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Centralizers for subsets of normed algebras

by

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Abstract. Let G be the set of invertible elements of a normed algebra A with an identity. For some but not all subsets H of G we have the following dichotomy. For $x \in A$ either $cxc^{-1} = x$ for all $c \in H$ or $\sup\{\|cxc^{-1}\| : c \in H\} = \infty$. In that case the set of $x \in A$ for which the sup is finite is the centralizer of H.

1. Introduction. Let A be a normed algebra with identity e, center Z and set G of invertible elements. For a subset H of G we set

$$\mathbf{F}(H) = \{x \in A : \sup\{\|cxc^{-1}\| : c \in H\} < \infty\}.$$

We determine subsets H for which, given $x \in A$, either $cxc^{-1} = x$ for all $c \in H$ or $x \notin \mathbf{F}(H)$. An equivalent statement is that $\mathbf{F}(H)$ is the centralizer of H, that is, the set of $y \in A$ where $yx = xy$ for all $x \in H$.

Our interest in this situation arose from a result not stated but which follows directly from arguments due to Le Page [6]. See [1, p. 42] where the exposition is readily modified to show the following.

THEOREM 1.1. *Let A be a unital Banach algebra* (*over the complex field*). *Then $x \in \mathbf{F}(G)$ if and only if $x \in Z$.*

Here, of course, Z is the centralizer of G. The proof of Theorem 1.1 depends heavily on complex variable theory via Liouville's theorem. Incidentally, Theorem 1.1 holds if G is replaced by G_1, the principal component of G. See [8, p. 14].

We study cases where A is not necessarily complete and G is replaced by subsets H. We describe one instance. Let $\mathbf{N}$ be the set of nilpotent elements of A. Then for each $v \in \mathbf{N}$ we have $e - v \in G$. Let H be the set of these $e - v$. We show that $\mathbf{F}(H)$ is the centralizer of H. It follows that, for $y \in A$, either $yv = vy$ for all $v \in \mathbf{N}$ or

$$\sup\{\|(e-v)y(e-v)^{-1}\| : v \in \mathbf{N}\} = \infty.$$

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A particular instance of $\mathbf{F}(H)$ has received some attention. Let $B(X)$ be the algebra of all bounded linear operators on a Hilbert space X and $V \in B(X)$ be invertible. Let B_V be the set of $T \in B(X)$ for which $\sup\{\|V^n T V^{-n}\| < \infty : n = 0, 1, \ldots\}$. The set B_V has been studied by several authors. We cite the paper [2] of Deddens and that of Herraro [3] where further references can be found.

2. On centralizers. Trivially, $\mathbf{F}(H)$ contains the centralizer of H. This set inequality can be proper as it is for a finite subset of G not contained in Z. More interesting instances where H is infinite are given below. Our main concern is to find H for which $\mathbf{F}(H)$ is the centralizer of H.

We let $\mathbf{I}$ represent the set of all integers, $\mathbf{I}^+$ those > 0 and $\mathbf{I}^-$ those < 0.

We turn attention to $\mathbf{N}$. For $w \in \mathbf{N}$ we say that the *order of nilpotency* of w is the largest integer n for which $w^n \neq 0$.

We make some preliminary calculations for use in the proof of Theorem 2.1.

LEMMA 2.1. *Let $v \in \mathbf{N}$ and $r \in \mathbf{I}^+$. Suppose that, for $y \in A$, $yv^k = v^k y$ whenever $k \geq 2$. Then*

$$(e-v)^r y(e-v)^{-r} = y + r(yv - vy) + r^2(v^2 y - vyv).$$

Proof. $(e-v)^r$ and $(e-v)^{-r}$ are finite sums where

$$(e-v)^r = e - rv + r(r-1)v^2/2 + \ldots,$$
$$(e-v)^{-r} = e + rv + r(r+1)v^2/2 + \ldots$$

since $(e-v)^{-r} = (e + \sum_{k=1}^{n} v^k)^r$ where v is nilpotent with order n. Note that $v^i y v^j = y v^{i+j} = v^{i+j} y$ if either $i \geq 2$ or $j \geq 2$. The expansion of $w = (e-v)^r y (e-v)^{-r}$ involves $v^i y v^j$ terms. Those for which $i + j \leq 2$ yield

$$w = y + r(yv - vy) + r^2(v^2 y - vyv) + \ldots$$

For the terms involving $v^i y v^j$ for $i + j = k$, with a particular $k \geq 3$, y factors out and these terms add up to yv^k times the coefficient of v^k in the expansion of $e = (e-v)^r(e-v)^{-r}$. That coefficient is zero. ∎

THEOREM 2.1. *Let H be the set of all $e - v$ for $v \in \mathbf{N}$. Then $\mathbf{F}(H)$ is the centralizer of H.*

Proof. Let $y \in \mathbf{F}(H)$. We must show that $vy = yv$ for all $v \in \mathbf{N}$. Of course, $(e-v)^r \in H$ for all $r \in \mathbf{I}$. Suppose first that v has order of nilpotency one. Then, for $r \in \mathbf{I}^+$, $(e-v)^r = e - rv$ and $(e-v)^{-r} = e + rv$. For $y \in \mathbf{F}(H)$, $\{(e-rv)y(e+rv)\}$ is a bounded sequence. Thus $\{r(yv - vy) - r^2 vyv\}$ is a bounded sequence and so

$$(yv - vy) - rvyv \to 0$$

as $r \to \infty$. Hence $vyv = 0$ and $yv = vy$.

Next let $w \in H$ have order of nilpotency two. Then, by the above, $yw^2 = w^2 y$. We have (see Lemma 2.1) $(e-w)^r y (e-w)^{-r} = y + r(yw - wy) + r^2(w^2 y - wyw)$.

Hence $\{r(yw - wy) + r^2(w^2 y - wyw\}$ is a bounded sequence. Thus $yw - wy + r(w^2 y - wyw) \to 0$ as $r \to \infty$ so that $yw = wy$.

Assume now that, for an integer $k \geq 2$, y permutes with all $v \in \mathbf{N}$ with order of nilpotency $\leq k$. Let $w \in \mathbf{N}$ have order of nilpotency $k+1$. Then $yw^j = w^j y$ for all $j \geq 2$. By Lemma 2.1 we have

$$(e-w)^r y(e-w)^{-r} = y + r(yw - wy) + r^2(w^2 y - wyw).$$

As above this implies that $yw = wy$. This inductive argument completes the proof. ∎

We turn our attention to a normed algebra B over the complex field which has no non-zero nilpotent one-sided ideals. Let $\mathbf{S}$ denote the socle of B [4, p. 64]. Each minimal right ideal of B is of the form pB where p is an idempotent. We call such an idempotent a *minimal idempotent*. Let $\mathbf{P}$ denote the set of minimal idempotents of B.

THEOREM 2.2. *The centralizer of $\mathbf{S}$ is the same as the centralizer of $\mathbf{P}$.*

Proof. Let y be in the centralizer of $\mathbf{P}$. We must show that y is in the centralizer of $\mathbf{S}$. Let $p \in \mathbf{P}$, $x \in B$, $x \neq 0$. There exists a complex number λ with $pxp = \lambda p$.

Suppose first that $\lambda \neq 0$. Then $(px)^2 = \lambda px$ so that $q = \lambda^{-1} px$ is an idempotent. Inasmuch as $qB = pB$ we see that $q \in \mathbf{P}$. Hence $ypx = pxy$.

Suppose that $pxp = 0$. Set $w = x + \alpha p$ where α is a scalar. Then $pwp \neq 0$ and thus

$$yp(x + \alpha p) = p(x + \alpha p)y \quad \text{for all } \alpha \neq 0.$$

Therefore $ypx = pxy$ so that, finally, y is in the centralizer of $\mathbf{S}$. ∎

We return to the study of A, any normed algebra, and make some preliminary calculations involving an idempotent p. Let λ be a scalar, $\lambda \neq 0$, $\lambda \neq 1$. We show that either $py = yp$ or

$$\sup\{\|(e - \lambda p)^n y (e - \lambda p)^{-n}\| : n \in \mathbf{I}\} = \infty.$$

Let $p \neq 0$, $p \neq e$ be an idempotent in A and let λ be a scalar, $\lambda \neq 0$, $\lambda \neq 1$. Let $q = e - p$ so that $(e - \lambda p) = q + (1 - \lambda)p$ where $qp = pq = 0$. For any $k \in \mathbf{I}^+$ we have

$$(1) \qquad (e - \lambda p)^k = q + (1-\lambda)^k p, \qquad (e - \lambda p)^{-k} = q + (1-\lambda)^{-k} p.$$

Thus $(e - \lambda p)^k$ and $(e - \lambda p)^{-k}$ are of the form $e - \alpha p$ for a scalar $\alpha \neq 1$. Let $y \in A$. For each $n \in \mathbf{I}$ we set

$$\phi(n, y) = (e - \lambda p)^n y (e - \lambda p)^{-n}.$$

Then

(2) $$\phi(n,y) = qyq + pyp + (1-\lambda)^n pyq + (1-\lambda)^{-n} qyp.$$

We set $H_+ = \{(e-\lambda p)^n : n \in \mathbf{I}^+\}$ and $H_- = \{(e-\lambda p)^n : n \in \mathbf{I}^-\}$.

THEOREM 2.3. *If* $|1-\lambda| > 1$ *then* $\mathbf{F}(H_+) = \{y \in A : py = pyp\}$ *and* $\mathbf{F}(H_-) = \{y \in A : yp = pyp\}$. *If* $0 < |1-\lambda| < 1$ *then* $\mathbf{F}(H_+) = \{y \in A : yp = pyp\}$ *and* $\mathbf{F}(H_-) = \{y : py = pyp\}$.

Proof. Suppose $|1-\lambda| > 1$. Then (2) shows that $y \in \mathbf{F}(H_+)$ if and only if $pyq = 0$ or $py = pyp$. The other statements follow from (2) in the same way. ∎

In every case considered in Theorem 2.3, the centralizers of H_+ and H_- are all $\{y \in A : py = yp\}$. Let A be the algebra of all two-by-two matrices and let p and y be given by

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $pyp = py$ (resp. yp) if and only if $b = 0$ (resp. $c = 0$) and $py = yp$ if and only if $b = c = 0$. Therefore, if $|1-\lambda| > 1$, $\mathbf{F}(H_+)$ properly contains the centralizer of H_+.

COROLLARY 2.1. *Let* λ *be any scalar*, $\lambda \neq 0$, $\lambda \neq 1$, *and* $H = \{(e-\lambda p]^n : n \in \mathbf{I}\}$. *Then* $\mathbf{F}(H) = \{y : py = yp\}$.

Proof. This follows immediately from Theorem 2.3. ∎

There are important classes of Banach algebras which are the closure of the linear span of its idempotents. As is well known this holds for von Neumann algebras [7, p. 119].

COROLLARY 2.2. *Let* A *be a unital Banach algebra which is the closed linear span of its idempotents. Let* H *be any open subgroup of* G. *Then* $\mathbf{F}(H) = Z$.

Proof. For each idempotent $p \neq 0$, $p \neq e$ there exists a scalar λ, $0 < |1-\lambda| \neq 1$, so that $e - \lambda p \in H$. Then H contains $(e-\lambda p)^n$ for all $n \in \mathbf{I}$. We apply Corollary 2.1 to see that $py = yp$ for $y \in \mathbf{F}(H)$.

3. A generalization of Theorem 1.1.

The setting for Theorem 1.1 is a Banach algebra A over the complex field. The completeness of A is essential for the proof in [1, p. 42]. Here we show that the requirement of completeness can be relaxed. It is sufficient that, in the normed algebra A, the invertible elements form an open subset. In the language first used by Kaplansky in [5], A should be a Q-algebra. Examples of incomplete normed Q-algebras abound.

For our work here, A will represent a unital normed Q-algebra over the complex field with completion A^c. Let G (resp. G^c) be the set of invertible elements in A (resp. A^c).

LEMMA 3.1. $G^c \cap A = G$.

Proof. Clearly, $G \subset G^c \cap A$. Suppose $x \in A$ where $x \notin G$. Then either x has no left inverse or no right inverse. Say x has no right inverse, so that $V = \{xw : w \in A\}$ is a proper right ideal in A. Then V is contained in a maximal proper right ideal M of A by [4, p. 6]. As A is a Q-algebra, M is closed in A. Therefore for some $d > 0$, $\|e - xw\| \geq d$ for all $w \in A$ and hence $\|e - xy\| \geq d$ for all $y \in A^c$. Consequently, x has no right inverse in A^c. ∎

THEOREM 3.1. *In the normed* Q *algebra* A *we have* $\mathbf{F}(G) = Z$.

Proof. For any $y \in \mathbf{F}(G)$ we have $\sup\{\|cyc^{-1}\| : c \in G\} < K$ for some real K. Let $w \in G^c$, $w = \lim x_n$ with each $x_n \in A$. As G^c is open in A^c we may suppose that each $x_n \in G^c \cap A$. By the preceding lemma, each $x_n \in G$. Therefore $\|x_n y x_n^{-1}\| < K$ for all $n = 1, 2, \ldots$ so that $\|wyw^{-1}\| \leq K$ for all $w \in G^c$. By Theorem 1.1, y is in the center of A^c and so is in the center of A. ∎

4. On F(H) in Banach algebras.

Henceforth A represents a Banach algebra. We retain our previous notation where H is a subset of G. For $c \in H$ we set $T_c(x) = cxc^{-1}$ and

$$M(x) = \sup\{\|T_c(x)\| : c \in H\}.$$

THEOREM 4.1. *Either* $\mathbf{F}(H) = A$ *or the complement of* $\mathbf{F}(H)$ *is dense in* A.

Proof. Suppose that $\mathbf{F}(H) \neq A$. For each positive integer n set $\Phi_n = \{x \in A : M(x) \leq n\}$. Each Φ_n is closed in A. We show that Φ_n cannot contain a non-empty open subset Γ of A. Suppose otherwise. Let $x_0 \in \Gamma$. For some $\varepsilon > 0$ we have $w \in \Gamma$ if $\|w - x_0\| \leq \varepsilon$. Let $y \in A$, $\|y\| = 1$. Then $x_0 + ty \in \Gamma$ for $|t| \leq \varepsilon$. For each $c \in H$ we have $\|T_c(x_0 + ty)\| \leq n$ for all t with $|t| \leq \varepsilon$. Then $\varepsilon\|T_c(y)\| \leq \|T_c(x_0 + \varepsilon y)\| + \|T_c(x_0)\| \leq 2n$. Hence $y \in F(H)$ so that $\mathbf{F}(H) = A$ contrary to our assumption.

Next let Ω_n be the complement of Φ_n. Each Ω_n is a dense open subset of A. By the Baire category theorem the intersection $\bigcap \Omega_n$ of all the sets Ω_n is dense in A. But $\bigcap \Omega_n = \{x \in A : M(x) = \infty\}$. ∎

Note that if we set $H_1 = H \cup \{e\}$ we have $\mathbf{F}(H_1) = \mathbf{F}(H)$. Therefore there is no loss of generality in assuming $e \in H$.

LEMMA 4.1. $\mathbf{F}(H)$ *is a subalgebra of* A *and* $M(x)$ *is a Banach algebra norm on* $\mathbf{F}(H)$.

Proof. Let $x, y \in \mathbf{F}(H)$ and $c \in H$. As $T_c(xy) = T_c(x)T_c(y)$ it is easy to see that $\mathbf{F}(H)$ is a subalgebra and that $M(x)$ is a normed algebra norm on $\mathbf{F}(H)$.

Clearly, $M(x) \ge \|x\|$ for all $x \in \mathbf{F}(H)$. Let $\{x_n\}$ be a Cauchy sequence in $\mathbf{F}(H)$ in the $M(x)$ norm. Then $\{x_n\}$ is a Cauchy sequence in A so there exists $y \in A$ where $\|x_n - y\| \to 0$. As $\{M(x_n)\}$ is a bounded sequence we have $M(x_n) \le K$, for a real K and all positive integers n. For $c \in H$ we have

$$\|T_c(y)\| \le \|T_c(y - x_n)\| + \|T_c(x_n)\| \le \|c\| \, \|c^{-1}\| \, \|y - x_n\| + K;$$

we let $n \to \infty$ to see that $y \in \mathbf{F}(H)$. ∎

THEOREM 4.2. *$\mathbf{F}(H)$ is a closed subset of A if and only if the norms $\|x\|$ and $M(x)$ are eqivalent norms on $\mathbf{F}(H)$.*

Proof. Suppose that $\mathbf{F}(H)$ is closed in A. Now $\sup\{\|T_c(x)\| : c \in H\}$ is finite for each $x \in \mathbf{F}(H)$. By the uniform boundedness theorem there exists a real number L so that $\|T_c(x)\| \le L\|x\|$ for all $x \in \mathbf{F}(H)$, $c \in H$. Then $M(x) \le L\|x\|$ on $\mathbf{F}(H)$ so that the two norms are equivalent there. The converse is clear. ∎

We have no example of a case where $\mathbf{F}(H)$ is not closed in A.

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On sharp reiteration theorems and weighted norm inequalities

by

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Abstract. We prove sharp end forms of Holmstedt's reiteration theorem which are closely connected with a general form of Gehring's Lemma. Reverse type conditions for the Hardy–Littlewood–Pólya order are considered and the maximal elements are shown to satisfy generalized Gehring conditions. The methods we use are elementary and based on variants of reverse Hardy inequalities for monotone functions.

1. Introduction. Given a fixed initial pair of compatible spaces, interpolation theory provides us with methods to construct scales of spaces with the interpolation property. The classical methods of interpolation all share the following reiteration principle: by iteration these constructions do not generate new spaces. Reiteration theorems thus play a central role in these theories. In particular, reiteration simplifies the process of identification of interpolation spaces. Holmstedt's reiteration formula [Ho], for the real method of interpolation, provides quantitative estimates and plays an important role in a variety of applications to classical analysis and approximation theory.

Let $\overline{A}$ be a pair of compatible Banach spaces, $0 < \theta_0 < \theta_1 < 1$, $0 < q_i \le \infty$, $i = 0, 1$, $\eta = \theta_1 - \theta_0$. Then Holmstedt's formula states that

$$K(t, f; \overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1}) \approx \Big\{ \int_0^{t^{1/\eta}} (s^{-\theta_0} K(s, f; \overline{A}))^{q_0} \frac{ds}{s} \Big\}^{1/q_0} + t \Big\{ \int_{t^{1/\eta}}^{\infty} (s^{-\theta_1} K(s, f; \overline{A}))^{q_1} \frac{ds}{s} \Big\}^{1/q_1}. \tag{1.1}$$

Holmstedt's formula is also valid if $\theta_0 = 0$ or $\theta_1 = 1$. For example, if $\theta_1 = 1$ we have

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