$

$$\text{(4.2)}\qquad |H(m,n,p,k,\lambda_{mn},\lambda_j)| \le \frac{C}{\lambda_{mn}^{1/2}\lambda_1^{1/2}\lambda_2},\qquad \lambda_1<\lambda_2;$$

$$\text{(4.3)}\qquad |H(m,n,p,k,\lambda_{mn},\lambda_j)| \le \frac{C}{\lambda_{mn}^{1/2}\lambda},\qquad \lambda_1=\lambda_2=\lambda.$$

Proof. We consider $\lambda_1>\lambda_2$ since the case $\lambda_1<\lambda_2$ can be studied analogously. It follows from the theory of Bessel functions [33, p. 479] that for any fixed n the function $J_{m+1/2}(\lambda_{mn}r)$ has $n+1$ intervals of monotonicity in the interval $[0,1]$. We shall denote them by $[0,r_1],[r_1,r_2],\ldots,[r_n,r_{n+1}]$. Then $J_{m+1/2}(\lambda_{mn}r)$ is increasing on $[0,r_1]$, decreasing on $[r_1,r_2]$, etc. Omitting the arguments of H and denoting by $H^{(i)}$ the integral over $[r_i,r_{i+1}]$ we can write $H=\sum_{i=1}^{n+1}H^{(i)}$.

Since $J_{m+1/2}(\lambda_{mn}r)\ge 0$ for $r\in[0,r_1]$ we can apply Bonnet's mean value theorem (see [23, p. 328]) to the corresponding integral to obtain

$$H^{(1)} = J_{m+1/2}(\lambda_{mn}r_1)\int_\eta^{r_1} J_{p+1/2}(\lambda_1 r)J_{k+1/2}(\lambda_2 r)\sqrt{r}\,dr,$$

where $\eta\in[0,r_1]$. In order to reveal the decay of $H^{(1)}$ with respect to λ_1 and λ_2 we make the change of variable $\xi=\lambda_1 r$ to get

$$\text{(4.4)}\qquad H^{(1)} = \frac{J_{m+1/2}(\lambda_{mn}r_1)}{\lambda_1^{3/2}}\int_{\lambda_1\eta}^{\lambda_1 r_1} J_{p+1/2}(\xi)J_{k+1/2}\Big(\frac{\lambda_2}{\lambda_1}\xi\Big)\sqrt{\xi}\,d\xi.$$

Note that for $\nu\ge 0$ and integer $l\ge 0$ (see [25, p. 226], [33, p. 199])

$$\text{(4.5)}\qquad |J_\nu(x)|\le \frac{C}{\sqrt{x}},\qquad x>0,$$

$$\text{(4.6)}\qquad J_{l+1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin(x-l\pi/2)+O\Big(\frac{1}{x^{3/2}}\Big),\qquad x\to\infty.$$

The convergence of the integral in (4.4) as $\lambda_1\to\infty$ follows from (4.6).

If $\eta=0$, then we choose $A>0$ sufficiently large and represent the integral in (4.4) as $\int_0^A+\int_A^{\lambda_1 r_1}$. From the uniform boundedness of the Bessel functions it follows that $|\int_0^A|\le C$, and the asymptotics (4.6) yields $|\int_A^{\lambda_1 r_1}|\le C\sqrt{\lambda_1/\lambda_2}$. Hence

$$\text{(4.7)}\qquad |H^{(1)}|\le \frac{C}{\lambda_{mn}^{1/2}\lambda_2^{1/2}\lambda_1}.$$

If $\eta>0$, then a better estimate can be obtained. Indeed, we can write

$$\int_\eta^{r_1} J_{p+1/2}(\lambda_1 r)J_{k+1/2}(\lambda_2 r)\sqrt{r}\,dr = \frac{2}{\pi\sqrt{\lambda_1\lambda_2}}\Big[\widetilde{H}+O\Big(\frac{1}{\lambda_1\lambda_2}\Big)\Big],$$

$$\widetilde{H}=\int_\eta^{r_1}\frac{\sin(\lambda_1 r-p\pi/2)\sin(\lambda_2 r-k\pi/2)}{\sqrt{r}}\,dr.$$

Setting $\Lambda_-=\lambda_1-\lambda_2=\lambda_1(1-\lambda_2/\lambda_1)>0$ (note that $\Lambda_-\to\infty$ as $\lambda_1\to\infty$) and $\Lambda_+=\lambda_1+\lambda_2$ we get

$$\begin{aligned}\widetilde{H} &= \int_\eta^{r_1}\frac{\cos[\Lambda_- r-(p-k)\pi/2]}{\sqrt{r}}\,dr-\int_\eta^{r_1}\frac{\cos[\Lambda_+ r-(p+k)\pi/2]}{\sqrt{r}}\,dr\\ &= \cos[(p-k)\pi/2]\int_\eta^{r_1}\frac{\cos(\Lambda_- r)}{\sqrt{r}}\,dr+\sin[(p-k)\pi/2]\int_\eta^{r_1}\frac{\sin(\Lambda_- r)}{\sqrt{r}}\,dr\\ &\quad -\cos[(p+k)\pi/2]\int_\eta^{r_1}\frac{\cos(\Lambda_+ r)}{\sqrt{r}}\,dr-\sin[(p+k)\pi/2]\int_\eta^{r_1}\frac{\sin(\Lambda_+ r)}{\sqrt{r}}\,dr.\end{aligned}$$

Consider, for example, the first of these integrals. Making the change of variable $\zeta=\Lambda_- r$ we obtain

$$\Big|\int_\eta^{r_1}\frac{\cos(\Lambda_- r)}{\sqrt{r}}\,dr\Big| = \frac{1}{\sqrt{\Lambda_-}}\Big|\int_{\Lambda_-\eta}^{\Lambda_- r_1}\frac{\cos(\zeta)}{\sqrt{\zeta}}\,d\zeta\Big| = \sqrt{\frac{2\pi}{\Lambda_-}}\,|C(\Lambda_- r_1)-C(\Lambda_-\eta)| \le \frac{C}{\Lambda_-}\le\frac{C}{\lambda_1}$$

as $\lambda_1\to\infty$. The other integrals can be treated analogously. Therefore, taking into account (4.5) we get

$$\text{(4.8)}\qquad |H^{(1)}|\le\frac{C}{\lambda_{mn}^{1/2}\lambda_2^{1/2}\lambda_1^{3/2}}.$$

Next, we estimate the integrals $H^{(i)}$ with $i\ge 2$. Assume, for example, that on $[r_i,r_{i+1}]$ the function $J_{m+1/2}(\lambda_{mn}r)$ is decreasing. Then, by Bonnet's theorem and (4.6), there exists $\eta_i\in[r_i,r_{i+1}]$ such that

$$\begin{aligned}H^{(i)} &= J_{m+1/2}(\lambda_{mn}r_i)\int_{r_i}^{\eta_i}J_{p+1/2}(\lambda_1 r)J_{k+1/2}(\lambda_2 r)\sqrt{r}\,dr\\ &\quad + J_{m+1/2}(\lambda_{mn}r_{i+1})\int_{\eta_i}^{r_{i+1}}J_{p+1/2}(\lambda_1 r)J_{k+1/2}(\lambda_2 r)\sqrt{r}\,dr\end{aligned}$$

$$= \frac{2}{\pi}\Big\{\frac{J_{m+1/2}(\lambda_{mn}r_i)}{\sqrt{\lambda_1\lambda_2}}$$
$$\times\Big[\int_{r_i}^{\eta_i}\frac{\sin(\lambda_1 r - p\pi/2)\sin(\lambda_2 r - k\pi/2)}{\sqrt{r}}\,dr + O\Big(\frac{1}{\lambda_1\lambda_2}\Big)\Big]$$
$$+\frac{J_{m+1/2}(\lambda_{mn}r_{i+1})}{\sqrt{\lambda_1\lambda_2}}$$
$$\times\Big[\int_{\eta_i}^{r_{i+1}}\frac{\sin(\lambda_1 r - p\pi/2)\sin(\lambda_2 r - k\pi/2)}{\sqrt{r}}\,dr + O\Big(\frac{1}{\lambda_1\lambda_2}\Big)\Big]\Big\}.$$

Reducing these integrals to the Fresnel integrals we can show that as $\lambda_1,\lambda_2\to\infty$, $\lambda_1>\lambda_2$,

$$|H^{(i)}| \le \frac{C}{\lambda_{mn}^{1/2}\lambda_1^{3/2}\lambda_2^{1/2}}. \tag{4.9}$$

Now (4.1) follows from (4.7)–(4.9). The estimate (4.2) is established in an analogous way.

Let $\lambda_1=\lambda_2=\lambda$. Applying the first mean value theorem for integrals we deduce that there exists $\xi_1\in(0,1)$ such that

$$|H| \le |J_{m+1/2}(\lambda_{mn}\xi_1)|\Big|\int_0^1 |J_{p+1/2}(\lambda r)|\cdot|J_{k+1/2}(\lambda r)|\sqrt{r}\,dr \le \frac{C}{\sqrt{\lambda_{mn}}\,\lambda}.$$

The estimate (4.3) is established. ∎

Now we should study the integral

$$I_{pkm} = (Y_pY_k, Y_m)_S = \int_S Y_p(Q)Y_k(Q)Y_m(Q)\,dS_Q$$

from the point of view of obtaining the decay estimate in m tracing the dependence on p and k at the same time. The following two propositions provide two different estimates of this integral.

LEMMA 3. *For all integers $p,k,m\ge 0$ there exists a constant C independent of p,k,m such that*

$$|I_{pkm}| \le \frac{C}{\sqrt{(m+1)(p+1)(k+1)}}.$$

Proof. In the chosen coordinate system with the north pole at the point P we have

$$I_{pkm} = \int_S P_p[\cos\gamma(P,Q)]P_k[\cos\gamma(P,Q)]P_m[\cos\gamma(P,Q)]\,dS_Q \tag{4.10}$$

$$= \int_0^{2\pi} d\varphi\int_0^{\pi} P_p(\cos\theta)P_k(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta$$
$$= 2\pi\int_{-1}^{1} P_p(x)P_k(x)P_m(x)\,dx.$$

If $p=k=m=0$, then the required estimate follows from the equality $P_0(x)=1$. If $p,k,m\ge 1$, then, by the first theorem of Stieltjes (see (2.7)), we have

$$|I_{pkm}| \le \frac{2\pi}{\sqrt{pkm}}\Big(\frac{4\sqrt{2}}{\sqrt{\pi}}\Big)^3\int_{-1}^{1}\frac{dx}{(1-x^2)^{3/4}} \le \frac{C}{\sqrt{(m+1)(p+1)(k+1)}}. \ ∎$$

LEMMA 4. *For all integers $p,k,m\ge 0$,*

$$|I_{pkm}| \le \frac{16}{(2m+1)\sqrt{(m+1)}}\Big\{\sqrt{p+1}\ln(p+1)+\sqrt{k+1}\ln(k+1)$$
$$+\frac{\ln 2}{2}(\sqrt{p+1}+\sqrt{k+1})+2\sqrt{\pi}\Big\}.$$

Proof. Integrating by parts in the representation (4.10) and applying (2.10), (2.12) we get

$$I_{pkm} = -2\pi\int_{-1}^{1}\Big(\int_{-1}^{x}P_m(\xi)\,d\xi\Big)[P_p'(x)P_k(x)+P_p(x)P_k'(x)]\,dx$$
$$= -\frac{2\pi}{2m+1}\int_{-1}^{1}[P_{m+1}(x)-P_{m-1}(x)][P_p'(x)P_k(x)+P_p(x)P_k'(x)]\,dx.$$

First, we consider the integral containing $P_p'(x)$. This derivative can be represented as

$$P_p'(x) = -p(p+1)\frac{\int_{-1}^{x}P_p(\xi)\,d\xi}{1-x^2} = -\frac{p(p+1)}{2}[G_p^{(1)}(x)-G_p^{(2)}(x)],$$
$$G_p^{(1)}(x) = \frac{\int_{-1}^{x}P_p(\xi)\,d\xi}{1+x},\quad G_p^{(2)}(x) = \frac{\int_{x}^{1}P_p(\xi)d\xi}{1-x},\quad -1<x<1.$$

By l'Hospital's rule and (2.12),

$$\lim_{x\to-1}G_p^{(1)}(x) = (-1)^p,\quad G_p^{(1)}(1)=0,\quad \lim_{x\to 1}G_p^{(2)}(x)=-1,\quad G_p^{(2)}(-1)=0.$$

Using Lagrange's mean value theorem for the intervals $[-1,x]$ and $[x,1]$ we deduce that

$$G_p^{(1)}(x) = P_p(\widetilde{\eta}),\quad \widetilde{\eta}\in(-1,x);\qquad G_p^{(2)}(x) = P_p(\overline{\eta}),\quad \overline{\eta}\in(x,1).$$

Therefore, $|G_p^{(1,2)}(x)|\le 1$ for $x\in[-1,1]$ and all $p\ge 0$.

Next, we estimate the integral

$$\bar{I}_{pkm} = \frac{p(p+1)\pi}{2m+1} \int_{-1}^{1} [P_{m+1}(x) - P_{m-1}(x)][G_p^{(1)}(x) - G_p^{(2)}(x)] P_k(x)\, dx.$$

To this end we take some positive $\delta_1(p) < 1$, divide the interval $[-1, 1]$ into three parts $[-1, -1+\delta_1]$, $[-1+\delta_1, 1-\delta_1]$, and $[1-\delta_1, 1]$, and denote by I_1, I_2, and I_3 the corresponding integrals. Then $\bar{I}_{pkm} = I_1 + I_2 + I_3$. By the second theorem of Stieltjes (see (2.8)), we obtain

$$(4.11) \qquad |I_{1,3}| \le \frac{16\sqrt{\pi}\, p(p+1)\delta_1(p)}{\sqrt{m+1}\,(2m+1)}.$$

It remains to estimate the integral

$$I_2 = \frac{p(p+1)\pi}{2m+1} \int_{-1+\delta_1}^{1-\delta_1} [P_{m+1}(x) - P_{m-1}(x)] P_k(x) \frac{\int_{-1}^{x} P_p(\xi)\, d\xi}{1-x^2}\, dx.$$

Using the inequality (2.11) for $\int_{-1}^{x} P_p(\xi)\, d\xi$ and (2.8) for the difference in square brackets we get

$$(4.12) \qquad |I_2| \le \frac{8\sqrt{p+1}\,\widetilde{I}(\delta_1)}{\sqrt{m+1}\,(2m+1)}, \qquad \widetilde{I}(\delta_1) = \int_{-1+\delta_1}^{1-\delta_1} \frac{dx}{1-x^2} = \ln\left|\frac{2-\delta_1}{\delta_1}\right|.$$

Choosing $\delta_1(p) = 1/(p+1)^2$ and combining (4.11) and (4.12) we obtain

$$|\bar{I}_{pkm}| \le \frac{16}{(2m+1)\sqrt{m+1}} \left[\sqrt{p+1}\ln(p+1) + \frac{\ln 2}{2}\sqrt{p+1} + 2\sqrt{\pi}\right].$$

Using symmetry we replace p by k and deduce an analogous estimate for the integral containing $P_k'(x)$ in the representation of I_{pkm}. Adding these inequalities we establish the required estimate. ∎

REMARK 4.1. In the special case $m = 0$ we have

$$(4.13) \qquad I_{pk0} = 2\pi \int_{-1}^{1} P_p(x) P_k(x)\, dx = \begin{cases} 0, & p \ne k, \\ 4\pi/(2p+1), & p = k. \end{cases}$$

Analogous results hold if $p = 0$ or $k = 0$. If $p = k = m \ge 1$, then by the first theorem of Stieltjes (see (2.7)) we get

$$|I_{pkm}| = 2\pi \left| \int_0^{\pi} [P_m(\cos\theta)]^3 \sin\theta\, d\theta \right| \le 4\pi \left(\frac{4\sqrt{2}}{\sqrt{\pi}}\right)^3 \frac{1}{m^{3/2}}.$$

LEMMA 5. *If $f(r, Q)$ satisfies Assumptions* A, *then for all integers $m \ge 0$, $n \ge 1$,*

$$(4.14) \qquad |\widehat{f}_{mn}| \le \frac{C}{\lambda_{mn}^{3/2}\sqrt{m+1}}.$$

Proof. First, we consider $m = 0, 1$. By Lemma 1, we have

$$|\widehat{f}_{mn}| \le C\lambda_{mn} \int_0^{2\pi} d\varphi \int_0^{\pi} |P_m(\cos\theta)| \sin\theta \left| \int_0^1 r^2 j_m(\lambda_{mn} r) f(r,\theta,\varphi)\, dr \right| \le \frac{C}{\lambda_{mn}^{3/2}}.$$

Next, we examine $m \ge 2$. Setting $z = \cos\theta$ we introduce the function

$$\wp_m^{(2)}(z) = \int_{-1}^{z} d\xi \int_{-1}^{\xi} P_m(\eta)\, d\eta = \int_{-1}^{z} (z-\xi) P_m(\xi)\, d\xi$$

and observe that $\wp_m^{(2)}(1) = 0$. It follows from (2.10) that

$$\wp_m^{(2)}(z) = \frac{1}{2m+1}\left[\int_{-1}^{z} P_{m+1}(\xi)\, d\xi - \int_{-1}^{z} P_{m-1}(\xi)\, d\xi\right]$$
$$= \frac{1}{2m+1}\left[\frac{P_{m+2}(z) - P_m(z)}{2m+3} - \frac{P_m(z) - P_{m-2}(z)}{2m-1}\right], \qquad m \ge 2.$$

Hence

$$|\wp_m^{(2)}(z)| \le \frac{C}{m^{5/2}}, \qquad m \ge 2.$$

Introduce the integral

$$\langle f\rangle(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta,\varphi)\, d\varphi = F(r, \cos\theta) = F(r, z).$$

It is the mean value of the function $f(r,\theta,\varphi)$ along the parallel, all of whose points have colatitude θ. Each plane characterized by the condition $\theta = \text{const}$ has a distance $z = \cos\theta$ from the center of the unit sphere S.

Now we study the integral

$$\Gamma_m(r) = \int_0^{\pi} F(r, \cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \int_{-1}^{1} F(r, z) P_m(z)\, dz.$$

Integrating two times by parts we get

$$\Gamma_m(r) = \int_{-1}^{1} \wp_m^{(2)}(z) \partial_z^2 F(r, z)\, dz = \int_0^{\pi} \wp_m^{(2)}(\cos\theta) D_\theta^2 F(r, \cos\theta) \sin\theta\, d\theta.$$

Then, by Lemma 1, we obtain

$$|\widehat{f}_{mn}| \le C\lambda_{mn}(2m+1)\int_0^{2\pi} d\varphi \int_0^{\pi} |\wp_m^{(2)}(\cos\theta)|\,d\theta$$
$$\times \left|\int_0^1 r^2 j_m(\lambda_{mn} r)\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right) f(r,\theta,\varphi)\,dr\right|$$
$$\le \frac{C}{\lambda_{mn}^{3/2}\sqrt{m+1}}. \ ■$$

LEMMA 6. *For the functions $\widehat{v}_{mn}^{(N)}(t)$ defined by (3.6) the following estimate holds for integers $m \ge 0$, $n \ge 1$, $N \ge 0$, and real $t \ge 0$:*

$$|\widehat{v}_{mn}^{(N)}(t)| \le c^N(N+1)^{-2}\lambda_{mn}^{-(\alpha-1/2)}(m+1)^{-1/2}\exp(-\lambda_{01}^{\alpha}t). \tag{4.15}$$

Proof. Since the function $\phi(r,\theta,\varphi)$ satisfies Assumptions A, its eigenfunction expansion coefficients $\widehat{\phi}_{mn}$ satisfy (4.10). We use induction on N. For $N = 0$ and sufficiently small ε we have

$$|\widehat{v}_{mn}^{(N)}(t)| \le \varepsilon|\widehat{\phi}_{mn}|\exp(-\lambda_{mn}^{\alpha}t) \le \lambda_{mn}^{-3/2}(m+1)^{-1/2}\exp(-\lambda_{01}^{\alpha}t).$$

We assume that (4.11) is valid for $\widehat{v}_{mn}^{(k)}(t)$ with $0 \le k \le N-1$ and prove that it holds for $k = N$. To this end we shall need the inequality (see [19, p. 181])

$$j^{-2}(N+1-j)^{-2} \le 2^2(N+1)^{-2}[j^{-2} + (N+1-j)^{-2}], \quad 1 \le j \le N.$$

By (3.6), for $N \ge 1$ we have

$$|\widehat{v}_{mn}^{(N)}(t)| \le \int_0^t \exp[-\lambda_{mn}^{\alpha}(t-\tau)] \sum_{p,k\ge 0;\, q,s\ge 1} |a(m,n,p,q,k,s)| \tag{4.16}$$
$$\times \left|\sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(\tau)\widehat{v}_{ks}^{(N-j)}(\tau)\right| d\tau$$
$$\le cL_{mn}^{(\alpha)}(t)\Psi^{(N)} \sum_{p,k\ge 0;\, q,s\ge 1} |a(m,n,p,q,k,s)|$$
$$\times \lambda_{pq}^{\alpha-1/2}\lambda_{ks}^{\alpha-1/2}(p+1)^{-1/2}(k+1)^{-1/2},$$
$$L_{mn}^{(\alpha)}(t) = \exp(-\lambda_{mn}^{\alpha}t)\int_0^t \exp[(\lambda_{mn}^{\alpha} - 2\lambda_{01}^{\alpha})\tau]\,d\tau,$$
$$\Psi^{(N)} = \sum_{j=1}^{N} c^{j-1}c^{N-j}j^{-2}(N+1-j)^{-2} \le c^{N-1}(N+1)^{-2},$$

and the coefficients $a(m,n,p,q,k,s)$ are defined by (3.4). By means of Lemmas 2 and 4 we see that for $n \ge 1$ and $m,p,k \ge 0$, $q \ge q_0 > 0$, $s \ge s_0 > 0$

(q_0, s_0 being sufficiently large),

$$|a(m,n,p,q,k,s)| \le C\frac{\lambda_{mn}^{1/2}}{\sqrt{m+1}}\Big[\sqrt{p+1}\ln(p+1) + \sqrt{k+1}\ln(k+1)$$
$$+ \frac{\ln 2}{2}(\sqrt{p+1} + \sqrt{k+1})/2 + 2\sqrt{\pi}\Big]$$
$$\times \begin{cases} \lambda_{pq}^{-1}\lambda_{ks}^{-1/2}, & \lambda_{pq} > \lambda_{ks}; \\ \lambda_{ks}^{-1}\lambda_{pq}^{-1/2}, & \lambda_{pq} < \lambda_{ks}; \\ \lambda_{pq}^{-1}, & p = k,\ q = s. \end{cases}$$

Taking the major terms with respect to p and k in square brackets in the last estimate we get

$$|\widehat{v}_{mn}^{(N)}(t)| \le cL_{mn}^{(\alpha)}(t)\Psi^{(N)}\Bigg\{\sum_{\substack{p,q,k,s:\\ \lambda_{pq}>\lambda_{ks}}} \frac{\sqrt{p+1}\ln(p+1) + \sqrt{k+1}\ln(k+1)}{\lambda_{pq}^{\alpha-1/2}\lambda_{ks}^{\alpha-1/2}\sqrt{p+1}\sqrt{k+1}\,\lambda_{pq}\lambda_{ks}^{1/2}}$$
$$+ \sum_{\substack{p,q,k,s:\\ \lambda_{pq}<\lambda_{ks}}} \frac{\sqrt{p+1}\ln(p+1) + \sqrt{k+1}\ln(k+1)}{\lambda_{pq}^{\alpha-1/2}\lambda_{ks}^{\alpha-1/2}\sqrt{p+1}\sqrt{k+1}\,\lambda_{ks}\lambda_{pq}^{1/2}} + \sum_{p,q} \frac{\sqrt{p+1}\ln(p+1)}{\lambda_{pq}^{2(\alpha-1/2)}(p+1)\lambda_{pq}}\Bigg\}$$
$$\le cL_{mn}^{(\alpha)}(t)\Psi^{(N)}[\sigma_1(\alpha) + \sigma_2(\alpha) + \sigma_3(\alpha)] \le cL_{mn}^{(\alpha)}(t)\Psi^{(N)},$$

where

$$\sigma_1(\alpha) = \sum_{p,q} \frac{\ln(p+1)}{\lambda_{pq}^{\alpha+1/2}} \sum_{k,s} \frac{1}{\lambda_{ks}^{\alpha}\sqrt{k+1}},$$
$$\sigma_2(\alpha) = \sum_{p,q} \frac{1}{\lambda_{pq}^{\alpha}\sqrt{p+1}} \sum_{k,s} \frac{\ln(k+1)}{\lambda_{ks}^{\alpha+1/2}\sqrt{k+1}},$$
$$\sigma_3(\alpha) = \sum_{p,q} \frac{\sqrt{p+1}\ln(p+1)}{\lambda_{pq}^{2\alpha}(p+1)}.$$

Here we have used the inequalities $\lambda_{pq}^{-(\alpha+1/2)} < \lambda_{pq}^{-\alpha}\lambda_{ks}^{-1/2}$ for $\lambda_{pq} > \lambda_{ks}$ and $\lambda_{ks}^{-(\alpha+1/2)} < \lambda_{pq}^{-\alpha}\lambda_{ks}^{-1/2}$ for $\lambda_{pq} < \lambda_{ks}$. By Fubini–Tonelli's theorem (see [23]), if the iterated series $\sum_m\sum_n |a_{mn}|$ converges, then the double series $\sum_{m,n}|a_{mn}|$ converges and $\sum_{m,n} a_{mn} = \sum_m\sum_n a_{mn}$. The convergence of the iterated series

$$\sum_{p\ge 0}\ln(p+1)\sum_{q\ge 1}\frac{1}{\lambda_{pq}^{\alpha+1/2}} \quad\text{and}\quad \sum_{k\ge 0}\frac{1}{\sqrt{k+1}}\sum_{s\ge 1}\frac{1}{\lambda_{ks}^{\alpha}}$$

is established by means of the asymptotics (2.3) and comparison with the integrals

$$\int_A^\infty \ln(p+1)\,dp \int_B^\infty \frac{dq}{(p+2q)^{\alpha+1/2}}, \qquad \int_A^B \frac{dk}{\sqrt{k+1}} \int_B^\infty \frac{ds}{(k+2s)^\alpha}$$

with sufficiently large $A, B > 0$. It holds for $3/2 + \varepsilon_1 \le \alpha \le 2$ with some small $\varepsilon_1 > 0$.

Next, we prove that

$$L^{(\alpha)}_{mn}(t) \le C\frac{\exp(-\lambda^\alpha_{01}t)}{\lambda^\alpha_{mn}}. \tag{4.17}$$

(i) If $m = 0$, $n = 1$, then

$$L^{(\alpha)}_{01}(t) = \exp(-\lambda^\alpha_{01}t)\int_0^t \exp(-\lambda^\alpha_{01}\tau)\,d\tau \le \frac{\exp(-\lambda^\alpha_{01}t)}{\lambda^\alpha_{01}}.$$

(ii) For $m = 0, n \ge 2$ we have $\lambda^\alpha_{0n} - 2\lambda^\alpha_{01} = \pi^\alpha(n^\alpha - 2) > 0$ for $3/2 + \varepsilon_1 \le \alpha < \alpha_{\rm cr}$ with $\varepsilon_1 > 0$. Therefore,

$$\begin{aligned} L^{(\alpha)}_{0n}(t) &= \exp(-\lambda^\alpha_{0n}t)\int_0^t \exp[(\lambda^\alpha_{0n} - 2\lambda^\alpha_{01})\tau]\,d\tau \\ &= \exp(-\lambda^\alpha_{0n}t)\frac{\exp[(\lambda^\alpha_{0n} - 2\lambda^\alpha_{01})t]}{\lambda^\alpha_{0n} - 2\lambda^\alpha_{01}} \\ &\le \frac{\exp(-2\lambda^\alpha_{01}t)}{\lambda^\alpha_{0n} - 2\lambda^\alpha_{01}} \le \frac{\exp(-2\lambda^\alpha_{01}t)}{\lambda^\alpha_{0n}[1 - 2(\lambda_{01}/\lambda_{0n})^\alpha]} \le C\frac{\exp(-\lambda^\alpha_{01})t}{\lambda^\alpha_{0n}}. \end{aligned}$$

(iii) If $m = 1$, $n = 1$, then $\lambda_{11} \simeq 4.493$. Consequently, $\lambda^\alpha_{11} - 2\lambda^\alpha_{01} = \lambda^\alpha_{11} - 2\pi^\alpha = \pi^\alpha[(\lambda_{11}/\pi)^\alpha - 2] < 0$ for $3/2 + \varepsilon_1 \le \alpha < \alpha_{\rm cr}$, where $\alpha_{\rm cr} = \ln 2/\ln(\lambda_{11}/\pi) \simeq 1.937$; $\lambda^\alpha_{11} - 2\lambda^\alpha_{01} = 0$ for $\alpha = \alpha_{\rm cr}$; and $\lambda^\alpha_{11} - 2\lambda^\alpha_{01} > 0$ for $\alpha_{\rm cr} < \alpha \le 2$.

Therefore, for $3/2 + \varepsilon_1 \le \alpha < \alpha_{\rm cr}$,

$$\begin{aligned} L^{(\alpha)}_{11}(t) &= \exp(-\lambda^\alpha_{11}t)\int_0^t \exp[-(2\lambda^\alpha_{01} - \lambda^\alpha_{11})\tau]\,d\tau \\ &= \exp(-\lambda^\alpha_{11}t)\frac{1 - \exp[-(2\lambda^\alpha_{01} - \lambda^\alpha_{11})t]}{2\lambda^\alpha_{01} - \lambda^\alpha_{11}} \le C\frac{\exp(-\lambda^\alpha_{11}t)}{\lambda^\alpha_{11}}. \end{aligned}$$

If $\alpha = \alpha_{\rm cr}$, then

$$L^{(\alpha)}_{11}(t) = t\exp(-\lambda^\alpha_{11}t) = t\exp(-2\lambda^\alpha_{01}t) \le C\frac{\exp(-\lambda^\alpha_{01}t)}{\lambda^\alpha_{11}}.$$

Finally, for $\alpha_{\rm cr} < \alpha \le 2$ we can repeat the arguments of item (ii) to get

$$\begin{aligned} L^{(\alpha)}_{11}(t) &= \exp(-\lambda^\alpha_{11}t)\int_0^t \exp[(\lambda^\alpha_{11} - 2\lambda^\alpha_{01})\tau]\,d\tau \\ &\le \frac{\exp(-2\lambda^\alpha_{01}t)}{\lambda^\alpha_{11} - 2\lambda^\alpha_{01}} \le C\frac{\exp(-\lambda^\alpha_{01}t)}{\lambda^\alpha_{11}}. \end{aligned}$$

(iv) If $m \ge 1$, $n \ge 2$, then $\lambda^\alpha_{mn} - 2\lambda^\alpha_{01} \ge \pi[(\lambda_{12}/\pi)^\alpha - 2] > 0$ for all $3/2 + \varepsilon_1 \le \alpha \le 2$ since $\lambda_{12} \simeq 7.725$. Therefore, the same arguments as in (ii) yield

$$L^{(\alpha)}_{mn}(t) \le \frac{\exp(-2\lambda^\alpha_{01}t)}{\lambda^\alpha_{0n} - 2\lambda^\alpha_{01}} \le C\frac{\exp(-\lambda^\alpha_{01})t}{\lambda^\alpha_{0n}}.$$

Combining (4.16) and (4.17) we establish (4.15). ∎

COROLLARY 4.1. *For $N \ge 0$, $t > 0$, $m = 0$, $n \ge 2$ the following inequalities hold*:

$$|\widehat{v}^{(N)}_{mn}(t)| \le c^N(N+1)^{-2}\lambda_{mn}^{-(\alpha-1/2)}(m+1)^{-1/2}\exp(-2\lambda^\alpha_{01}t), \tag{4.18}$$

and for $m \ge 1$, $n \ge 1$,

$$|\widehat{v}^{(N)}_{mn}(t)| \le c^N(N+1)^{-2}\lambda_{mn}^{-(\alpha-1/2)}(m+1)^{-1/2} \times \begin{cases} \exp(-\lambda^\alpha_{11}t), & 3/2 + \varepsilon_1 \le \alpha < \alpha_{\rm cr}; \\ t\exp(-2\lambda^\alpha_{01}t), & \alpha = \alpha_{\rm cr}; \\ \exp(-2\lambda^\alpha_{01}t), & \alpha_{\rm cr} < \alpha \le 2. \end{cases}$$

Proof. First, we consider the case $m = 0$, $n \ge 2$. We use induction on N as before. For $N = 0$ and sufficiently small ε we have

$$|\widehat{v}^{(0)}_{0n}(t)| \le \lambda_{0n}^{-3/2}\exp(-\lambda^\alpha_{02}t) \le \lambda_{0n}^{-3/2}\exp(-2\lambda^\alpha_{01}t).$$

Next, we assume that (4.18) holds for all $\widehat{v}^{(j)}_{mn}(t)$ with $0 \le j \le N-1$ and prove it for $j = N$. According to (4.15), (4.16) the time decay of $\widehat{v}^{(N)}_{0n}(t)$ is determined by that term in the integrand that has the weakest decay in t, i.e., by $\sum_{j=1}^N \widehat{v}^{(j-1)}_{0q}(\tau)\widehat{v}^{(N-j)}_{0s}(\tau)$. Therefore, it is determined by the factor

$$L_{0n}(t) = \int_0^t \exp[-\lambda^\alpha_{0n}(t-\tau)][\exp(-2\lambda^\alpha_{01})\tau]\,d\tau \le c\frac{\exp(-2\lambda^\alpha_{01}t)}{\lambda^\alpha_{0n}}.$$

Thus, we obtain the estimate of item (ii) with the exponential multiplier $\exp(-2\lambda^\alpha_{01}t)$. Since $\lambda^\alpha_{mn} - 2\lambda^\alpha_{01} > 0$ as $m \ge 1$, $n \ge 2$, analogous considerations yield (4.18) for $m \ge 1$, $n \ge 2$.

In order to prove (4.19) we again apply induction on N. For $\widehat{v}^{(0)}_{mn}(t)$ the estimate is evident. Assuming that it holds for $\widehat{v}^{(j)}_{mn}(t)$ with $0 \le j \le N-1$, for the time factor in $\widehat{v}_{mn}^{(N)}(t)$ we have

$$L_{mn}(t) \le \exp(-\lambda_{mn}^\alpha t) \times \int_0^t \exp(\lambda_{mn}^\alpha \tau)[c_1 \exp(-2\lambda_{01}^\alpha)\tau + c_2\tau \exp[-(\lambda_{01}^\alpha + \lambda_{11}^\alpha)\tau]\, d\tau.$$

Since $\lambda_{mn}^\alpha - 2\lambda_{01}^\alpha \ge \lambda_{11}^\alpha - 2\lambda_{01}^\alpha$, we can repeat the considerations of item (iii) and arrive at the inequalities (4.19). ∎

5. Proof of Theorem 1

5.1. *Construction of solutions.* We seek mild solutions of (3.1) in the form of an eigenfunction expansion

$$u(r,\theta,\varphi,t) = \sum_{m\ge 0,\, n\ge 1} \widehat{u}_{mn}(t)\chi_{mn}(r,\theta,\varphi), \tag{5.1}$$

where

$$\widehat{u}_{mn}(t) = \frac{(u,\chi_{mn})_B(t)}{\|\chi_{mn}\|_B^2}, \qquad \chi_{mn}(r,\theta,\varphi) = j_m(\lambda_{mn} r) Y_m(\theta,\varphi).$$

Expanding the nonlinearity u^2 in a series of type (5.1) with $(u^2)^\wedge_{mn}(t)$ defined by (3.4) we substitute this expansion and (5.1) into (3.1) to obtain

$$\begin{aligned} \widehat{u}'_{mn}(t) + \lambda_{mn}^\alpha \widehat{u}_{mn}(t) &= \beta(u^2)^\wedge_{mn}(t), \quad t > 0,\\ \widehat{u}_{mn}(0) &= \varepsilon^2 \widehat{\phi}_{mn}, \end{aligned} \tag{5.2}$$

where $\widehat{\phi}_{mn}$ are the coefficients of the eigenfunction expansion of the initial function, namely:

$$\phi(r,\theta,\varphi) = \sum_{m\ge 0,\, n\ge 1} \widehat{\phi}_{mn}\chi_{mn}(r,\theta,\varphi), \qquad \widehat{\phi}_{mn} = \frac{(\phi,\chi_{mn})_B}{\|\chi_{mn}\|_B^2}.$$

Setting $\widehat{\Phi}_{mn} = \varepsilon\widehat{\phi}_{mn}$ we integrate the Cauchy problem (5.2) in t to get

$$\widehat{u}_{mn}(t) = \varepsilon\widehat{\Phi}_{mn} \exp(-\lambda_{mn}^\alpha t) + \int_0^t \exp[-\lambda_{mn}^\alpha (t-\tau)](u^2)^\wedge_{mn}(\tau)\, d\tau. \tag{5.3}$$

To solve this nonlinear integral equation we employ perturbation theory. Representing $\widehat{u}_{mn}(t)$ as a formal series in ε (see (3.5)) we substitute it into (5.3) and obtain the formula (3.6) for the series coefficients $\widehat{v}_{mn}^{(N)}(t)$. By Lemma 6, the estimates (4.15) are valid for $\widehat{v}_{mn}^{(N)}(t)$ with $m \ge 0$, $n \ge 1$ and, by Corollary 4.1, the inequalities (4.18) hold for these functions with $m \ge 1$, $n \ge 2$ and (4.19) for $m \ge 1$, $n \ge 1$.

Next, we prove that the formally constructed function (5.1), (5.3), (3.5), (3.6) is really a mild solution of (3.1) (i.e., solution of (3.2)) from the space $C^0([0,\infty), H^\kappa(B))$, $\kappa < \alpha - 1/2$. Choosing ε so that $\varepsilon \le \varepsilon_0 < 1/c$, where c is the constant in the estimates (4.15), and using (3.5) we deduce that for $m \ge 0$, $n \ge 1$,

$$|\widehat{u}_{mn}(t)| \le c\lambda_{mn}^{-(\alpha-1/2)}(m+1)^{-1/2}\exp(-\lambda_{01}^\alpha t). \tag{5.4}$$

By means of (2.2), (2.4), and the last estimate we can establish that the series

$$\|u(t)\|_\kappa^2 = \sum_{m\ge 0,\, n\ge 1} \lambda_{mn}^{2\kappa} |\widehat{u}_{mn}(t)|^2 \|Y_m\|_S^2 \|j_m\|_{(n)}^2$$

converges absolutely and uniformly with respect to $t \ge 0$ for $\kappa < \alpha - 1/2$. Indeed, applying the Fubini–Tonelli theorem we can prove the convergence of the iterated series $\sum_m \sum_n$ via comparing with the integral

$$\int_A^\infty \frac{dm}{(m+1)(2m+1)} \int_B^\infty \frac{dn}{(m+2n)^{2\alpha-2\kappa}}$$

with sufficiently large $A, B > 0$. The restriction $\kappa < \alpha - 1/2$ secures the convergence of the inner integral. ∎

5.2. *Uniqueness of solutions.* Assume that there exist two mild solutions $u^{(1)}$ and $u^{(2)}$ of the problem (3.1) from the class stated in the theorem. Then each of them can be expanded into a series (5.1), where the coefficients $\widehat{u}_{mn}^{(i)}(t)$, $i = 1, 2$, satisfy (5.3) and (5.4). Setting $w = u^{(1)} - u^{(2)}$ we expand it in a series of type (5.1) and get

$$\begin{aligned} w(r,\theta,\varphi,t) &= \sum_{m\ge 0,\, n\ge 1} \widehat{w}_{mn}(t)\chi_{mn}(r,\theta,\varphi),\\ \widehat{w}_{mn}(t) &= \int_0^t \exp[-\lambda_{mn}^\alpha (t-\tau)]\widetilde{F}_{mn}(\tau)\, d\tau,\\ \widetilde{F}_{mn}(t) &= [(u^{(1)})^2]^\wedge_{mn}(t) - [(u^{(2)})^2]^\wedge_{mn}(t). \end{aligned} \tag{5.5}$$

By (3.4), we can write

$$\widetilde{F}_{mn}(t) = \sum_{p,q,k,s} a(m,n,p,q,k,s)[\widehat{u}_{pq}^{(1)}\widehat{w}_{ks}(t) + \widehat{u}_{ks}^{(2)}\widehat{w}_{pq}(t)].$$

Using Lemmas 2 and 3 we can estimate these terms as follows:

$$\Big|\sum_{p,q,k,s} a(m,n,p,q,k,s)\widehat{u}_{pq}^{(1)}(t)\widehat{w}_{ks}(t)\Big| \le C\frac{(2m+1)\lambda_{mn}^{1/2}}{\sqrt{m+1}}(S_1+S_2+S_3),$$

where

$$S_1 = \sum_{\substack{p,q,k,s:\\ \lambda_{pq}>\lambda_{ks}}} \frac{|\widehat{u}^{(1)}_{pq}(t)|}{\sqrt{(p+1)(k+1)}} \cdot \frac{|\widehat{w}_{ks}(t)|}{\lambda_{pq}\lambda_{ks}^{1/2}},$$

$$S_2 = \sum_{\substack{p,q,k,s:\\ \lambda_{pq}>\lambda_{ks}}} \frac{|\widehat{u}^{(1)}_{pq}(t)|}{\sqrt{(p+1)(k+1)}} \cdot \frac{|\widehat{w}_{ks}(t)|}{\lambda_{pq}^{1/2}\lambda_{ks}},$$

$$S_3 = \sum_{k,s} \frac{|\widehat{u}^{(1)}_{ks}(t)|}{k+1} \cdot \frac{|\widehat{w}_{ks}(t)|}{\lambda_{ks}}.$$

We shall estimate only the sum S_1 since $S_{2,3}$ can be treated analogously. By means of the Cauchy–Schwarz inequality and (5.4), we have, for any $æ > 0$,

$$\begin{aligned} S_1 &\le C \sum_{k,s} \sqrt{\frac{2k+1}{k}} \frac{1}{\lambda_{ks}^{æ+1}} \cdot \frac{\lambda_{ks}^{æ}|\widehat{w}_{ks}(t)|}{\sqrt{2k+1}\,\lambda_{ks}^{1/2}} \sum_{p,q} \frac{|\widehat{u}^{(1)}_{pq}(t)|}{\sqrt{p+1}} \\ &\le C\Big(\sum_{k,s}\frac{1}{\lambda_{ks}^{2æ+2}}\Big)^{1/2}\Big(\sum_{k,s}\lambda_{ks}^{2æ}|\widehat{w}_{ks}(t)|^2\|Y_k\|^2\|j_k\|^2_{(s)}\Big)^{1/2} \\ &\quad \times \sum_{p,q}\frac{1}{\lambda_{pq}^{\alpha-1/2}(p+1)} \\ &\le C\|w(t)\|_\kappa. \end{aligned}$$

Thus, we get

$$|\widehat{w}_{mn}(t)| \le C\int_0^t \exp[-\lambda^\alpha_{mn}(t-\tau)]\|w(\tau)\|_\kappa\,d\tau.$$

Squaring both sides of the last inequality, multiplying the result by $\lambda^{2\kappa}_{mn}\|Y_m\|^2_S\|j_m\|^2_{(n)}$, and summing it over m,n we deduce that for $h>0$ and $t\in[0,h]$,

$$\|w(t)\|^2_\kappa \le C(\sup_{t\in[0,h]}\|w(t)\|^2_\kappa)\Xi(t) \quad \text{with} \quad \Xi(t) = \sum_{m,n}\frac{[1-\exp(-\lambda^\alpha_{mn}t)]^2}{\lambda^{2\alpha-2\kappa}_{mn}}.$$

The series $\Xi(t)$ converges absolutely and uniformly with respect to $t\in[0,h]$ if $\kappa<\alpha-1$. It is a nondecreasing function on $[0,h]$ and $\Xi(0)=0$. Therefore, for $0<\kappa<\alpha-1/2$,

$$(\sup_{t\in[0,h]}\|w(t)\|_\kappa)^2 \le C\Xi(t)(\sup_{t\in[0,h]}\|w(t)\|_\kappa)^2 \le C(h)(\sup_{t\in[0,h]}\|w(t)\|_\kappa)^2,$$

where $C(h) = C\Xi(h)$. The constant $C(h)$ can be made less than one by the appropriate choice of h. This contradiction allows one to establish the uniqueness for $t\in[0,h]$.

Finally, we consider the sequence of intervals $\{[T_k,T_{k+1}]\}_{k=1}^\infty$ with $T_k = kh$. Since

$$\int_{T_k}^t \exp[-\lambda^\alpha_{mn}(t-\tau)]\,d\tau = \frac{1-\exp[-\lambda^\alpha_{mn}(t-T_k)]}{\lambda^\alpha_{mn}}$$

we deduce that for $t\in[T_k,T_{k+1}]$,

$$(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_\kappa)^2 \le C\Xi(t-T_k)(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_\kappa)^2.$$

Setting $t=T_k+\eta$, $\eta\in[0,h]$, we have $\Xi(t-T_k)=\Xi(\eta)$. Since the inequality $C\Xi(\eta)\le C\Xi(h)<1$ has already been proved, we have established the uniqueness of solutions for all $t\ge 0$ and $0<\kappa<\alpha-1/2$. The proof of Theorem 1 is complete. ■

6. Proof of Corollary 3.1: regular perturbation series. We study the series (5.1) representing the solution in the domain $B_\delta\times[0,\infty)$. Note that the coefficients $\widehat{u}_{mn}(t)$ satisfy the inequalities (5.4) and $|j_m(\lambda_{mn}r)|\le C(\delta)/\lambda^{1/2}_{mn}$ for $r\ge\delta>0$. In the chosen coordinate system with the north pole at the point P we have $|Y_m(\theta,\varphi)| = |P_m(\cos\gamma(P,Q))|\le 1$ for all $m\ge 0$. Hence

$$\Big|\sum_{m\ge0,\,n\ge1}^\infty \widehat{u}_{mn}(t)j_m(\lambda_{mn}r)Y_m(\theta,\varphi)\Big| \le C(\delta)\sum_{m\ge0,\,n\ge1}^\infty \frac{1}{\lambda^\alpha_{mn}\sqrt{m+1}}<\infty.$$

Thus, the series (5.1) converges absolutely and uniformly with respect to $(r,\theta,\varphi,t)\in B_\delta\times[0,\infty)$, $\varepsilon\in[0,\varepsilon_0]$, and its sum is a continuous function in this domain. Changing the order of summation we get

$$\begin{aligned} u(r,\theta,\varphi,t) &= \sum_{m\ge0,\,n\ge1}^\infty\Big[\sum_{N=0}^\infty \varepsilon^{N+1}\widehat{v}^{(N)}_{mn}(t)\Big]j_m(\lambda_{mn}r)Y_m(\theta,\varphi) \\ &= \sum_{N=0}^\infty \varepsilon^{N+1}u^{(N)}(r,\theta,\varphi,t), \end{aligned} \tag{6.1}$$

$$u^{(N)}(r,\theta,\varphi,t) = \sum_{m\ge0,\,n\ge1}^\infty\sum_{N=0}^\infty \widehat{v}^{(N)}_{mn}(t)j_m(\lambda_{mn}r)Y_m(\theta,\varphi). \ ■$$

7. Proof of Theorem 2: long-time asymptotics. If $\alpha_{\rm cr}\le\alpha\le 2$, we represent the solution as

$$u(r,\theta,\varphi,t) = \widehat{u}_{01}(t)j_0(\lambda_{01}r) + R_1(r,t) + R_2(r,\theta,\varphi,t), \tag{7.1}$$

where

$$R_1(r,t)=\sum_{n=2}^{\infty}\widehat{u}_{0n}(t)j_0(\lambda_{0n}r),$$
$$R_2(r,\theta,\varphi,t)=\sum_{m,n\ge 1}\widehat{u}_{mn}(t)j_m(\lambda_{mn}r)Y_m(\theta,\varphi),$$

and we have taken into account that $Y_0(\theta,\varphi)=1$. For $3/2+\varepsilon_1\le\alpha<\alpha_{\rm cr}$ we single out two terms in the series (5.1), namely:

$$u(r,\theta,\varphi,t)=\widehat{u}_{01}(t)j_0(\lambda_{01}r)+\widehat{u}_{11}(t)j_1(\lambda_{11}r)Y_1(\theta,\varphi)+R_1(r,t)+\widetilde{R}_2(r,\theta,\varphi,t),\tag{7.2}$$

where

$$\widetilde{R}_2(r,\theta,\varphi,t)=\sum_{n=2}\widehat{u}_{1n}(t)j_1(\lambda_{1n}r)Y_1(\theta,\varphi)+\sum_{m\ge 2,\,n\ge 1}\widehat{u}_{mn}(t)j_m(\lambda_{mn}r)Y_m(\theta,\varphi)$$

and $R_1(r,t)$ is the same as above. Thanks to (3.15) the estimates of the type of (4.15), (4.18), (4,19) (without the factor $c^N(N+1)^{-2}$) also hold for $\widehat{u}_{mn}(t)$ if $\varepsilon<1/c$. If $\alpha_{\rm cr}\le\alpha\le 2$, we obtain a subtle asymptotic estimate of $\widehat{u}_{01}(t)$ and estimate the residual terms R_1 and R_2. If $3/2+\varepsilon_1\le\alpha<\alpha_{\rm cr}$, apart from using the asymptotics of $\widehat{u}_{01}(t)$, we also deduce a sophisticated estimate of $\widehat{u}_{11}(t)$, while R_1 and $\widetilde{R}_2$ contribute to the residual term of the asymptotics.

According to (3.5), we have

$$\widehat{u}_{m1}(t)=\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{v}_{m1}^{(N)}(t),\qquad m=0,1.$$

Adding and subtracting integrals from t to ∞ in the representations of $\widehat{v}_{m1}^{(N)}(t)$, $m=0,1$, we can write

$$\widehat{v}_{m1}^{(0)}(t)=A_m^{(0)}(\varepsilon)\exp(-\lambda_{m1}^{\alpha}t),$$
$$\widehat{v}_{m1}^{(N)}(t)=\exp(-\lambda_{m1}^{\alpha}t)[A_m^{(N)}(\varepsilon)+R_{m1}^{(N)}(t)],$$
$$A_m^{(0)}(\varepsilon)=\varepsilon\widehat{\phi}_{01},\qquad A_m^{(N)}(\varepsilon)=\int_0^{\infty}\exp(\lambda_{m1}^{\alpha}\tau)Q_{m1}^{(N)}(\widehat{v}(\tau),\varepsilon)\,d\tau,$$
$$R_{m1}^{(N)}(t)=\int_t^{\infty}\exp(\lambda_{m1}^{\alpha}\tau)Q_{m1}^{(N)}(\widehat{v}(\tau),\varepsilon)\,d\tau,\tag{7.3}$$
$$Q_{m1}^{(N)}(\widehat{v}(t),\varepsilon)=\sum_{p,k\ge 0;\,q,s\ge 1}a(m,1,p,q,s)\times\sum_{j=1}^{N}\widehat{v}_{pq}^{(j-1)}(t)\widehat{v}_{ks}^{(N-j)}(t),\qquad N\ge 1,$$

where $\widehat{v}_{m1}^{(N)}(t)$, $m=0,1$, $0\le j\le N-1$, are defined by (3.6), and $Q_{m1}^{(N)}$ depends on ε through its dependence on $\widehat{v}_{pq}^{(j-1)}(t)$ and $\widehat{v}_{ks}^{(N-j)}(t)$.

Next, we estimate the residual terms $R_{m1}^{(N)}(t)$, $m=0,1$. By means of Lemmas 3 and Corollary 4.1 we obtain

$$|R_{01}^{(N)}(t)|\le c\int_t^{\infty}\exp(\lambda_{01}^{\alpha}\tau)\exp(-2\lambda_{01}^{\alpha}\tau)\le c\exp(-\lambda_{01}^{\alpha}t),$$
$$\begin{aligned}|R_{11}^{(N)}(t)|&\le c\int_t^{\infty}\exp(\lambda_{11}^{\alpha}\tau)\{c_1\exp(-2\lambda_{01}^{\alpha}\tau)+c_2\tau\exp[-(\lambda_{01}^{\alpha}+\lambda_{11}^{\alpha})\tau]\}\,d\tau\\&\le\exp[-æ(\alpha)t],\end{aligned}\tag{7.4}$$

where $æ(\alpha)=2\lambda_{01}^{\alpha}-\lambda_{11}^{\alpha}>0$ for $3/2+\varepsilon_1\le\alpha<\alpha_{\rm cr}$. If $\alpha=\alpha_{\rm cr}$, then $|R_{11}^{(N)}(t)|\le ct$, and for $\alpha_{\rm cr}\le\alpha\le 2$, $|R_{11}^{(N)}(t)|\le c\exp[æ(\alpha)t]$. Therefore, the term containing $\widehat{u}_{11}(t)$ must be included in the remainder of the asymptotics for $\alpha_{\rm cr}\le\alpha\le 2$.

Thus, for all $t\ge 0$ and $m=0,1$,

$$\begin{aligned}&|\widehat{u}_{m1}(t)-A_m(\varepsilon)\exp(-\lambda_{m1}^{\alpha}t)|\le c\exp(-2\lambda_{01}^{\alpha}t),\\&A_m(\varepsilon)=\sum_{N=0}^{\infty}\varepsilon^{N+1}A_m^{(N)}(\varepsilon),\end{aligned}\tag{7.5}$$

where the series converges absolutely and uniformly with respect to $\varepsilon\in[0,\varepsilon_0]$. It follows from the estimates (4.18), (4.19) that

$$\|R_{1,2}(t)\|_{\kappa}\le c\exp(-2\lambda_{01}^{\alpha}t),\qquad\|\widetilde{R}_2(t)\|_{\kappa}\le c\exp(-2\lambda_{01}^{\alpha}t).\tag{7.6}$$

Combining (7.1)–(7.6), recalling that

$$j_0(\lambda_{01}r)=\sqrt{\frac{\pi}{2r}}J_{1/2}(\pi r)=\frac{\sin(\pi r)}{\sqrt{\pi}r},$$
$$j_1(\lambda_{11}r)=\sqrt{\frac{\pi}{2r}}J_{3/2}(\pi r)=\frac{1}{\sqrt{\lambda_{11}}r}\left[\frac{\sin(\lambda_{11}r)}{\lambda_{11}r}-\cos(\lambda_{11}r)\right]$$

and setting $\widetilde{A}_0(\varepsilon)=\sqrt{\pi}A_0(\varepsilon)$, $\widetilde{A}_1(\varepsilon)=\sqrt{\lambda_{11}}A_1(\varepsilon)$ we obtain (3.7). ■

Acknowledgements. The author wishes to thank Prof. P. Biler for useful discussions and moral support.

References

[1] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[2] C. Bardos, P. Penel, U. Frisch, and P. Sulem, *Modified dissipativity for a nonlinear evolution equation arising in turbulence*, Arch. Rational Mech. Anal. 71 (1979), 237–256.

[3] P. Biler, T. Funaki, and W. Woyczynski, *Fractal Burgers equations*, J. Differential Equations 148 (1998), 9–46.

[4] P. Biler, G. Karch, and W. Woyczynski, *Asymptotics for multifractal conservation laws*, Studia Math. 135 (1999), 231–252.

[5] H. Brezis, *Semilinear equations in* $\mathbb{R}^N$ *without conditions at infinity*, Appl. Math. Optim. 12 (1984), 271–282.

[6] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. 62 (1983), 73–97.

[7] A. Carasso, *Error bounds in nonsmooth image deblurring*, SIAM J. Math. Anal. 28 (1997), 656–668.

[8] M. Escobedo, O. Kavian, and H. Matano, *Large time behavior of solutions of a dissipative semilinear heat equation*, Comm. Partial Differential Equations 20 (1995), 1427–1452.

[9] M. Escobedo and E. Zuazua, *Long-time behavior for a convection-diffusion equation in higher dimensions*, SIAM J. Math. Anal. 28 (1999), 570–594.

[10] A. Gmira and L. Veron, *Large time behavior of the solutions of a semilinear parabolic eqaution in* $\mathbb{R}^N$, J. Differential Equations 53 (1984), 258–276.

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, 1981.

[12] L. Herraiz, *Asymptotic behavior of solutions of some semilinear parabolic problems*, Ann. Inst. H. Poincaré 16 (1999), 16–49.

[13] —, *A nonlinear parabolic problem in an exterior domain*, J. Differential Equations, to appear.

[14] E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions*, 6th ed., Teubner, Stuttgart, 1960.

[15] S. Kamin and L. A. Peletier, *Large time behavior of solutions of the heat equation with absorption*, Ann. Scuola Norm. Sup. Pisa 12 (1985), 393–408.

[16] S. Kamin and M. Ughi, *On the behavior as* $t \to \infty$ *of the solutions of the Cauchy problem for certain nonlinear parabolic equations*, J. Math. Anal. Appl. 128 (1987), 456–469.

[17] J. A. Mann, Jr. and W. Woyczynski, *Rough surfaces generated by nonlinear transport*, Invited paper, Symposium on Nonlinear Diffusion, TMS International Meeting, September 1997.

[18] C. Müller, *Spherical Harmonics*, Lecture Notes in Math. 17, Springer, New York, 1966.

[19] P. I. Naumkin and I. A. Shishmarëv, *Nonlinear Nonlocal Equations in the Theory of Waves*, Transl. Math. Monographs 133, Amer. Math. Soc., Providence, 1994.

[20] L. Oswald, *Isolated positive singularities for a nonlinear heat equation*, Houston J. Math. 14 (1988), 543–572.

[21] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos 7 (1997), 753–764.

[22] G. Sansone, *Orthogonal Functions*, Interscience, New York, 1962.

[23] K. R. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

[24] N. Sugimoto, *"Generalized" Burgers equations and fractional calculus*, in: Nonlinear Wave Motion, A. Jeffrey (ed.), Longman Sci., Harlow, 1989, 162–179.

[25] G. Tolstov, *Fourier Series*, Dover, New York, 1962.

[26] V. V. Varlamov, *On the Cauchy problem for the damped Boussinesq equation*, Differential Integral Equations 9 (1996), 619–634.

[27] —, *On the initial-boundary value problem for the damped Boussinesq equation*, Discrete Contin. Dynam. Systems 4 (1998), 431–444.

[28] —, *Long-time asymptotics of solutions of the second initial-boundary value problem for the damped Bousssinesq equation*, Abstract Appl. Anal. 2 (1998), 97–115.

[29] —, *On the damped Boussinesq equation in a circle*, Nonlinear Anal. 38 (1999), 447–470.

[30] —, *On the spatially two-dimensional Boussinesq equation in a circular domain*, ibid., to appear.

[31] —, *Nonlinear heat equation with a fractional Laplacian in a disk*, Colloq. Math. 81 (1999), 101–122.

[32] M. Vishik, *Asymptotic Behavior of Solutions of Evolutionary Equations*, Cambridge Univ. Press, Cambridge, 1992.

[33] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1966.

[34] C. E. Wayne, *Invariant manifolds for parabolic partial differential equations on unbounded domains*, Arch. Rational Mech. Anal. 138 (1997), 279–306.

[35] G. N. Webster, *Partial Differential Equations of Mathematical Physics*, 2nd ed., Hafner, New York, 1966.

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1082, U.S.A.
E-mail: varlamov@math.utexas.edu

Received November 2, 1999 (4425)
Revised version March 6, 2000