

- A. Jourani (1996), *Subdifferentiability and subdifferential monotonicity of  $\gamma$ -paraconvex functions*, Control Cybernet. 25, 721–737.
- D. Pallaschke and S. Rolewicz (1997), *Foundations of Mathematical Optimization*, Math. Appl. 388, Kluwer, Dordrecht.
- R. T. Rockafellar (1970), *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. 33, 209–216.
- R. T. Rockafellar (1980), *Generalized directional derivatives and subgradients of non-convex functions*, Canad. J. Math. 32, 257–280.
- S. Rolewicz (1999), *On  $\alpha(\cdot)$ -monotone multifunctions and differentiability of  $\gamma$ -paraconvex functions*, Studia Math. 133, 29–37.

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## On the complemented subspaces of the Schreier spaces

by

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**Abstract.** It is shown that for every  $1 \leq \xi < \omega$ , two subspaces of the Schreier space  $X^\xi$  generated by subsequences  $(e_{l_n}^\xi)$  and  $(e_{m_n}^\xi)$ , respectively, of the natural Schauder basis  $(e_n^\xi)$  of  $X^\xi$  are isomorphic if and only if  $(e_{l_n}^\xi)$  and  $(e_{m_n}^\xi)$  are equivalent. Further,  $X^\xi$  admits a continuum of mutually incomparable complemented subspaces spanned by subsequences of  $(e_n^\xi)$ . It is also shown that there exists a complemented subspace spanned by a block basis of  $(e_n^\xi)$ , which is not isomorphic to a subspace generated by a subsequence of  $(e_n^\zeta)$ , for every  $0 \leq \zeta \leq \xi$ . Finally, an example is given of an uncomplemented subspace of  $X^\xi$  which is spanned by a block basis of  $(e_n^\xi)$ .

**1. Introduction.** The Schreier families  $\{S_\xi\}_{\xi < \omega_1}$  of finite subsets of positive integers (the precise definition is given in the next section), introduced in [1], have played a central role in the development of modern Banach space theory. We mention the use of Schreier families in the construction of mixed Tsirelson spaces which are asymptotic  $\ell_1$  and arbitrarily distortable [3]. The distortion of mixed Tsirelson spaces has been extensively studied in [2]. In that paper, as well as in [14], the moduli  $(\delta_\alpha)_{\alpha < \omega_1}$  were introduced measuring the complexity of the asymptotic  $\ell_1$  structure of a Banach space. The definitions of those moduli also involve the Schreier families. Other applications can be found in [6] and [5] where the Schreier families form the main tool for determining the structure of those convex combinations of a weakly null sequence that tend to zero in norm, or are equivalent to the unit vector basis of  $c_0$ . For applications of the Schreier families in the construction of hereditarily indecomposable Banach spaces, we refer to [3] and [4].

A notion companion to the Schreier families is that of the Schreier spaces. These are Banach spaces whose norm is related to a corresponding Schreier family. More precisely, for every countable ordinal  $\xi$ , we define a norm  $\|\cdot\|_\xi$  on  $c_{00}$ , the space of finitely supported real-valued sequences, in the following

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manner: Given  $x = (x(n)) \in c_{00}$  define

$$\|x\|_{\xi} = \sup_{F \in \mathcal{S}_{\xi}} \sum_{n \in F} |x(n)|.$$

$X^{\xi}$ , the Schreier space of order  $\xi$ , is the completion of  $c_{00}$  under the norm  $\|\cdot\|_{\xi}$ .  $X^0 = c_0$ , the Banach space of null sequences.  $X^1$  was first considered by Schreier [15] in order to provide an example of a weakly null sequence without Cesàro summable subsequence. It is proven in [1] that the natural Schauder basis  $(e_n^{\xi})$  of  $X^{\xi}$  is 1-unconditional and shrinking.  $X^1$  has been studied in [13] where it is shown that every quotient of  $X^1$  is  $c_0$ -saturated. That is, every infinite-dimensional subspace contains a further subspace isomorphic to  $c_0$ .

Given  $M$ , an infinite subset of  $\mathbb{N}$ , we let  $X_M^{\xi}$  denote the closed linear subspace of  $X^{\xi}$  spanned by the subsequence  $(e_n^{\xi})_{n \in M}$ . For an element  $x \in X^{\xi}$ ,  $x = \sum_{n \in \mathbb{N}} a_n e_n^{\xi}$ , we set  $\|x\|_0 = \sup_{n \in \mathbb{N}} |a_n|$ . The main result of this paper is the following

**THEOREM 1.1.** *Let  $L = (l_n)$ ,  $M = (m_n)$  be infinite subsets of  $\mathbb{N}$ , and let  $\xi < \omega$ . The following are equivalent:*

1. *There exist a bounded linear operator  $T : X_L^{\xi} \rightarrow X_M^{\xi}$  and  $\delta > 0$  such that  $\|T(e_l^{\xi})\|_0 > \delta$  for all  $l \in L$ .*
2.  *$(e_{l_n}^{\xi})$  dominates  $(e_{m_n}^{\xi})$  for every  $\zeta \leq \xi$ .*
3.  *$(e_{l_n}^{\xi})$  dominates  $(e_{m_n}^{\xi})$ .*

We recall here that a basic sequence  $(x_n)$  in some Banach space  $X$  is said to *dominate* the basic sequence  $(y_n)$  in the Banach space  $Y$  if there exists a constant  $C > 0$  so that  $\|\sum_{i=1}^n a_i y_i\| \leq C \|\sum_{i=1}^n a_i x_i\|$  for every  $n \in \mathbb{N}$  and all scalar sequences  $(a_i)_{i=1}^n$ . Equivalently,  $(x_n)$  dominates  $(y_n)$  if there exists a bounded linear operator  $T$  from the closed linear span of  $(x_n)$  into the closed linear span of  $(y_n)$  so that  $T(x_n) = y_n$  for every  $n \in \mathbb{N}$ . The sequences  $(x_n)$  and  $(y_n)$  are *equivalent* if each one of them dominates the other.

Combining Theorem 1.1 with Lemma 3.14, we obtain

**COROLLARY 1.2.** *Let  $\xi < \omega$  and  $L = (l_n)$ ,  $M = (m_n)$  be infinite subsets of  $\mathbb{N}$ .*

1. *If  $X_L^{\xi}$  is isomorphic to a subspace of  $X_M^{\xi}$  then  $(e_{l_n}^{\xi})$  dominates  $(e_{m_n}^{\xi})$ . Consequently,  $X_L^{\xi}$  is isomorphic to  $X_M^{\xi}$  if, and only if,  $(e_{l_n}^{\xi})$  is equivalent to  $(e_{m_n}^{\xi})$ .*
2. *If  $X_L^{\xi}$  is isomorphic to  $X_M^{\xi}$ , then  $X_L^{\zeta}$  is isomorphic to  $X_M^{\zeta}$  for every  $\zeta \leq \xi$ .*

3. *Suppose that  $(e_{l_n}^{\xi})$  dominates a permutation of  $(e_{m_n}^{\xi})$ . Then  $(e_{l_n}^{\xi})$  dominates  $(e_{m_n}^{\xi})$ .*

Theorem 1.1 combined with elementary descriptive set theory yields our next result on the structure of the subsequences of  $(e_n^{\xi})$ ,  $\xi < \omega$ . We recall here that the Banach spaces  $X$  and  $Y$  are *incomparable* if neither of them is isomorphic to a closed linear subspace of the other. In what follows,  $[\mathbb{N}]$  denotes the set of all infinite subsets of  $\mathbb{N}$ .

**THEOREM 1.3.** *For every  $1 \leq \xi < \omega$  there exists  $\mathcal{A} \subset [\mathbb{N}]$  (depending on  $\xi$ ) of cardinality equal to the continuum and with the following property: for every pair  $(L, M)$  of distinct elements of  $\mathcal{A}$ , the spaces  $X_L^{\xi}$  and  $X_M^{\xi}$  are incomparable.*

Corollary 3.15 provides another application of Theorem 1.1:  $X_N^{\xi}$  is not primary, for every  $N \in [\mathbb{N}]$  and all  $1 \leq \xi < \omega$ .

The proofs of the aforementioned results are given in the third section of our paper. In the fourth section we deal with complemented subspaces of  $X^{\xi}$  spanned by block bases of  $(e_n^{\xi})$ . Proposition 4.3 and Corollary 4.4 establish that for every semi-normalized weakly null sequence in  $X^{\xi}$ ,  $\xi < \omega$ , there exist  $\zeta \leq \xi$  and a subsequence which is equivalent to a subsequence of  $(e_n^{\zeta})$ .

Further, we show that there exists a block basis of  $(e_n^{\xi})$  spanning a complemented subspace of  $X^{\xi}$  which is not isomorphic to  $X_M^{\zeta}$ , for all  $0 \leq \zeta \leq \xi$  and every infinite subset  $M$  of  $\mathbb{N}$ . We also show that there exists a block basis of  $(e_n^{\xi})$  spanning a subspace which is not complemented in  $X^{\xi}$ .

The problem of the isomorphic classification of the complemented subspaces of  $X^{\xi}$ , even for block subspaces, seems rather difficult.

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**2. Preliminaries.** We shall make use of standard Banach space facts and terminology as may be found in [11]. In this section we shall review some of the necessary concepts. We shall also review two important hierarchies, the Schreier hierarchy [1] and the repeated averages hierarchy [6]. Finally we shall state some fundamental results from descriptive set theory which will be widely used in what follows. For a detailed study of descriptive set theory we refer to [9].

We first indicate some special notation that we will be using. A sequence  $(x_n)_{n=1}^{\infty}$  of elements of an arbitrary set will be conveniently denoted by

$(x_n)$ . Given  $M$ , a subset of  $\mathbb{N}$ ,  $[M]^{<\omega}$  denotes the set of all finite subsets of  $M$ , while  $[M]$  stands for the set of all infinite subsets of  $M$ . If  $M \in [\mathbb{N}]$ , then the notation  $M = (m_n)$  indicates that  $M = \{m_1 < m_2 < \dots\}$ . Let  $E, F$  be finite sets of integers. We shall adopt the notation  $E < F$  to mean  $\max E < \min F$ . If  $x = (x(n))$  belongs to  $c_{00}$ , the space of finitely supported real-valued sequences, and  $F \in [\mathbb{N}]^{<\omega}$ , then  $x(F) = \sum_{n \in F} x(n)$ , and  $|x|(F) = \sum_{n \in F} |x(n)|$ .

All Banach spaces considered throughout this paper are real.  $\ell_1$  denotes the Banach space of absolutely summable sequences under the norm given by the sum of the absolute values of the coordinates.  $c_0$  is the Banach space of null sequences under the norm given by the maximum of the absolute values of the coordinates. By the term *subspace* of a Banach space we shall always mean a closed linear subspace. A subspace  $Y$  of the Banach space  $X$  is said to be *complemented* if it is the range of a bounded linear projection on  $X$ .

We next recall that if  $(x_n)$  is a sequence in some normed linear space, then the sequence  $(y_n)$  is called a *block subsequence* (resp. *convex block subsequence*) of  $(x_n)$  if there exist sets  $F_i \subset \mathbb{N}$  with  $F_1 < F_2 < \dots$  and a sequence  $(a_i)$  of scalars (resp. non-negative scalars such that  $\sum_{n \in F_i} a_n = 1$ , for every  $i \in \mathbb{N}$ ) such that for every  $i \in \mathbb{N}$ ,  $y_i = \sum_{n \in F_i} a_n x_n$ . We then denote by  $\text{supp } y_i$  the *support* of  $y_i$ , that is, the set  $\{n \in F_i : |a_n| > 0\}$ . We shall also adopt the notation  $y_1 < y_2 < \dots$  to indicate that  $(y_n)$  is a block subsequence of  $(x_n)$ . If  $(x_n)$  is Schauder basic, then  $(y_n)$  will be called a *block basis* (resp. *convex block basis*) of  $(x_n)$ .

Next we review the definition and some basic properties of the Schreier families  $\{S_\xi\}_{\xi < \omega_1}$  (cf. [1]). The Schreier families are defined by transfinite induction as follows:  $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ . Suppose  $S_\zeta$  has been defined for every  $\zeta < \xi$ . If  $\xi$  is a successor ordinal, say  $\xi = \zeta + 1$ , we set

$$S_\xi = \left\{ \bigcup_{i=1}^n F_i : n \in \mathbb{N}, n \leq \min F_1, \right. \\ \left. F_1 < \dots < F_n, F_i \in S_\zeta, i \leq n \right\} \cup \{\emptyset\}.$$

If  $\xi$  is a limit ordinal, let  $(\xi_n)$  be a preassigned increasing sequence of successor ordinals whose limit is  $\xi$ . We set

$$S_\xi = \bigcup_{n=1}^{\infty} \{F \in S_{\xi_n} : n \leq \min F\} \cup \{\emptyset\}.$$

Given  $M \in [\mathbb{N}]$  we denote by  $S_\xi[M]$  the family  $\{F : F \in S_\xi, F \subset M\}$ .

An important property shared by the Schreier families is that they are *spreading*: If  $\{p_1, \dots, p_k\} \in S_\xi$ ,  $p_1 < \dots < p_k$ , and  $q_1 < \dots < q_k$  are so that  $p_i \leq q_i$  for all  $i \leq k$ , then  $\{q_1, \dots, q_k\} \in S_\xi$ .

Of particular interest are the maximal (under inclusion) members of  $S_\xi$ . The following lemma concerning those sets is proved in [8].

LEMMA 2.1. Let  $M \in [\mathbb{N}]$  and  $\xi < \omega_1$ . Then there exists a (necessarily) unique sequence  $\{F_n^\xi(M)\}_{n=1}^\infty$  of successive maximal  $S_\xi$  sets so that  $M = \bigcup_{n=1}^\infty F_n^\xi(M)$ .

REMARK. The following stability properties of  $\{F_n^\xi(M)\}_{n=1}^\infty$  are easily verified:

1. If  $k_1 < k_2 < \dots$  and  $N = \bigcup_{n=1}^\infty F_{k_n}^\xi(M)$ , then  $F_n^\xi(N) = F_{k_n}^\xi(M)$  for all  $n \in \mathbb{N}$ .
2. Let  $M = (m_i)$  and  $N = (n_i)$  be infinite subsets of  $\mathbb{N}$ . Assume that for some  $p \in \mathbb{N}$ ,  $m_i = n_i$  for all  $i \leq p$ . If  $F_k^\xi(M)$  is contained in  $\{m_i : i \leq p\}$ , then  $F_i^\xi(M) = F_i^\xi(N)$  for all  $i \leq k$ .

We shall make use of the following

LEMMA 2.2. Let  $M \in [\mathbb{N}]$ ,  $L \in [M]$  and  $\xi < \omega$ . Then  $\max F_1^\xi(M) \leq \max F_1^\xi(L)$ .

Proof. Suppose  $L = (l_i)$  and  $M = (m_i)$ . We prove the assertion by induction on  $\xi$ . The case  $\xi = 0$  is trivial. Assume now that  $\xi \geq 1$  and that the assertion holds for  $\xi - 1$  and all  $P, Q$  with  $Q \in [P]$ .

For an arbitrary  $P \in [\mathbb{N}]$ , we set  $P_1 = P$  and

$$P_i = \{p \in P : p > \max F_{i-1}^{\xi-1}(P)\}, \quad i \geq 2.$$

We observe that  $F_i^{\xi-1}(P) = F_1^{\xi-1}(P_i)$  for all  $i \in \mathbb{N}$ . We also have  $F_1^\xi(P) = \bigcup_{i=1}^{p_1} F_1^{\xi-1}(P_i)$ , where  $p_1 = \min P_1$ . It follows now, by the induction hypothesis, that  $L_i \in [M_i]$  for all  $i \in \mathbb{N}$ . Therefore,

$$\max F_1^\xi(M) = \max F_1^{\xi-1}(M_{m_1}) \leq \max F_1^{\xi-1}(L_{m_1}) \leq \max F_1^\xi(L)$$

as  $m_1 \leq l_1$ . ■

We now pass to the definition of the repeated averages hierarchy introduced in [6]. We let  $(e_n)$  denote the unit vector basis of  $c_{00}$ . For every countable ordinal  $\xi$  and every  $M \in [\mathbb{N}]$ , we define a convex block subsequence  $(\xi_n^M)_{n=1}^\infty$  of  $(e_n)$  by transfinite induction on  $\xi$  in the following manner: If  $\xi = 0$ , then  $\xi_n^M = e_{m_n}$ , for all  $n \in \mathbb{N}$ , where  $M = (m_n)$ . Assume that  $(\zeta_n^M)_{n=1}^\infty$  has been defined for all  $\zeta < \xi$  and  $M \in [\mathbb{N}]$ . Let  $\xi = \zeta + 1$ . Set

$$\xi_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \zeta_i^M$$

where  $m_1 = \min M$ . Suppose that  $\xi_1^M < \dots < \xi_n^M$  have been defined. Let

$$M_n = \{m \in M : m > \max \text{supp } \xi_n^M\} \quad \text{and} \quad k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_i^{M_n}.$$

If  $\xi$  is a limit ordinal, let  $(\xi_n + 1)$  be the sequence of ordinals associated with  $\xi$  in the definition of  $S_\xi$ , and let  $M \in [\mathbb{N}]$ . Define

$$\xi_1^M = [\xi_{m_1} + 1]_1^M$$

where  $m_1 = \min M$ . Suppose that  $\xi_1^M < \dots < \xi_n^M$  have been defined. Let

$$M_n = \{m \in M : m > \max \text{supp } \xi_n^M\} \quad \text{and} \quad k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = [\xi_{k_n} + 1]_1^{M_n}.$$

The inductive definition of  $(\xi_n^M)_{n=1}^\infty$ ,  $M \in [\mathbb{N}]$ , is now complete.

The following properties are established in [6].

- P1**  $(\xi_n^M)_{n=1}^\infty$  is a convex block subsequence of  $(e_n)$  such that, for all  $M \in [\mathbb{N}]$  and  $\xi < \omega_1$ , we have  $M = \bigcup_{n=1}^\infty \text{supp } \xi_n^M$ .
- P2**  $\text{supp } \xi_n^M \in S_\xi$  for all  $M \in [\mathbb{N}]$ ,  $\xi < \omega_1$  and  $n \in \mathbb{N}$ .
- P3** If  $M, N \in [\mathbb{N}]$ ,  $\xi < \omega_1$ , and  $\text{supp } \xi_i^M = \text{supp } \xi_i^N$  for  $i \leq k$ , then  $\xi_i^M = \xi_i^N$  for  $i \leq k$ .
- P4** If  $\xi < \omega_1$ ,  $\{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$ , and  $\{L_k : k \in \mathbb{N}\} \subset [\mathbb{N}]$  are such that  $\text{supp } \xi_{n_i}^{L_i} < \text{supp } \xi_{n_{i+1}}^{L_{i+1}}$  for all  $i \in \mathbb{N}$ , then letting  $L = \bigcup_{i=1}^\infty \text{supp } \xi_{n_i}^{L_i}$ , we have  $\xi_i^L = \xi_{n_i}^{L_i}$  for all  $i \in \mathbb{N}$ .

Properties **P3** and **P4** are called *stability properties* of the hierarchy  $\{(\xi_n^M)_{n=1}^\infty : M \in [\mathbb{N}]\}$ . It is easily seen, by induction, that  $\text{supp } \xi_n^M = F_n^\xi(M)$  for every  $\xi < \omega$ , all  $M \in [\mathbb{N}]$  and  $n \in \mathbb{N}$ .

In the next lemma we show that for  $\xi < \omega$  and  $M \in [\mathbb{N}]$  the sequence  $(\xi_n^M)$ , considered as a sequence in  $X^\xi$ , is equivalent to the unit vector basis of  $c_0$ . Moreover, the equivalence constant depends only on  $\xi$ .

**LEMMA 2.3.**  $\|\sum_{i=1}^n \xi_i^M\|_\xi \leq \xi + 1$  for every  $M \in [\mathbb{N}]$ ,  $n \in \mathbb{N}$ , and  $\xi < \omega$ .

**Proof.** We use induction on  $\xi$ . The case  $\xi = 0$  is trivial. Assume the assertion holds for  $\xi - 1$ . Let  $G \in S_\xi$ . We shall show that  $\sum_{i=1}^n \xi_i^M(G) \leq \xi + 1$ , for every  $M \in [\mathbb{N}]$  and  $n \in \mathbb{N}$ . To this end choose successive members  $G_1 < \dots < G_l$ , of  $S_{\xi-1}$  so that  $l \leq \min G$  and  $G = \bigcup_{i=1}^l G_i$ . Let  $\{i_1, \dots, i_p\}$  be an enumeration of the set  $\{i \leq n : F_i^\xi(M) \cap G \neq \emptyset\}$ . We define

$$L = \bigcup_{t=1}^p F_{i_t}^\xi(M) \cup \{m \in M : m > \max F_{i_p}^\xi(M)\}$$

and observe that  $F_{i_t}^\xi(M) = \bigcup_{j=r_{t-1}+1}^{r_t} F_j^{\xi-1}(L)$  for all  $t \leq p$ , where  $r_0 = 0 < r_1 < \dots < r_p$  are chosen so that  $r_t - r_{t-1} = \min F_{i_t}^\xi(M)$  for all  $t \leq p$ .

Therefore,

$$\xi_{i_t}^M = \frac{1}{\min F_{i_t}^\xi(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi - 1)_j^L, \quad t \leq p,$$

and thus

$$\begin{aligned} \sum_{t=2}^p \xi_{i_t}^M(G) &= \sum_{t=2}^p \sum_{s=1}^l \frac{1}{\min F_{i_t}^\xi(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi - 1)_j^L(G_s) \\ &= \sum_{s=1}^l \sum_{t=2}^p \frac{1}{\min F_{i_t}^\xi(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi - 1)_j^L(G_s) \\ &\leq \sum_{s=1}^l \frac{1}{\min F_{i_2}^\xi(M)} \left( \sum_{t=2}^p \sum_{j=r_{t-1}+1}^{r_t} (\xi - 1)_j^L(G_s) \right) \\ &\leq \sum_{s=1}^l \frac{1}{\min F_{i_2}^\xi(M)} \xi \quad \text{by the induction hypothesis} \\ &= \frac{l}{\min F_{i_2}^\xi(M)} \xi \\ &\leq \xi \quad \text{as } l \leq \min G \leq \max F_{i_1}^\xi(M) < \min F_{i_2}^\xi(M). \end{aligned}$$

Finally,  $\sum_{t=1}^p \xi_{i_t}^M(G) \leq 1 + \xi$  and hence  $\sum_{i=1}^n \xi_i^M(G) \leq 1 + \xi$ . We conclude, since  $G \in S_\xi$  was arbitrary, that  $\|\sum_{i=1}^n \xi_i^M\|_\xi \leq \xi + 1$ , as claimed. ■

Let now  $M \in [\mathbb{N}]$ . By identifying elements of  $[M]$  with their indicator functions,  $[M]$  can be endowed with the topology of pointwise convergence. It is not difficult to see that  $[M]$  is then homeomorphic to a  $G_\delta$  subset of the Cantor set  $\{0, 1\}^\mathbb{N}$ , and thus it is a zero-dimensional Polish space. Further,  $[M]$  is perfect (that is, contains no isolated points) and every compact subset of  $[M]$  is nowhere dense. It is then a classical result that  $[M]$ , endowed with the topology of pointwise convergence, is homeomorphic to the space of irrational numbers with the ordinary topology. It is worthwhile to note here that the family

$$\{W(p_1, \dots, p_k) : k \in \mathbb{N}, p_1 < \dots < p_k, p_i \in M, i \leq k\}$$

where  $W(p_1, \dots, p_k) = \{L \in [M], L = (l_i) : l_i = p_i, i \leq k\}$ , forms a basis of clopen subsets for the topology of pointwise convergence in  $[M]$ .

**3. Proofs of the main results.** This section is devoted to the proofs of Theorems 1.1 and 1.3.

**DEFINITION 3.1.** Let  $\xi < \omega_1$  and  $A \in [\mathbb{N}]^{<\omega}$ . We set

$$\tau_\xi(A) = \max\{n \in \mathbb{N} : A \cap F_n^\xi(A \cup \{m \in \mathbb{N} : m > \max A\}) \neq \emptyset\}.$$



We observe that  $\tau_\xi(A)$  remains invariant if  $\{m \in \mathbb{N} : m > \max A\}$  is replaced by  $\{m \in M : m > \max A\}$ ,  $M \in [\mathbb{N}]$ , in Definition 3.1. The quantity  $\tau_\xi(A)$  is important for our purposes since it will enable us to state a criterion for determining whether or not the sequence  $(e_{l_n}^\xi)$  dominates  $(e_{m_n}^\xi)$ , where  $L = (l_n)$  and  $M = (m_n)$  belong to  $[\mathbb{N}]$ . Our next lemma describes some permanence properties of  $\tau_\xi(A)$ .

LEMMA 3.2. Let  $\xi < \omega_1$  and  $A, B$  belong to  $[\mathbb{N}]^{<\omega}$ .

1. If  $A \subset B$  then  $\tau_\xi(A) \leq \tau_\xi(B)$ .
2. If  $A < B$  then  $\tau_\xi(A \cup B) \leq \tau_\xi(A) + \tau_\xi(B)$ .
3. If  $A = \{a_1 < \dots < a_n\}$ ,  $B = \{b_1 < \dots < b_n\}$ ,  $n \in \mathbb{N}$ , and  $a_i \leq b_i$  for  $i \leq n$ , then  $\tau_\xi(B) \leq \tau_\xi(A)$ .
4. Assume that  $A = \bigcup_{i=1}^n A_i$ ,  $B = \bigcup_{i=1}^n B_i$ , where  $n \in \mathbb{N}$ , and  $A_1 < \dots < A_n$ ,  $B_1 < \dots < B_n$  are maximal members of  $S_\xi$ . If  $\min A_i \leq \min B_i$  for all  $i \leq n$ , then  $\tau_{\xi+1}(B) \leq \tau_{\xi+1}(A)$ .
5. Assume that  $A = \bigcup_{i=1}^n A_i$  for some  $n \in \mathbb{N}$ . Then

$$\tau_\xi(A) \leq \left( \sum_{i=1}^n \tau_\xi(A_i) \right) (\xi + 1) + 1$$

for any  $\xi < \omega$ .

Proof. The first two properties are immediate consequences of Definition 3.1. The third property follows because  $S_\xi$  is spreading. Let us show that 4 holds. This is accomplished by induction on  $n$ . The case  $n = 1$  is easy because  $\tau_{\xi+1}(B) = \tau_{\xi+1}(A) = 1$ . Assuming the assertion true for all  $k < n$ , we set  $k_1 = \min A_1$  and  $l_1 = \min B_1$ . In case  $l_1 \geq n$ , we obtain  $B \in S_{\xi+1}$ . Thus  $\tau_{\xi+1}(B) = 1$  and hence the assertion holds.

Next suppose that  $l_1 < n$ . It follows that  $\bigcup_{i=1}^{k_1} A_i$  and  $\bigcup_{i=1}^{l_1} B_i$  are maximal  $S_{\xi+1}$  sets. On the other hand, because  $n - l_1 < n$ , the induction hypothesis yields  $\tau_{\xi+1}(\bigcup_{i=l_1+1}^n B_i) \leq \tau_{\xi+1}(\bigcup_{i=l_1+1}^n A_i)$ . But also  $k_1 \leq l_1$ , and so property 1 yields that  $\tau_{\xi+1}(\bigcup_{i=l_1+1}^n A_i) \leq \tau_{\xi+1}(\bigcup_{i=k_1+1}^n A_i)$ . The proof is complete since  $\tau_{\xi+1}(B) = 1 + \tau_{\xi+1}(\bigcup_{i=l_1+1}^n B_i)$ , while  $\tau_{\xi+1}(A) = 1 + \tau_{\xi+1}(\bigcup_{i=k_1+1}^n A_i)$ .

We now prove 5. Let  $k = \tau_\xi(A)$  and  $M = A \cup \{m \in \mathbb{N} : m > \max A\}$ . Denote  $\sum_{j=1}^{k-1} \xi_j^M$  by  $x$ . By Lemma 2.3,  $\|x\|_\xi \leq \xi + 1$ . Hence

$$k - 1 = x(A) \leq \sum_{i=1}^n x(A_i) \leq \left( \sum_{i=1}^n \tau_\xi(A_i) \right) \|x\|_\xi \leq \left( \sum_{i=1}^n \tau_\xi(A_i) \right) (\xi + 1),$$

from which the result follows. ■

DEFINITION 3.3. Let  $\xi < \omega_1$  and  $L = (l_n)$ ,  $M = (m_n)$  belong to  $[\mathbb{N}]$ .

Define

$$d_\xi(L, M) = \sup\{\tau_\xi(\phi^{-1}A) : A \in S_\xi[M]\},$$

where  $\phi : L \rightarrow M$  is the natural bijection  $\phi(l_n) = m_n$  for all  $n \in \mathbb{N}$ .

The reason we introduced the quantity  $d_\xi(L, M)$  is justified by our next lemma.

LEMMA 3.4. Let  $\xi < \omega$  and  $L = (l_n)$ ,  $M = (m_n)$  belong to  $[\mathbb{N}]$ . Then  $(e_{l_n}^\xi)$  dominates  $(e_{m_n}^\xi)$  if and only if  $d_\xi(L, M)$  is finite.

Proof. Suppose first that  $d_\xi(L, M) = p < \infty$ . Let  $(a_i)_{i=1}^n$  be scalars and choose  $F \in S_\xi[M]$  so that  $\sum_{i \in H} |a_i| = \|\sum_{i=1}^n a_i e_{m_i}^\xi\|$ , where we have set  $H = \{i \leq n : m_i \in F\}$ . It follows, by our assumption, that we can find successive  $S_\xi[L]$  sets  $G_1 < \dots < G_p$  so that  $\phi^{-1}F \subset \bigcup_{j=1}^p G_j$ . We now set  $H_j = \{i \in H : l_i \in G_j\}$  for all  $j \leq p$ . It is clear that  $H = \bigcup_{j=1}^p H_j$ , and moreover,  $\{l_i : i \in H_j\}$  belongs to  $S_\xi[L]$ . Finally,

$$\sum_{i \in H} |a_i| = \sum_{j=1}^p \sum_{i \in H_j} |a_i| \leq p \left\| \sum_{i=1}^n a_i e_{l_i}^\xi \right\|.$$

Thus,  $\|\sum_{i=1}^n a_i e_{m_i}^\xi\| \leq p \|\sum_{i=1}^n a_i e_{l_i}^\xi\|$ .

Conversely, assume that  $(e_{l_n}^\xi)$   $C$ -dominates  $(e_{m_n}^\xi)$  and let  $F \in S_\xi[M]$ . Suppose that  $\tau_\xi(\phi^{-1}F) = k$ . It follows that there exist successive maximal  $S_\xi[L]$  sets  $G_1 < \dots < G_{k-1}$  so that  $\bigcup_{i=1}^{k-1} G_i \subset \phi^{-1}F$ . Put

$$Q = \bigcup_{i=1}^{k-1} G_i \cup \{l \in L : l > \max G_{k-1}\}.$$

We may write  $\xi_i^Q = \sum_{j \in G_i} a_j^i e_j^\xi$  with  $\sum_{j \in G_i} a_j^i = 1$  for all  $i \leq k - 1$ . If we apply Lemma 2.3, we obtain

$$C(\xi + 1) \geq C \left\| \sum_{i=1}^{k-1} \xi_i^Q \right\|_\xi = C \left\| \sum_{i=1}^{k-1} \sum_{j \in G_i} a_j^i e_j^\xi \right\| \geq \left\| \sum_{i=1}^{k-1} \sum_{j \in G_i} a_j^i e_{\phi(j)}^\xi \right\| \geq k - 1,$$

as  $\bigcup_{i=1}^{k-1} \{\phi(j) : j \in G_i\} \subset F$  and  $\sum_{j \in G_i} a_j^i = 1$ . Hence  $k \leq C(\xi + 1) + 1$ , which implies that  $d_\xi(L, M) \leq C(\xi + 1) + 1$  as  $F$  was an arbitrary  $S_\xi[M]$  set. ■

We shall next show that  $(e_n^\xi)$  has “many” non-equivalent subsequences.

LEMMA 3.5. Let  $1 \leq \xi < \omega$ ,  $N \in [\mathbb{N}]$  and set

$$\mathcal{D} = \{(L, M) \in [N] \times [N] : d_\xi(L, M) = d_\xi(M, L) = \infty\}.$$

Then  $\mathcal{D}$  is a  $G_\delta$  dense subset of  $[N] \times [N]$ .

**Proof.** By Baire's theorem, it suffices to show that the sets  $\{(L, M) \in [N] \times [N] : d_\xi(L, M) < \infty\}$  and  $\{(L, M) \in [N] \times [N] : d_\xi(M, L) < \infty\}$  are first category  $F_\sigma$  subsets of  $[N] \times [N]$ . Indeed, we may write

$$\{(L, M) : d_\xi(L, M) < \infty\} = \bigcup_{n=1}^{\infty} \{(L, M) : d_\xi(L, M) \leq n\}.$$

It is easy to see that each set in the union is closed in  $[N] \times [N]$  and thus it remains to show that  $\{(L, M) \in [N] \times [N] : d_\xi(L, M) \leq n\}$  has empty interior in  $[N] \times [N]$ . If that were not the case, choose non-empty basic clopen subsets  $\mathcal{U}$  and  $\mathcal{V}$  of  $[N]$  so that  $\mathcal{U} \times \mathcal{V}$  is contained in  $\{(L, M) \in [N] \times [N] : d_\xi(L, M) \leq n\}$ . There exist  $p_1 < \dots < p_k$  in  $N$  so that  $\mathcal{V} = W(p_1, \dots, p_k)$ . Fix  $L \in \mathcal{U}$ ,  $L = (l_i)$ . If  $M \in [N]$  with  $\min M > p_k$ , let  $P = \{p_1, \dots, p_k\} \cup M$ . Since  $d_\xi(L, P) \leq n$ , it follows that if  $L_k = (l_{k+i})_{i=1}^{\infty}$ , then  $d_\xi(L_k, M) \leq n$ . By Lemma 3.4 this implies that  $(e_i^\xi)_{i \in L_k}$  is equivalent to the unit vector basis of  $\ell_1$ , which is absurd. Arguing similarly, we also find that  $\{(L, M) \in [N] \times [N] : d_\xi(M, L) < \infty\}$  is a first category  $F_\sigma$  subset of  $[N] \times [N]$ . ■

We also need the following result which is a special case of a theorem by Mycielski [12] and Kuratowski [10] (cf. also [9]).

**PROPOSITION 3.6.** *Let  $K$  be a perfect Polish space and  $G$  a  $G_\delta$  dense subset of  $K \times K$ . There exists a subset  $C$  of  $K$  homeomorphic to the Cantor set such that  $C \times C \setminus \Delta \subset G$  (here  $\Delta$  is the diagonal of  $K \times K$ ).*

This result may be found in [9] (p. 129, Theorem 19.1) but we shall include a proof to be thorough.

**LEMMA 3.7.** *Let  $K$  be Polish and  $G$  be an open dense subset of  $K \times K$ . Let  $(A_i)_{i=1}^n$  ( $n \geq 2$ ) be a finite sequence of open non-empty subsets of  $K$ . Then for every  $\varepsilon > 0$  there exist open non-empty subsets  $(B_i)_{i=1}^n$  of  $K$  so that  $\text{diam } B_i < \varepsilon$  for all  $i \leq n$  and  $\bar{B}_i \times \bar{B}_j \subset G \cap (A_i \times A_j)$  for all  $i \neq j$  in  $\{1, \dots, n\}$ .*

**Proof.** By induction on  $n$ . Suppose first that  $n = 2$ . Since  $(A_1 \times A_2) \cap G \neq \emptyset$ , there exist open non-empty subsets  $C_1, C_2$  of  $K$ , with diameters smaller than  $\varepsilon$ , so that  $\bar{C}_1 \times \bar{C}_2 \subset G \cap (A_1 \times A_2)$ . Further,  $(C_2 \times C_1) \cap G \neq \emptyset$  and thus there exist open non-empty subsets  $B_1, B_2$  of  $K$  so that  $\bar{B}_2 \times \bar{B}_1 \subset G \cap (C_2 \times C_1)$ . Of course,  $B_1$  and  $B_2$  satisfy the conclusion of the lemma for  $n = 2$ .

Next assume  $n > 2$  and that the result holds for  $n - 1$ . We can therefore choose open non-empty subsets  $(C_i)_{i=1}^{n-1}$  of  $K$ , with diameters smaller than  $\varepsilon$ , so that  $\bar{C}_i \times \bar{C}_j \subset G \cap (A_i \times A_j)$ , for all  $i \neq j$  in  $\{1, \dots, n - 1\}$ . Next, set  $A_{n,0} = A_n$  and choose, as in the case  $n = 2$ , non-empty open subsets  $(B_i)_{i=1}^{n-1}, (A_{n,i})_{i=1}^{n-1}$  of  $K$ , with diameters smaller than  $\varepsilon$ , so that  $\bar{B}_i \times \bar{A}_{n,i} \subset$

$G \cap (C_i \times A_{n,i-1})$  and  $\bar{A}_{n,i} \times \bar{B}_i \subset G \cap (A_{n,i-1} \times C_i)$ , for all  $i \leq n - 1$ . Set  $B_n = A_{n,n-1}$  and it is easy to check that  $(B_i)_{i=1}^n$  is the desired sequence. ■

**Proof of Proposition 3.6.** Since  $K$  contains no isolated points,  $\Delta$  is nowhere dense in  $K \times K$ . Hence,  $G \cap (K \times K \setminus \Delta)$  is a  $G_\delta$  dense subset of  $K \times K$ . We shall therefore assume, without loss of generality, that  $G \cap \Delta = \emptyset$ . Now let  $(G_n)$  be a decreasing sequence of open dense subsets of  $K \times K$  whose intersection is  $G$ . We can assume that  $G_n \cap \Delta = \emptyset$  for all  $n \in \mathbb{N}$ .

We shall construct a collection  $\{U_\alpha : \alpha \in \{0, 1\}^{\mathbb{N}}, n \in \mathbb{N}\}$  of open non-empty subsets of  $K$  so that the following properties are satisfied for every  $n \in \mathbb{N}$ :

- (i)  $\bar{U}_\alpha \cap \bar{U}_\beta = \emptyset$  whenever  $\alpha \neq \beta$  in  $\{0, 1\}^{\mathbb{N}}$ .
- (ii)  $\bar{U}_\alpha \subset U_\beta$  for all  $\alpha \in \{0, 1\}^n$  and every  $\beta \in \{0, 1\}^m$  ( $m < n$ ) which is an initial segment of  $\alpha$ .
- (iii)  $\text{diam } U_\alpha < 1/n$  for every  $\alpha \in \{0, 1\}^n$ .
- (iv)  $\bar{U}_\alpha \times \bar{U}_\beta \subset G_n$  whenever  $\alpha \neq \beta$  in  $\{0, 1\}^n$ .

Once this is accomplished, we let

$$C = \left\{x \in K : \exists \alpha \in \{0, 1\}^{\mathbb{N}}, \{x\} = \bigcap_{n=1}^{\infty} U_{\alpha|n}\right\},$$

where  $\alpha|n = (a_1, \dots, a_n)$  if  $\alpha = (a_i) \in \{0, 1\}^{\mathbb{N}}$ . It is a standard result that  $C$  is homeomorphic to the Cantor set. Property (iv) yields that  $C$  satisfies the conclusion of Proposition 3.6.

The construction is done by induction on  $n \in \mathbb{N}$ . For  $n = 1$  choose open non-empty subsets  $W_0$  and  $W_1$  of  $K$  so that  $W_0 \times W_1 \subset G_1$ . Then  $W_0$  and  $W_1$  are disjoint since  $G_1 \cap \Delta = \emptyset$ . If we apply Lemma 3.7, for  $\varepsilon = 1$ , to the dense open subset  $G_1$  and the open sets  $W_0$  and  $W_1$ , we shall find non-empty open subsets  $U_0, U_1$  of  $K$  having properties (i)–(iv) for  $n = 1$ .

Now suppose that for every  $k \leq n$  we have constructed a collection  $\{U_\alpha : \alpha \in \{0, 1\}^k\}$  of open non-empty subsets of  $K$  whose members share properties (i)–(iv) for  $k$ . Let  $\{d_1, \dots, d_p\}$ ,  $p = 2^n$ , be an enumeration of  $\{0, 1\}^n$ . Another application of Lemma 3.7 yields non-empty open subsets  $W_{j0}$  and  $W_{j1}$  of  $K$ ,  $j \leq p$ , so that  $\bar{W}_{jr} \times \bar{W}_{js} \subset (U_{d_j} \times U_{d_s}) \cap G_{n+1}$  for every  $j \leq p$  and all pairs  $(r, s)$  of distinct elements of  $\{0, 1\}$ . It follows, since  $G_{n+1} \cap \Delta = \emptyset$ , that  $W_{j0} \cap W_{j1} = \emptyset$ . According to the induction hypothesis  $\bar{U}_{d_j} \cap \bar{U}_{d_i} = \emptyset$  for all  $i \neq j$  in  $\{1, \dots, p\}$ , and thus  $\bar{W}_{jr} \cap \bar{W}_{is} = \emptyset$  for all  $(j, r) \neq (i, s)$  in  $\{1, \dots, p\} \times \{0, 1\}$ .

We next apply Lemma 3.7, for  $\varepsilon = 1/(n + 1)$ , to the family  $\{W_{jr} : (j, r) \in \{1, \dots, p\} \times \{0, 1\}\}$  and the dense open subset  $G_{n+1}$ . We obtain a collection  $\{U_\alpha : \alpha \in \{0, 1\}^{n+1}\}$  of non-empty open subsets of  $K$  so that  $\bar{U}_\alpha \times \bar{U}_\beta \subset (W_{jr} \times W_{is}) \cap G_{n+1}$  whenever  $\alpha = (d_j, r)$ ,  $\beta = (d_i, s)$  and  $(j, r) \neq (i, s)$  in  $\{1, \dots, p\} \times \{0, 1\}$ . Evidently,  $\{U_\alpha : \alpha \in \{0, 1\}^{n+1}\}$  has

properties (i)–(iv). The inductive step and hence the proof of the proposition are now complete. ■

Assuming we have proved Theorem 1.1 and Corollary 1.2, let us now show how to derive Theorem 1.3 from our previous results.

*Proof of Theorem 1.3.* Let  $\mathcal{D}$  be as in the statement of Lemma 3.5, where we have taken  $N = \mathbb{N}$ . We can apply Proposition 3.6 for the space  $[\mathbb{N}]$  and the set  $\mathcal{D}$  to obtain  $\mathcal{A} \subset [\mathbb{N}]$  homeomorphic to the Cantor set and such that  $\mathcal{A} \times \mathcal{A} \setminus \Delta \subset \mathcal{D}$ . Lemma 3.4 and Corollary 1.2 yield that  $\mathcal{A}$  is the desired subset of  $[\mathbb{N}]$ . ■

We shall next pass to the proof of Theorem 1.1. We start with some necessary lemmas.

LEMMA 3.8. Let  $G \in [\mathbb{N}]^{<\omega}$  and  $\xi < \omega$ . The following are equivalent:

1.  $G$  is a member (resp. maximal member) of  $S_\xi$ .
2. For every  $0 \leq \zeta \leq \xi$  there exist  $n \in \mathbb{N}$  and successive members (resp. maximal members)  $G_1 < \dots < G_n$  of  $S_\zeta$  so that  $G = \bigcup_{i=1}^n G_i$  and  $\{\min G_i : i \leq n\}$  is a member (resp. maximal member) of  $S_{\xi-\zeta}$ .
3. There exist  $0 \leq \zeta \leq \xi$ ,  $n \in \mathbb{N}$  and successive members (resp. maximal members)  $G_1 < \dots < G_n$  of  $S_\zeta$  so that  $G = \bigcup_{i=1}^n G_i$  and  $\{\min G_i : i \leq n\}$  is a member (resp. maximal member) of  $S_{\xi-\zeta}$ .

PROOF. We show that all three conditions are equivalent for the members of  $S_\xi$ .

1 $\Rightarrow$ 2. Induction on  $\xi$ . If  $\xi = 0$  the assertion is trivial. Suppose now  $\xi \geq 1$  and that the assertion holds for  $\xi - 1$ . Let  $\zeta \leq \xi$ . If  $\zeta = \xi$ , the assertion is again trivial. So assume  $\zeta < \xi$ . Choose  $H_1 < \dots < H_p$  in  $S_{\xi-1}$  so that  $p \leq \min H_1$  and  $G = \bigcup_{i=1}^p H_i$ . The induction hypothesis yields that for each  $i \leq p$  there exist  $H_{i1} < \dots < H_{ir_i}$  in  $S_\zeta$  so that  $\{\min H_{ij} : j \leq r_i\}$  belongs to  $S_{\xi-\zeta-1}$  and  $H_i = \bigcup_{j=1}^{r_i} H_{ij}$ . Let  $\{G_1, \dots, G_n\}$  be an enumeration of the set  $\{H_{ij} : j \leq r_i, i \leq p\}$  so that  $G_1 < \dots < G_n$ . Note that  $\{\min G_i : i \leq n\} = \bigcup_{i=1}^p \{\min H_{ij} : j \leq r_i\}$  and so it is a member of  $S_{\xi-\zeta}$  as  $p \leq \min H_{11} = \min H_1$ .

2 $\Rightarrow$ 3. This implication is trivial.

3 $\Rightarrow$ 1. Again induction on  $\xi$ . If  $\xi = 0$  the assertion is trivial. Suppose now  $\xi \geq 1$  and that the assertion holds for  $\xi - 1$ . Let  $\zeta \leq \xi$ . If  $\zeta = \xi$ , the assertion is again trivial. So assume  $\zeta < \xi$ . We first apply 1 $\Rightarrow$ 2 for the set  $\{\min G_i : i \leq n\} \in S_{\xi-\zeta}$  to obtain  $H_1 < \dots < H_p$  in  $S_{\xi-\zeta-1}$  so that  $\{\min G_i : i \leq n\} = \bigcup_{i=1}^p H_i$  and  $\{\min H_i : i \leq p\} \in S_1$ . Set  $I_i = \{j \leq n : \min G_j \in H_i\}$ ,  $i \leq p$ . Then  $\bigcup_{j \in I_i} G_j \in S_{\xi-1}$  by the induction hypothesis since  $\{\min G_j : j \in I_i\} = H_i$ , which belongs to  $S_{\xi-\zeta-1}$ . Finally,  $G = \bigcup_{i=1}^p (\bigcup_{j \in I_i} G_j) \in S_\xi$ , as  $\min G = \min H_1 \geq p$ . The latter inequality holds because  $\{\min H_i : i \leq p\} \in S_1$ .

The proof for the case of maximal Schreier sets requires only minor modifications. Namely, all the sets which belong to an appropriate class  $S_\alpha$ ,  $\alpha \leq \xi$ , and appear in the previous arguments, can be taken to be maximal members of  $S_\alpha$ . ■

LEMMA 3.9. Let  $\xi < \omega_1$ . Suppose that  $L = (l_i)$ ,  $M = (m_i)$  belong to  $[\mathbb{N}]$  and satisfy  $l_i < m_i < l_{i+1}$  for every  $i \in \mathbb{N}$ . Then  $(e_{l_i}^\xi)$  is 2-equivalent to  $(e_{m_i}^\xi)$ . That is,  $\|\sum_{i=1}^n a_i e_{l_i}^\xi\| \leq \|\sum_{i=1}^n a_i e_{m_i}^\xi\| \leq 2\|\sum_{i=1}^n a_i e_{l_i}^\xi\|$  for every  $n \in \mathbb{N}$  and all scalar sequences  $(a_i)_{i=1}^\infty$ .

We omit the easy proof and pass to

LEMMA 3.10. Let  $\xi < \omega$  and  $0 \leq \zeta \leq \xi$ . Let  $L \in [\mathbb{N}]$  and set  $q_n = \min F_n^\zeta(L)$  for all  $n \in \mathbb{N}$ . Then  $(\zeta_n^L)$ , considered as a sequence in  $X^\xi$ , is  $12(\zeta + 1)$ -equivalent to  $(e_{q_n}^{\xi-\zeta})$ .

PROOF. Let  $n \in \mathbb{N}$  and  $(a_i)_{i=1}^n$  be scalars. Choose  $G \subset \{q_1, \dots, q_n\}$ ,  $G \in S_{\xi-\zeta}$ , such that  $\sum_{i \in I} |a_i| = \|\sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta}\|$ , where  $I = \{i \leq n : q_i \in G\}$ . It follows that  $H = \bigcup_{i \in I} F_i^\zeta(L) \in S_\xi$ , by Lemma 3.8. Hence,  $\|\sum_{i=1}^n a_i \zeta_i^L\|_\xi \geq \sum_{i \in I} |a_i|$  and thus  $\|\sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta}\| \leq \|\sum_{i=1}^n a_i \zeta_i^L\|_\xi$ .

We next show that  $\|\sum_{i=1}^n a_i \zeta_i^L\|_\xi \leq 12(\zeta + 1)\|\sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta}\|$ . Let  $G \in S_\xi[L]$  be maximal and put  $G_i = F_i^\zeta(L) \cap G$  for all  $i \leq n$ . We apply Lemma 3.8 to find  $p \in \mathbb{N}$  and maximal members  $H_1 < \dots < H_p$  of  $S_\zeta$  so that  $G = \bigcup_{j=1}^p H_j$  and  $\{\min H_j : j \leq p\}$  is a maximal member of  $S_{\xi-\zeta}$ .

We claim that each of the  $G_i$ 's can intersect at most two of the  $H_j$ 's. Indeed, assume that for some  $i$  and  $j_1 < j_2 < j_3$  we had  $G_i \cap H_{j_r} \neq \emptyset$  for all  $r \leq 3$ . Then  $H_{j_2} \subset G_i$  because  $H_{j_2} \subset [\min G_i, \max G_i]$ . Thus,  $H_{j_2} \subset F_i^\zeta(L)$  and hence  $H_{j_2} = F_i^\zeta(L)$ , by the maximality of  $H_{j_2}$ . It follows that  $H_{j_2} = G_i$ , which is a contradiction as  $H_{j_1} \cap H_{j_2} = \emptyset$ . Therefore our claim holds and evidently, for each  $i \leq n$ ,  $G_i$  intersects either exactly one of the  $H_j$ 's, or exactly two (consecutive)  $H_j$ 's. We can thus partition  $\{1, \dots, n\}$  in the following two subsets:

$$I_1 = \{i \leq n : G_i \subset H_j \text{ for some } j \leq p\},$$

$$I_2 = \{i \leq n : \exists j_1 < j_2 \leq p, G_i \subset H_{j_1} \cup H_{j_2}, G_i \cap H_{j_r} \neq \emptyset, r \leq 2\}.$$

Let  $k_i = \max F_i^\zeta(L)$  for all  $i \leq n$ . We now need the following

CLAIM. Suppose that  $T_i \subset F_i^\zeta(L)$  for all  $i \leq n$ . Assume also that for each  $i \leq n$  there exists  $j \leq p$  so that  $T_i \subset H_j$ . Then

$$\left\| \sum_{i=1}^n a_i \zeta_i^L \left( \bigcup_{m=1}^n T_m \right) \right\| \leq (\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi-\zeta} \right\|.$$

Once the claim is established we finish the proof as follows: Observe that our claim yields

$$\left| \sum_{i \in I_1} a_i \zeta_i^L(G) \right| \leq (\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi - \zeta} \right\|.$$

On the other hand, if  $i \in I_2$  there exist  $A_i < B_i$  so that  $G_i = A_i \cup B_i$  and each element of  $\{A_i, B_i\}$  is contained in some  $H_j$ . Our claim then yields that

$$\left| \sum_{i \in I_2} a_i \zeta_i^L(G) \right| \leq 2(\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi - \zeta} \right\|.$$

Therefore,  $|\sum_{i=1}^n a_i \zeta_i^L(G)| \leq 3(\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi - \zeta} \right\|$ . It now follows, since  $G$  was arbitrary, that  $\|\sum_{i=1}^n a_i \zeta_i^L\|_\xi \leq 6(\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi - \zeta} \right\|$ . The desired estimate now follows from Lemma 3.9.

We proceed to prove our claim. Let  $R_j = \{i \leq n : T_i \neq \emptyset, T_i \subset H_j\}$ ,  $j \leq p$ , and choose  $i_j \in R_j$  such that  $\max_{r \in R_j} |a_r| = |a_{i_j}|$ . Then

$$\begin{aligned} \left| \sum_{i=1}^n a_i \zeta_i^L \left( \bigcup_{m=1}^n T_m \right) \right| &= \left| \sum_{i=1}^n a_i \zeta_i^L(T_i) \right| = \left| \sum_{j=1}^p \sum_{i \in R_j} a_i \zeta_i^L(T_i) \right| \\ &\leq \sum_{j=1}^p \left| \left( \sum_{i \in R_j} a_i \zeta_i^L \right) \left( \bigcup_{m \in R_j} T_m \right) \right| \\ &\leq \sum_{j=1}^p \left\| \sum_{i \in R_j} a_i \zeta_i^L \right\|_\zeta, \quad \text{because } \bigcup_{m \in R_j} T_m \in S_\zeta \\ &\leq \sum_{j=1}^p (\zeta + 1) \max_{i \in R_j} |a_i| \quad \text{by Lemma 2.3} \\ &= \sum_{j=1}^p (\zeta + 1) |a_{i_j}| \leq (\zeta + 1) \left\| \sum_{i=1}^n a_i e_{k_i}^{\xi - \zeta} \right\|. \end{aligned}$$

The last inequality holds since  $T_{i_j} \subset H_j$  implies that  $\min H_j \leq k_{i_j}$  for all  $j \leq p$  and thus  $\{k_{i_j} : j \leq p\} \in S_{\xi - \zeta}$ . ■

We recall here that a bounded sequence  $(x_n)$  in some Banach space is said to be an  $\ell_1^\xi$ -spreading model,  $\xi < \omega_1$ , provided that there exists a constant  $C > 0$  so that  $\|\sum_{i \in F} a_i x_i\| \geq C \sum_{i \in F} |a_i|$  for every  $F \in S_\xi$  and all scalars  $(a_i)_{i \in F}$ .

REMARK. It is easy to see that every subsequence of  $(e_n^\xi)$  is an  $\ell_1^\xi$ -spreading model in  $X^\xi$ . However, Lemma 2.3 implies that no subsequence of  $(e_n^\xi)$  is an  $\ell_1^{\xi+1}$ -spreading model in  $X^\xi$ .

PROPOSITION 3.11. Suppose  $L = (l_i)$ ,  $M = (m_i)$  belong to  $[N]$  and that  $\xi < \omega$ . Assume further that there exist a map  $\psi : L \rightarrow M$  and a bounded linear operator  $T : X_L^\xi \rightarrow X_M^\xi$  so that  $T(e_l^\xi) = e_{\psi(l)}^\xi$  for every  $l \in L$ . Then there exists an integer  $D > 0$  so that  $\tau_\zeta(\psi^{-1}F) \leq D$  for every  $F \in S_\zeta[M]$  and all  $0 \leq \zeta \leq \xi$ .

PROOF. We first note that  $\psi^{-1}F \in [L]^{<\infty}$  for every  $F \in [M]^{<\infty}$ . Indeed, if that were not the case, we would find  $m \in M$  and  $N \in [L]$  so that  $\psi(l) = m$  for every  $l \in N$ . It follows that  $T(e_l^\xi) = e_m^\xi$  for all  $l \in N$ , contradicting our assumption that  $T$  is bounded.

Fix  $0 \leq \zeta \leq \xi$ . Our first task is to show that  $\sup_n \tau_\zeta(\psi^{-1}F_n^\zeta(P)) < \infty$  for every  $P \in [M]$ . Suppose instead that  $\sup_n \tau_\zeta(\psi^{-1}F_n^\zeta(P)) = \infty$  for some  $P \in [M]$ . We claim that there exist a sequence of positive integers,  $(n_i)$ , and a sequence of successive maximal  $S_{\zeta+1}[L]$  sets,  $(G_i)$ , so that letting  $q_i = \min G_i$  for all  $i \in \mathbb{N}$  we get

$$G_i \setminus \{q_i\} \subset \psi^{-1}F_{n_i}^\zeta(P) \quad \text{for all } i \in \mathbb{N}.$$

Indeed, choose  $n_1$  so that  $\tau_\zeta(\psi^{-1}F_{n_1}^\zeta(P)) > l_1$ . Put  $q_1 = l_1$ . Because  $\psi^{-1}F_{n_1}^\zeta(P)$  contains at least  $l_1$  successive maximal  $S_\zeta[L]$  sets, it is clear that there exists  $H_1 \subset \psi^{-1}F_{n_1}^\zeta(P)$ ,  $q_1 < \min H_1$ , so that  $G_1 = \{q_1\} \cup H_1$  is a maximal  $S_{\zeta+1}[L]$  set.

Put  $l_{t_1} = \max G_1$  and  $w_1 = \tau_\zeta(\{l_1, \dots, l_{t_1}\})$ . We can find  $n_2 > n_1$  so that  $\tau_\zeta(\psi^{-1}F_{n_2}^\zeta(P)) > w_1 + l_{t_1+1}$ . Now,  $\{l \in L : l \geq l_{t_1+1}\} \cap \psi^{-1}F_{n_2}^\zeta(P)$  must contain at least  $l_{t_1+1}$  successive maximal  $S_\zeta[L]$  sets. If not, then  $\tau_\zeta(\{l \in L : l \geq l_{t_1+1}\} \cap \psi^{-1}F_{n_2}^\zeta(P)) \leq l_{t_1+1}$  and thus  $\tau_\zeta(\psi^{-1}F_{n_2}^\zeta(P)) \leq l_{t_1+1} + w_1$ , by Lemma 3.2. But this contradicts the choice of  $n_2$ .

We set  $q_2 = l_{t_1+1}$  and arguing as we did in the case  $i = 1$ , we can find  $H_2 \subset \psi^{-1}F_{n_2}^\zeta(P)$ ,  $q_2 < \min H_2$ , so that  $G_2 = \{q_2\} \cup H_2$  is a maximal  $S_{\zeta+1}[L]$  set. We next put  $l_{t_2} = \max G_2$  and continue in the same fashion to obtain sequences  $(n_i)$ ,  $(G_i)$  with the desired properties.

Let  $Q = \bigcup_{i=1}^\infty G_i$ . Clearly,  $Q \in [L]$  and  $F_i^{\zeta+1}(Q) = G_i$  for all  $i \in \mathbb{N}$ . We now set  $R = \bigcup_{i=1}^\infty F_{n_i}^\zeta(P)$ . Then  $R \in [M]$  and  $F_i^\zeta(R) = F_{n_i}^\zeta(P)$  for all  $i \in \mathbb{N}$ . We observe that if  $q \in G_i \setminus \{q_i\}$ , then  $T(e_q^\xi) = e_m^\xi$  for some  $m \in F_i^\zeta(R)$ .

Next write  $(\zeta + 1)_i^Q = a_i e_{q_i}^\xi + (1 - a_i) u_i$  for all  $i \in \mathbb{N}$ . Here,  $u_i$  is a convex combination of the vectors  $(e_q^\xi)_{q \in G_i \setminus \{q_i\}}$  and  $0 < a_i < 1$ . Evidently,  $\lim_i a_i = 0$ . Observe that  $Tu_i$  is a convex combination of the vectors  $(e_m^\xi)_{m \in F_i^\zeta(R)}$  and thus  $\|Tu_i\|_\xi = 1$  for all  $i \in \mathbb{N}$ .

It must be the case that  $\zeta < \xi$  for if not, Lemma 2.3 yields  $\lim_i \|u_i\|_\xi = 0$ . On the other hand  $\|Tu_i\|_\xi = 1$  for all  $i \in \mathbb{N}$ . Hence  $T$  is not bounded, contrary to our assumption. Therefore,  $\zeta < \xi$  and so  $\|u_i\|_\xi = 1$  for all  $i \in \mathbb{N}$ .



Recall that  $\lim_i \|(1-a_i)Tu_i\|_\xi = 1$  and  $(1-a_i)Tu_i$  is supported by  $F_i^\zeta(R)$ . Using Lemma 3.8, it is easy to check that  $((1-a_i)Tu_i)$  is an  $\ell_1^{\xi-\zeta}$ -spreading model in  $X_M^\xi$ , and consequently, since  $\lim_i a_i = 0$ , also  $(T[(\zeta+1)_i^Q])$  is an  $\ell_1^{\xi-\zeta}$ -spreading model in  $X_M^\xi$ . We conclude, as  $T$  is bounded, that  $((\zeta+1)_i^Q)$  is an  $\ell_1^{\xi-\zeta}$ -spreading model in  $X^\xi$ . However, if we apply Lemma 3.10 we deduce that  $(e_{q_n}^{\xi-\zeta-1})$  is an  $\ell_1^{\xi-\zeta-1}$ -spreading model in  $X^{\xi-\zeta-1}$ . But this contradicts the Remark after Lemma 3.10. Hence,  $\sup_n \tau_\zeta(\psi^{-1}F_n^\zeta(P)) < \infty$  for every  $P \in [M]$ . It follows that

$$[M] = \bigcup_{k=1}^{\infty} \{P \in [M] : \tau_\zeta(\psi^{-1}F_n^\zeta(P)) \leq k, \forall n \in \mathbb{N}\}.$$

It is easily seen that every set in the union is closed in  $[M]$ . Baire's theorem now yields  $k_\zeta \in \mathbb{N}$  and  $r_1^\zeta < \dots < r_{s_\zeta}^\zeta$  in  $M$  so that if  $P \in [M]$ ,  $P = (p_i)$ , and  $p_i = r_i^\zeta$ ,  $i \leq s_\zeta$ , then  $\tau_\zeta(\psi^{-1}F_n^\zeta(P)) \leq k_\zeta$  for all  $n \in \mathbb{N}$ . It now follows that there exists  $m_0^\zeta \in M$  so that if  $F \in S_\zeta[M]$ ,  $\min F > m_0^\zeta$ , then  $\tau_\zeta(\psi^{-1}F) \leq k_\zeta$ .

Finally, choose  $D_\zeta \in \mathbb{N}$  so that  $\tau_\zeta(\psi^{-1}F) \leq D_\zeta$  for every  $F \in S_\zeta[M]$ ,  $\max F \leq m_0^\zeta$ . Part 5 of Lemma 3.2 now yields  $\tau_\zeta(\psi^{-1}F) \leq (D_\zeta + k_\zeta)(\zeta + 1) + 1$  for every  $F \in S_\zeta[M]$ . To complete the proof we need only take  $D = \max\{(D_\zeta + k_\zeta)(\zeta + 1) + 1 : \zeta \leq \xi\}$ . ■

**PROPOSITION 3.12.** *Suppose  $L = (l_i)$ ,  $M = (m_i)$  belong to  $[\mathbb{N}]$  and that  $\xi < \omega$ . Assume further that there exist a map  $\psi : L \rightarrow M$  and an integer  $D > 0$  so that  $\tau_\zeta(\psi^{-1}F) \leq D$  for every  $F \in S_\zeta[M]$  and all  $0 \leq \zeta \leq \xi$ . Then there exist integer constants  $E_\zeta$ ,  $0 \leq \zeta \leq \xi$ , so that  $\tau_\zeta(\phi^{-1}F) \leq E_\zeta$  for every  $F \in S_\zeta[M]$  and all  $0 \leq \zeta \leq \xi$  ( $\phi : L \rightarrow M$  is the natural bijection  $\phi(l_i) = m_i$ ).*

**Proof.** If  $\zeta = 0$  it is trivial ( $E_0 = 1$ ). Suppose the assertion holds for some  $\zeta \leq \xi - 1$ . We show  $E_{\zeta+1} = [(\zeta+1)E_\zeta + 1][(2D+1)(\zeta+2) + 1]$  works for  $\zeta+1$ . Let  $F \in S_{\zeta+1}[M]$ ,  $F = \{m_{i_1}, \dots, m_{i_p}\}$ . Our hypothesis yields that  $\{l_{i_1}, \dots, l_{i_p}\}$  is contained in the union of  $E_\zeta m_{i_1}$   $S_\zeta[L]$  sets and so  $\tau_\zeta(\{l_{i_1}, \dots, l_{i_p}\}) \leq (\zeta+1)E_\zeta m_{i_1} + 1$  by part 5 of Lemma 3.2. Choose  $q_1 \in \mathbb{N}$  so that the set  $\{l_{i_1+j} : 0 \leq j \leq q_1\}$  is the union of exactly  $[(\zeta+1)E_\zeta + 1]m_{i_1}$  successive maximal  $S_\zeta[L]$  sets.

**CLAIM.**  $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q_1\}) \leq E_{\zeta+1}$ .

Once our claim is proven, we apply Lemmas 2.2 and 3.2 (parts 4 and 1) to conclude that  $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq p-1\}) \leq E_{\zeta+1}$  and  $\tau_{\zeta+1}(\{l_{i_1}, \dots, l_{i_p}\}) \leq E_{\zeta+1}$ .

To prove the claim we choose  $q < q_1$  so that the set  $\{l_{i_1+j} : 0 \leq j \leq q\}$  is the union of exactly  $m_{i_1}$  successive maximal  $S_\zeta[L]$  sets. Our task now is

to show that  $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q\}) \leq (2D+1)(\zeta+2) + 1$ . The claim will then follow by applying parts 4 and 2 of Lemma 3.2.

We first observe that if  $0 \leq j_0 \leq q$  is chosen so that  $l_{i_1+j} \leq m_{i_1+j}$  for all  $j \leq j_0$ , then  $\{l_{i_1+j} : 0 \leq j \leq j_0\}$  is contained in the union of  $2D$   $S_{\zeta+1}[L]$  sets.

Indeed,  $\{m_{i_1+j} : 0 \leq j \leq j_0\}$  belongs to  $S_{\zeta+1}[M]$ , by part 3 of Lemma 3.2 and the fact that  $\tau_\zeta(\{l_{i_1+j} : 0 \leq j \leq j_0\}) \leq m_{i_1}$ . It now follows that  $\Psi = \{\psi(l_{i_1+j}) : 0 \leq j \leq j_0, \psi(l_{i_1+j}) \geq m_{i_1}\}$  belongs to  $S_{\zeta+1}[M]$ . To see this let  $\{m_{t_0} < \dots < m_{t_k}\}$ , where  $k \leq j_0$  and  $i_1 \leq t_0$ , be an enumeration of  $\Psi$ . Then  $m_{t_j} \geq m_{i_1+j}$  for every  $0 \leq j \leq k$ . Since  $\{m_{i_1+j} : 0 \leq j \leq k\}$  belongs to  $S_{\zeta+1}[M]$ , which is spreading, we conclude that  $\Psi$  belongs to  $S_{\zeta+1}[M]$ . Our hypothesis (for  $\zeta+1$ ) yields that  $\psi^{-1}(\Psi)$  is contained in the union of  $D$   $S_{\zeta+1}[L]$ -sets, hence so is its subset  $\{l_{i_1+j} : 0 \leq j \leq j_0, \psi(l_{i_1+j}) \geq m_{i_1}\}$ . On the other hand, the cardinality of  $\{\psi(l_{i_1+j}) : 0 \leq j \leq j_0, \psi(l_{i_1+j}) < m_{i_1}\}$  is at most  $i_1 - 1$ . Our hypothesis (for  $\zeta = 0$ ) now yields that the cardinality of the set  $\{l_{i_1+j} : \psi(l_{i_1+j}) < m_{i_1}, 0 \leq j \leq j_0\}$  is at most  $D(i_1 - 1)$ . We deduce, since  $l_{i_1} > i_1 - 1$ , that  $\{l_{i_1+j} : \psi(l_{i_1+j}) < m_{i_1}, 0 \leq j \leq j_0\}$  is contained in the union of  $D$   $S_1[L]$  sets. Hence,  $\{l_{i_1+j} : 0 \leq j \leq j_0\}$  is contained in the union of  $2D$   $S_{\zeta+1}[L]$  sets.

Next set  $j_1 = \min\{j : 0 \leq j \leq q \text{ and } l_{i_1+j} > m_{i_1+j}\}$ . If  $j_1$  does not exist, then  $l_{i_1+j} \leq m_{i_1+j}$  for all  $0 \leq j \leq q$ . We deduce, by our previous observation for  $j_0 = q$ , that  $\{l_{i_1+j} : 0 \leq j \leq q\}$  is contained in the union of  $2D$   $S_{\zeta+1}[L]$  sets.

If  $j_1$  does exist, then  $\{l_{i_1+j} : 0 \leq j < j_1\}$  is contained in the union of  $2D$   $S_{\zeta+1}[L]$  sets. Indeed, this is obvious if  $j_1 = 0$ . If  $j_1 \geq 1$  the assertion follows from our previous observation by taking  $j_0 = j_1 - 1$ . Finally,  $\{l_{i_1+j} : j_1 \leq j \leq q\}$  belongs to  $S_{\zeta+1}[L]$ , since  $l_{i_1+j_1} > m_{i_1+j_1} \geq m_{i_1}$  and  $\{l_{i_1+j} : j_1 \leq j \leq q\}$  is contained in the union of  $m_{i_1}$  successive maximal  $S_\zeta[L]$  sets. Thus,  $\{l_{i_1+j} : 0 \leq j \leq q\}$  is contained in the union of  $2D+1$   $S_{\zeta+1}[L]$  sets.

Concluding, in any case the set  $\{l_{i_1+j} : 0 \leq j \leq q\}$  is contained in the union of  $2D+1$   $S_{\zeta+1}[L]$  sets. Hence, applying part 5 of Lemma 3.2, we find that  $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q\}) \leq (2D+1)(\zeta+2) + 1$ , as desired. ■

**PROPOSITION 3.13.** *Let  $\xi < \omega_1$  and  $L = (l_i)$ ,  $M = (m_i)$  be in  $[\mathbb{N}]$ . Suppose that there exist  $\delta > 0$  and a bounded linear operator  $T : X_L^\xi \rightarrow X_M^\xi$  such that  $\|T(e_l^\xi)\|_0 > \delta$  for every  $l \in L$ . Then there exist a map  $\psi : L \rightarrow M$  and a bounded linear operator  $R : X_L^\xi \rightarrow X_M^\xi$  such that  $R(e_l^\xi) = e_{\psi(l)}^\xi$  for every  $l \in L$ .*

**Proof.** Following [11], given two infinite matrices  $(a_{ij})$  and  $(d_{ij})$ , we shall call  $(d_{ij})$  a *block diagonal* of  $(a_{ij})$  if there exist increasing sequences

$(r_k), (s_k)$  of positive integers so that

$$d_{ij} = \begin{cases} a_{ij} & \text{if } (i, j) \in \bigcup_{k=1}^{\infty} [r_k, r_{k+1}) \times [s_k, s_{k+1}), \\ 0 & \text{otherwise.} \end{cases}$$

We represent  $T$  as an infinite matrix  $(a_{ij})$ . Then  $T(e_{l_i}^\xi) = \sum_{j=1}^{\infty} a_{ij} e_{m_j}^\xi$  for every  $i \in \mathbb{N}$ . Because  $\|T(e_{l_i}^\xi)\|_0 > \delta$ , for every  $i \in \mathbb{N}$  there exists  $j \in \mathbb{N}$  such that  $|a_{ij}| > \delta$ . We can thus define a map  $\psi : L \rightarrow M$  so that if  $\psi(l_i) = m_j$ , then  $|a_{ij}| > \delta$ . Observe that  $\psi^{-1}\{m_j\}$  is finite for all  $j \in \mathbb{N}$ , since  $(T(e_{l_i}^\xi))$  is weakly null in  $X_M^\xi$ . In particular,  $\psi(L) \in [M]$ . Let  $(m_{k_j})_{j=1}^{\infty}$  be the increasing enumeration of  $\psi(L)$ . Given  $x = \sum_{i=1}^{\infty} \lambda_i e_{l_i} \in X_L^\xi$ , we set  $S(x) = \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} \lambda_i a_{ik_j}) e_{m_{k_j}}^\xi$ . It follows, since  $T$  is bounded and  $(e_n^\xi)$  is unconditional, that  $S$  is a well defined bounded linear operator from  $X_L^\xi$  into  $X_{\psi(L)}^\xi$ . Moreover, the matrix representation  $(c_{ij})$  of  $S$  with respect to the bases  $(e_{l_i}^\xi)$  and  $(e_{m_{k_j}}^\xi)$  is given by  $c_{ij} = a_{ik_j}$  for all positive integers  $i, j$ .

We next consider the matrix  $(b_{ij})$  given by

$$b_{ij} = \begin{cases} c_{ij} & \text{if } \psi(l_i) = m_{k_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exists a unique non-zero entry in every row of the matrix  $(b_{ij})$ , while each column contains only finitely many non-zero entries. We can thus find a permutation  $p$  of  $\mathbb{N}$  so that the matrix  $(b_{p(i)j})$  is a block diagonal of  $(c_{p(i)j})$ . Since  $(c_{p(i)j})$  represents the bounded linear operator  $S : X_L^\xi \rightarrow X_{\psi(L)}^\xi$  with respect to the bases  $(e_{l_{p(i)}}^\xi)$  and  $(e_{m_{k_j}}^\xi)$ , and  $(b_{p(i)j})$  is a block diagonal of  $(c_{p(i)j})$ , Proposition 1.c.8 of [11] yields that  $(b_{p(i)j})$  also represents a bounded linear operator from  $X_L^\xi$  into  $X_{\psi(L)}^\xi$  with respect to the bases  $(e_{l_{p(i)}}^\xi)$  and  $(e_{m_{k_j}}^\xi)$ . Consequently,  $(b_{ij})$  represents a bounded linear operator  $W : X_L^\xi \rightarrow X_{\psi(L)}^\xi$  with respect to the bases  $(e_{l_i}^\xi)$  and  $(e_{m_{k_j}}^\xi)$  which evidently satisfies  $W(e_{l_i}^\xi) = a_{ik_j} e_{\psi(l_i)}^\xi$  (where  $\psi(l_i) = m_{k_j}$ ) for all  $i \in \mathbb{N}$ . Because  $|a_{ik_j}| > \delta$ , if  $\psi(l_i) = m_{k_j}$ , and  $(e_n^\xi)$  is unconditional, we conclude that there exists a bounded linear operator  $R : X_L^\xi \rightarrow X_M^\xi$  such that  $R(e_{l_i}^\xi) = e_{\psi(l_i)}^\xi$  for every  $i \in L$ . ■

We are now ready for the

*Proof of Theorem 1.1.*  $3 \Rightarrow 1$  and  $2 \Rightarrow 3$  are immediate. To prove that 1 implies 2 we first apply Proposition 3.13 to obtain a map  $\psi : L \rightarrow M$  and a bounded linear operator  $R : X_L^\xi \rightarrow X_M^\xi$  such that  $R(e_{l_i}^\xi) = e_{\psi(l_i)}^\xi$  for every  $i \in L$ . Propositions 3.11 and 3.12 will then yield a constant  $E > 0$  such that  $\tau_\zeta(\phi^{-1}F) \leq E$  for every  $F \in S_\zeta[M]$  and  $0 \leq \zeta \leq \xi$  (where  $\phi : L \rightarrow M$  is the natural bijection). The result now follows from Lemma 3.4. ■

To obtain Corollary 1.2 we shall need the following

LEMMA 3.14. *Let  $\xi < \omega$  and  $s = (u_n)$  be a bounded block basis of  $(e_n^\xi)$  such that  $\lim_n \|u_n\|_0 = 0$ . Then for every  $N \in [\mathbb{N}]$  and  $0 \leq \zeta \leq \xi$  there exists  $M \in [N]$  so that  $\lim_n \|\zeta_n^M \cdot s\|_\zeta = 0$ . (Given  $\mu = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$ , we denote by  $\mu \cdot s$  the vector  $\sum_{i=1}^{\infty} a_i u_i$ , which of course belongs to  $c_{00}$ .)*

*Proof.* If  $\zeta=0$ , the assertion follows from the fact that  $\lim_n \|u_n\|_0=0$ . Assume now that  $\zeta \leq \xi - 1$  and that the assertion holds for  $\zeta$ . Let  $N \in [\mathbb{N}]$  and  $\varepsilon > 0$ . We will find  $Q \in [N]$  so that  $\|(\zeta + 1)_1^Q \cdot s\|_{\zeta+1} < \varepsilon$ . Once this is accomplished, we can choose  $(Q_i) \subset [N]$  so that  $\|(\zeta + 1)_1^{Q_i} \cdot s\|_{\zeta+1} < \varepsilon_i$ , where  $\lim_i \varepsilon_i = 0$  and  $F_1^{\zeta+1}(Q_1) < F_1^{\zeta+1}(Q_2) < \dots$ . Letting  $M = \bigcup_{i=1}^{\infty} F_1^{\zeta+1}(Q_i)$ , we obtain  $(\zeta + 1)_1^{Q_i} = (\zeta + 1)_i^M$  for all  $i \in \mathbb{N}$ , and thus  $\lim_i \|(\zeta + 1)_i^M \cdot s\|_{\zeta+1} = 0$ .

We now pass to the construction of  $Q$ . By the induction hypothesis we can choose a sequence  $(P_i) \subset [N]$  sharing the following properties:

1.  $F_1^\zeta(P_1) < F_1^\zeta(P_2) < \dots$
2.  $\min F_1^\zeta(P_1) > (2 + 2b)/\varepsilon$ , where  $b$  is chosen so that  $\|u_n\|_\xi \leq b$  for every  $n \in \mathbb{N}$ .
3.  $\|\zeta_1^{P_i} \cdot s\|_\zeta < 1/(2^i k_{i-1})$  for all  $i \geq 2$ , where  $k_i = \max \text{supp}(\zeta_1^{P_i} \cdot s)$ .

Put  $Q = \bigcup_{i=1}^{\infty} F_1^\zeta(P_i)$ . We are going to show that  $\|\sum_{i=1}^n \zeta_i^Q \cdot s\|_{\zeta+1} \leq 2 + 2b$  for every  $n \in \mathbb{N}$ . Note that  $\zeta_i^Q = \zeta_1^{P_i}$  and  $F_i^\zeta(Q) = F_1^\zeta(P_i)$  for all  $i \in \mathbb{N}$ .

Let  $G \in S_{\zeta+1}$ . Let  $\{i_1, \dots, i_p\}$  be an enumeration of the set  $\{i \leq n : \text{supp}(\zeta_i^Q \cdot s) \cap G \neq \emptyset\}$ . Choose  $l \leq \min G$  and  $G_1 < \dots < G_l$  in  $S_\zeta$  so that  $G = \bigcup_{i=1}^l G_i$ . Then  $|\langle \zeta_{i_1}^Q \cdot s, G \rangle| \leq b$ . Further,

$$\begin{aligned} \left| \sum_{t=2}^p \zeta_{i_t}^Q \cdot s(G) \right| &= \left| \sum_{j=1}^l \sum_{t=2}^p (\zeta_{i_t}^Q \cdot s)(G_j) \right| \leq \sum_{j=1}^l \sum_{t=2}^p |(\zeta_{i_t}^Q \cdot s)(G_j)| \\ &\leq \sum_{j=1}^l \sum_{t=2}^p \|\zeta_{i_t}^Q \cdot s\|_\zeta \leq \sum_{j=1}^l \sum_{t=2}^p \frac{1}{2^{i_t} k_{i_t-1}} \\ &\leq \frac{l}{k_{i_1}} \leq 1, \quad \text{since } l \leq \min G \leq k_{i_1}. \end{aligned}$$

Therefore,  $|\sum_{i=1}^n (\zeta_i^Q \cdot s)(G)| = |\sum_{t=1}^p (\zeta_{i_t}^Q \cdot s)(G)| \leq b + 1$ . It follows that  $\|\sum_{i=1}^n \zeta_i^Q \cdot s\|_{\zeta+1} \leq 2 + 2b$ . If we take  $n = \min F_1^\zeta(P_1)$ , we conclude that  $\|(\zeta + 1)_1^Q \cdot s\|_{\zeta+1} < \varepsilon$ . ■

*Proof of Corollary 1.2.* Let  $T : X_L^\xi \rightarrow X_M^\xi$  be an isomorphic embedding. We apply Theorem 1.1 to show that  $(e_{l_n}^\xi)$  dominates  $(e_{m_n}^\xi)$ . Indeed, we

need only check that  $\inf_{i \in L} \|T(e_i^\xi)\|_0 > 0$ . If that were not the case, let  $(x_i)$  be a subsequence of  $(T(e_i^\xi))$  such that  $\lim_i \|x_i\|_0 = 0$ . By a standard perturbation result we can assume, without loss of generality, that for some block basis  $(u_i)$  of  $(e_i^\xi)$  and a null sequence of positive scalars  $(\varepsilon_i)$  we have  $\|x_i - u_i\|_\xi < \varepsilon_i$  for all  $i \in \mathbb{N}$ . It follows that also  $\lim_i \|u_i\|_0 = 0$ , and thus Lemma 3.14 yields  $N \in [\mathbb{N}]$  so that  $\lim_i \|\xi_i^N \cdot s\|_\xi = 0$ , where  $s = (u_i)$ . But then  $\lim_i \|\xi_i^N \cdot x\|_\xi = 0$  as well ( $x = (x_i)$ ). This is a contradiction because  $(x_i)$  is equivalent to a subsequence of  $(e_i^\xi)$  and thus it is an  $\ell_1^\xi$ -spreading model. Hence,  $(e_{i_n}^\xi)$  dominates  $(e_{m_n}^\xi)$ , completing the proof of part 1. Parts 2 and 3 are immediate consequences of Theorem 1.1. ■

We recall that a Banach space  $X$  is said to be *primary* if, for every bounded linear projection  $P$  on  $X$ , either  $PX$  or  $(I - P)X$  is isomorphic to  $X$ .

**COROLLARY 3.15.**  $X_N^\xi$  is not primary, for every  $N \in [\mathbb{N}]$  and all  $1 \leq \xi < \omega$ .

**Proof.** We first let  $\mathcal{F} = \{(L, M) \in [N] \times [N] : L \cup M = N, L \cap M = \emptyset\}$ . Then  $\mathcal{F}$  is easily seen to be closed in  $[N] \times [N]$  and thus it is a Polish space. We next set  $\mathcal{G} = \{(L, M) \in \mathcal{F} : d_\xi(N, L) = d_\xi(N, M) = \infty\}$ . Arguing as we did in the proof of Lemma 3.5 we deduce that  $\mathcal{G}$  is a  $G_\delta$  dense subset of  $\mathcal{F}$ . If  $(L, M) \in \mathcal{G}$  then  $X_N^\xi = X_L^\xi \oplus X_M^\xi$ . However, Theorem 1.1 implies that  $X_N^\xi$  is not isomorphic to a subspace of either  $X_L^\xi$  or  $X_M^\xi$ . ■

**4. Subspaces spanned by block bases.** In this section we investigate subspaces of  $X^\xi$  spanned by block bases of  $(e_n^\xi)$ . We first show that there exists a block basis of  $(e_n^\xi)$  spanning a complemented subspace of  $X^\xi$  which is not isomorphic to  $X_M^\xi$ , for every  $M \in [\mathbb{N}]$  and all  $0 \leq \zeta \leq \xi$ .

**LEMMA 4.1.** Let  $x_1 < \dots < x_p$  be a finite block basis of  $(e_n)$ , the unit vector basis of  $c_{00}$ . Let  $G_1 < \dots < G_q$  be finite subsets of  $\mathbb{N}$  and  $(a_i)_{i=1}^p$  be scalars. Assume that there exists  $C > 0$  such that  $|(\sum_{i \in I} a_i x_i)(\bigcup_{j \in J} G_j)| \leq C$  whenever  $I \subset \{1, \dots, p\}$  and  $J \subset \{1, \dots, q\}$  satisfy one of the following two conditions:

1.  $I = \bigcup_{j \in J} I_j$ ,  $I_{j_1} < I_{j_2}$  if  $j_1 < j_2$  and  $I_j = \{i \in I : \text{supp } x_i \cap G_j \neq \emptyset\}$  for all  $j \in J$ .
2.  $J = \bigcup_{i \in I} J_i$ ,  $J_{i_1} < J_{i_2}$  if  $i_1 < i_2$  and  $J_i = \{j \in J : \text{supp } x_i \cap G_j \neq \emptyset\}$  for all  $i \in I$ .

Then  $|(\sum_{i=1}^p a_i x_i)(\bigcup_{j=1}^q G_j)| \leq 3C$ .

**Proof.** Given  $j \leq q$ , we let  $T_j = \{i \leq p : \text{supp } x_i \cap G_j \neq \emptyset\}$ . We also let  $J = \{j \leq q : T_j \neq \emptyset\}$  and  $J_1 = \{j \in J : |T_j| = 1\}$ . Set  $J_2 = J \setminus J_1$ . Given

$j \in J_2$  we let  $s_j = \min T_j$  and  $t_j = \max T_j$ . We observe that  $s_{j_1} < t_{j_1} \leq s_{j_2}$  for every  $j_1 < j_2$  in  $J_2$ .

Next, we define a map  $\sigma : J_1 \rightarrow \{1, \dots, p\}$  so that  $\{\sigma(j)\} = T_j$  for every  $j \in J_1$ . Note that  $\sigma(J_1)$  and  $J_1$  satisfy condition 2 and therefore

$$\left| \left( \sum_{i=1}^p a_i x_i \right) \left( \bigcup_{j \in J_1} G_j \right) \right| = \left| \left( \sum_{i \in \sigma(J_1)} a_i x_i \right) \left( \bigcup_{j \in J_1} G_j \right) \right| \leq C.$$

Suppose now that  $J_2 = \{j_1, \dots, j_k\}$  and put  $J_3 = \{j_r : r \leq k, r \text{ is odd}\}$  and  $J_4 = \{j_r : r \leq k, r \text{ is even}\}$ . It follows that  $\bigcup_{j \in J_m} T_j$  and  $J_m$ ,  $m \in \{3, 4\}$ , satisfy condition 1 and thus

$$\left| \left( \sum_{i=1}^p a_i x_i \right) \left( \bigcup_{j \in J_m} G_j \right) \right| = \left| \left( \sum_{i \in \bigcup_{j \in J_m} T_j} a_i x_i \right) \left( \bigcup_{j \in J_m} G_j \right) \right| \leq C, \quad m \in \{3, 4\}.$$

Hence,  $|(\sum_{i=1}^p a_i x_i)(\bigcup_{j \in J_2} G_j)| \leq 2C$ . The assertion follows since

$$\left( \sum_{i=1}^p a_i x_i \right) \left( \bigcup_{j=1}^q G_j \right) = \left( \sum_{i=1}^p a_i x_i \right) \left( \bigcup_{j \in J_1} G_j \right) + \left( \sum_{i=1}^p a_i x_i \right) \left( \bigcup_{j \in J_2} G_j \right). \quad \blacksquare$$

**LEMMA 4.2.** Let  $1 \leq \zeta \leq \xi < \omega$  and  $(x_n)$  be a block basis of  $(e_n^\xi)$  so that for some  $b > 0$ ,  $\|x_n\|_\xi < b$ , for every  $n \in \mathbb{N}$ . Let  $k_n = \max \text{supp } x_n$  for every  $n \in \mathbb{N}$ , and suppose that  $\|x_n\|_{\zeta-1} < 1/2^{k_n-1}$  for every  $n \geq 2$ . Then  $|(\sum_{i=1}^n a_i x_i)(H)| \leq (2+b) \max_{i \leq n} |a_i|$  for every  $H \in S_\zeta$ ,  $n \in \mathbb{N}$  and all scalar sequences  $(a_i)_{i=1}^\infty$ .

**Proof.** Let  $H \in S_\zeta$  and put  $i_0 = \min\{i \leq n : \text{supp } x_i \cap H \neq \emptyset\}$ . We may write  $H = \bigcup_{j=1}^r H_j$ , where  $r \leq \min H$  and  $H_1 < \dots < H_r$  belong to  $S_{\zeta-1}$ . Note that  $\min H \leq k_{i_0}$ . We also observe that  $|x_i(H)| \leq r \|x_i\|_{\zeta-1}$  and hence

$$\begin{aligned} \left| \sum_{i=i_0+1}^n a_i x_i(H) \right| &\leq \sum_{i=i_0+1}^n |a_i| r \|x_i\|_{\zeta-1} \\ &\leq r (\max_{i \leq n} |a_i|) \sum_{i=i_0+1}^\infty \frac{1}{2^{k_i-1}} \leq 2 \max_{i \leq n} |a_i|. \end{aligned}$$

Finally,  $|x_{i_0}(H)| \leq \|x_{i_0}\|_\xi < b$  and so  $|(\sum_{i=1}^n a_i x_i)(H)| \leq (2+b) \max_{i \leq n} |a_i|$ , as desired. ■

Our next proposition is a partial generalization of Lemma 3.10.

**PROPOSITION 4.3.** Let  $\xi < \omega$  and  $(x_n)$  be a semi-normalized block basis of  $(e_n^\xi)$ . Set  $\zeta = \min\{\alpha \leq \xi : \inf_n \|x_n\|_\alpha > 0\}$ . Then there exists a subsequence of  $(x_n)$  which is equivalent to a subsequence of  $(e_n^{\xi-\zeta})$ .

**Proof.** Choose  $\delta > 0$ ,  $b > 0$  so that  $\delta < \|x_n\|_\zeta$  and  $\|x_n\|_\xi < b$  for every  $n \in \mathbb{N}$ . Assume first that  $\zeta \geq 1$ . Then we choose inductively  $n_1 < n_2 < \dots$

so that  $\|x_{n_i}\|_{\zeta-1} < 1/2^{k_{i-1}}$  for every  $i \geq 2$ , where  $k_i = \max \text{supp } x_{n_i}$ . For every  $i \in \mathbb{N}$  we can find  $F_i \in S_\zeta$ ,  $F_i \subset \text{supp } x_{n_i}$ , so that  $|x_{n_i}|(F_i) > \delta$ . Put  $m_i = \min F_i$ . We are going to show that  $(x_{n_i})$  is equivalent to  $(e_{m_i}^{\xi-\zeta})$ . To this end let  $k \in \mathbb{N}$  and  $(a_i)_{i=1}^k$  be scalars. We first show that  $\|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\| \leq \delta^{-1} \|\sum_{i=1}^k a_i x_{n_i}\|_\xi$ . Indeed, if  $G \subset \{m_1, \dots, m_k\}$  belongs to  $S_{\xi-\zeta}$  then set  $A = \{i \leq k : m_i \in G\}$ . We have the estimate

$$\sum_{i \in A} |a_i| \leq \delta^{-1} \sum_{i \in A} |a_i| |x_{n_i}|(F_i) \leq \delta^{-1} \left\| \sum_{i=1}^k a_i x_{n_i} \right\|_\xi$$

as  $\bigcup_{i \in A} F_i \in S_\xi$ , by Lemma 3.8.

Next, let  $G \in S_\xi$ . Lemma 3.8 yields  $G_1 < \dots < G_q$  in  $S_\zeta$  with  $\{\min G_j : j \leq q\}$  belonging to  $S_{\xi-\zeta}$  and such that  $G = \bigcup_{j=1}^q G_j$ . We shall apply Lemma 4.1 in order to estimate  $|(\sum_{i=1}^k a_i x_{n_i})(\bigcup_{j=1}^q G_j)|$ . Let  $I \subset \{1, \dots, k\}$  and  $J \subset \{1, \dots, q\}$  satisfy condition 1 of Lemma 4.1. Then  $I_j = \{i \in I : \text{supp } x_{n_i} \cap G_j \neq \emptyset\}$  for every  $j \in J$ . We choose  $i_j \in I_j$  such that  $|a_{i_j}| = \max_{i \in I_j} |a_i|$  for every  $j \in J$ . Fix  $j_0 \in J$ . Then

$$\begin{aligned} \left| \left( \sum_{i \in I_0} a_i x_{n_i} \right) \left( \bigcup_{j \in J} G_j \right) \right| &= \left| \sum_{i \in I_{j_0}} a_i x_{n_i}(G_{j_0}) \right| \leq (2+b) \max_{i \in I_{j_0}} |a_i| \\ &= (2+b) |a_{i_{j_0}}|, \quad \text{by Lemma 4.2.} \end{aligned}$$

Hence  $|(\sum_{i \in I} a_i x_{n_i})(\bigcup_{j \in J} G_j)| \leq (2+b) \sum_{j \in J} |a_{i_j}|$ .

Note also that  $\{m_{i_j} : j \in J \setminus \{\min J\}\}$  belongs to  $S_{\xi-\zeta}$ . This is so since  $\text{supp } x_{n_i} \cap G_j \neq \emptyset$  whenever  $i \in I_j$  and  $j \in J$ , and thus  $\min G_{j_1} < \min \text{supp } x_{n_i} \leq m_i$  for every  $i \in I_{j_2}$  and  $j_1 < j_2$  in  $J$ . In particular,  $\min G_{j_1} < m_{i_{j_2}}$  when  $j_1 < j_2$  in  $J$ . Since  $S_{\xi-\zeta}$  is spreading we see that  $\{m_{i_j} : j \in J \setminus \{\min J\}\}$  belongs to  $S_{\xi-\zeta}$ . It now follows that  $\sum_{j \in J \setminus \{\min J\}} |a_{i_j}| \leq \|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$  and hence

$$\left| \left( \sum_{i \in I} a_i x_{n_i} \right) \left( \bigcup_{j \in J} G_j \right) \right| \leq 2(2+b) \left\| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta} \right\|.$$

We now assume that  $I \subset \{1, \dots, k\}$  and  $J \subset \{1, \dots, q\}$  satisfy condition 2 of Lemma 4.1. Then  $J_i = \{j \in J : \text{supp } x_{n_i} \cap G_j \neq \emptyset\}$  for all  $i \in I$ . An argument similar to that in the preceding paragraph yields that  $\{m_i : i \in I \setminus \{\min I\}\}$  belongs to  $S_{\xi-\zeta}$ . It follows that  $\sum_{i \in I} |a_i| \leq 2 \|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$ . Finally,

$$\begin{aligned} \left| \left( \sum_{i \in I} a_i x_{n_i} \right) \left( \bigcup_{j \in J} G_j \right) \right| &= \left| \sum_{i \in I} a_i x_{n_i} \left( \bigcup_{j \in J_i} G_j \right) \right| \leq b \sum_{i \in I} |a_i| \text{ as } \bigcup_{j \in J_i} G_j \in S_\xi \\ &\leq 2b \left\| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta} \right\|. \end{aligned}$$

We deduce from Lemma 4.1 that

$$\left| \left( \sum_{i=1}^k a_i x_{n_i} \right) \left( \bigcup_{j=1}^q G_j \right) \right| \leq 6(2+b) \left\| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta} \right\|,$$

and hence  $\|\sum_{i=1}^k a_i x_{n_i}\|_\xi \leq 12(2+b) \|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$ .

To complete the proof we need to consider the case  $\zeta = 0$ . We now choose  $m_n \in \text{supp } x_n$  such that  $|x_n|(\{m_n\}) > \delta$  for all  $n \in \mathbb{N}$ . We are going to show that  $(x_n)$  is equivalent to  $(e_{m_n}^\xi)$ . Arguing as we did in the case  $\zeta \geq 1$  we find that  $\|\sum_{i=1}^k a_i e_{m_i}^\xi\| \leq \delta^{-1} \|\sum_{i=1}^k a_i x_{n_i}\|_\xi$  for every  $k \in \mathbb{N}$  and all scalar sequences  $(a_i)_{i=1}^k$ .

Next let  $G \in S_\xi$  and put  $I = \{i \leq k : \text{supp } x_i \cap G \neq \emptyset\}$ . Then

$$\begin{aligned} \left| \sum_{i=1}^k a_i x_i(G) \right| &\leq \sum_{i \in I} |a_i| |x_i(G)| \leq b \sum_{i \in I} |a_i| \\ &\leq 2b \left\| \sum_{i=1}^k a_i e_{m_i}^\xi \right\| \quad \text{as } \{m_i : i \in I \setminus \{\min I\}\} \in S_\xi. \end{aligned}$$

Hence  $\|\sum_{i=1}^k a_i x_i\|_\xi \leq 4b \|\sum_{i=1}^k a_i e_{m_i}^\xi\|$ . ■

As an immediate consequence of Proposition 4.3 we obtain

**COROLLARY 4.4.** *For every semi-normalized weakly null sequence in  $X^\xi$ ,  $\xi < \omega$ , there exist  $\zeta \leq \xi$  and a subsequence which is equivalent to a subsequence of  $(e_n^\zeta)$ .*

**LEMMA 4.5.** *Let  $1 \leq \xi < \omega$  and  $(F_n)$  be a sequence of successive members of  $S_\xi$  satisfying the following requirements:*

1.  $(\tau_{\xi-1}(F_n))$  increases to  $\infty$ .
2.  $\sup_n \frac{\min F_n}{\tau_{\xi-1}(F_{n+k})} > k$  for every  $k \in \mathbb{N}$ .

*Let  $(u_n)$  be a convex block basis of  $(e_n^\xi)$  such that  $\text{supp } u_n = F_n$  for every  $n \in \mathbb{N}$ . Assume furthermore that  $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$ . Then the closed linear span of  $(u_n)$  in  $X^\xi$  is not isomorphic to  $X_M^\zeta$ , for every  $\zeta \leq \xi$  and  $M \in [\mathbb{N}]$ .*

**Proof.**  $(u_n)$  is normalized in  $X^\xi$  since  $F_n \in S_\xi$  for every  $n \in \mathbb{N}$ . We let  $X$  denote the closed linear span of  $(u_n)$  in  $X^\xi$ . Because  $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$ , we deduce from Proposition 4.3 that every semi-normalized block basis of  $(u_n)$  admits a subsequence equivalent to the unit vector basis of  $c_0$ . Indeed, let  $(v_n)$ ,  $v_n = \sum_{i \in G_n} b_i u_i$ , be a semi-normalized block basis of  $(u_n)$ . Note that  $(b_n)$  is bounded since  $(v_n)$  is. But also,  $\lim_n \sum_{i \in G_n} \|u_i\|_{\xi-1} = 0$ , since  $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$ . Therefore  $\lim_n \|v_n\|_{\xi-1} = 0$  and hence Proposition 4.3 (for  $\zeta = \xi$ ) yields a subsequence of  $(v_n)$  equivalent to the unit basis of  $c_0$ .



It follows that every semi-normalized weakly null sequence in  $X$  admits a subsequence equivalent to the unit vector basis of  $c_0$ . That is,  $X$  has property (S) (see [7]). However,  $X_M^\zeta$  fails property (S) when  $\zeta \geq 1$  and  $M \in [\mathbb{N}]$ . Thus,  $X_M^\zeta$  is not isomorphic to a subspace of  $X$ , for every  $1 \leq \zeta \leq \xi$  and  $M \in [\mathbb{N}]$ .

To complete the proof we show that  $X$  is not isomorphic to  $c_0$ . This is accomplished by showing that for every  $k \in \mathbb{N}$  there exists  $n \in \mathbb{N}$  so that  $(u_{n+i})_{i=1}^k$  is isometrically equivalent to the unit vector basis of  $\ell_1^k$ . In particular,  $X$  contains uniformly complemented  $\ell_1^k$ 's. It is a well known fact that  $c_0$  fails this property.

We let  $k \in \mathbb{N}$  and choose  $n \in \mathbb{N}$  according to 2 so that

$$\frac{\min F_n}{\tau_{\xi-1}(F_{n+k})} > k.$$

Condition 1 now yields that  $\sum_{i=1}^k \tau_{\xi-1}(F_{n+i}) < \min F_n$  and thus  $\bigcup_{i=1}^k F_{n+i} \in S_\xi$ . Hence

$$\left\| \sum_{i=1}^k a_i u_{n+i} \right\|_\xi \geq \sum_{i=1}^k |a_i| u_{n+i}(F_{n+i}) = \sum_{i=1}^k |a_i|$$

for every scalar sequence  $(a_i)_{i=1}^k$ . Therefore  $(u_{n+i})_{i=1}^k$  is isometrically equivalent to the unit vector basis of  $\ell_1^k$ . ■

**PROPOSITION 4.6.** *Let  $1 \leq \xi < \omega$ . There exists a normalized convex block basis  $(u_n)$  of  $(e_n^\xi)$  so that letting  $F_n = \text{supp } u_n$  for every  $n \in \mathbb{N}$ , the following hold:*

1.  $\tau_{\xi-1}(F_n) = n^2$  and  $\min F_n > k(n+k)^2$  for every  $n$  and  $k$  in  $\mathbb{N}$  such that  $k < n$ .
2.  $X = [u_n : n \in \mathbb{N}]$  is not isomorphic to  $X_M^\zeta$  for every  $\zeta \leq \xi$  and  $M \in [\mathbb{N}]$ .
3.  $X$  is complemented in  $X^\xi$ .

**Proof.** We inductively choose a sequence of successive integer intervals  $(F_n)$  such that for every  $n \in \mathbb{N}$ ,

$$\min F_n > \max\{k(n+k)^2 : k < n\} \quad \text{and} \quad \tau_{\xi-1}(F_n) = n^2.$$

We now let, for every  $n \in \mathbb{N}$ ,

$$M_n = F_n \cup \{m \in \mathbb{N} : m > \max F_n\} \quad \text{and} \quad u_n = \frac{1}{n^2} \sum_{i=1}^{n^2} (\xi-1)_i^{M_n}.$$

Condition 1 is an immediate consequence of the inductive construction. This condition implies that in fact  $F_n \in S_\xi$  for every  $n \in \mathbb{N}$ , and thus  $(u_n)$  is indeed a normalized convex block basis of  $(e_n^\xi)$ . We also deduce from

Lemma 2.3 that  $\|u_n\|_{\xi-1} \leq \xi/n^2$  and so  $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$ . Hence condition 2 holds in view of Lemma 4.5.

It remains to establish that  $X$  is complemented in  $X^\xi$ . To this end we define a map  $P : c_{00} \rightarrow c_{00}$  by

$$P(x) = \sum_{i=1}^\infty x(F_i) u_i \quad \text{for all } x \in c_{00}.$$

Clearly,  $P$  is well defined and linear. It is also clear that  $P(u_i) = u_i$  for every  $i \in \mathbb{N}$ . Our objective is to show that  $P$  is bounded with respect to the  $\|\cdot\|_\xi$ -norm on  $c_{00}$ , for then  $P$  will extend to a bounded linear projection on  $X^\xi$  with range equal to  $X$ . To achieve our goal it suffices to show that if  $G \in S_\xi$  is maximal, then  $(\sum_{i=1}^p x(F_i) u_i)(G) \leq 18\xi$  for every  $p \in \mathbb{N}$  and  $x \in c_{00}$ ,  $\|x\|_\xi \leq 1$ , with  $x(\{i\}) \geq 0$ ,  $i \in \mathbb{N}$ .

According to condition 1 of our hypothesis, for every  $i \in \mathbb{N}$  there exist successive  $S_{\xi-1}$  sets  $F_{i1} < \dots < F_{ii^2}$  so that  $F_i = \bigcup_{k=1}^{i^2} F_{ik}$ , and  $\{\min F_{ik} : k \leq i^2\} \in S_1$ . Next let  $q = \min G$  and choose maximal members  $G_1 < \dots < G_q$  of  $S_{\xi-1}$  so that  $G = \bigcup_{j=1}^q G_j$ . Of course,  $\{\min G_j : j \leq q\}$  is maximal in  $S_1$ . We shall apply Lemma 4.1. Let  $I \subset \{1, \dots, p\}$  and  $J \subset \{1, \dots, q\}$  satisfy condition 1 of Lemma 4.1. Recall that  $I_j = \{i \in I : \text{supp } u_i \cap G_j \neq \emptyset\}$ ,  $j \in J$ . For each  $j \in J$  we choose  $i_j \in I_j$  and  $k_j \leq i_j^2$  such that  $x(F_{i_j k_j}) = \max_{k \leq i_j^2, i \in I_j} x(F_{ik})$ . We have the estimate

$$\begin{aligned} \left( \sum_{i \in I} x(F_i) u_i \right) \left( \bigcup_{j \in J} G_j \right) &= \sum_{j \in J} \sum_{i \in I_j} x(F_i) u_i(G_j) \\ &= \sum_{j \in J} \sum_{i \in I_j} x(F_i) \frac{1}{i^2} \sum_{k=1}^{i^2} (\xi-1)_k^{M_i}(G_j) \\ &\leq \sum_{j \in J} \sum_{i \in I_j} x(F_{i_j k_j}) \sum_{k=1}^{i^2} (\xi-1)_k^{M_i}(G_j) \\ &\leq \sum_{j \in J} x(F_{i_j k_j}) \sum_{i \in I_j} \sum_{k=1}^{i^2} (\xi-1)_k^{M_i}(G_j) \\ &\leq \sum_{j \in J} x(F_{i_j k_j}) \xi \quad \text{by Lemma 2.3} \\ &\leq \xi x \left( \bigcup_{j \in J} F_{i_j k_j} \right) \leq 2\xi. \end{aligned}$$

since  $F_i = \bigcup_{k=1}^{i^2} F_{ik}$

The last inequality holds because  $\|x\|_\xi \leq 1$  and  $\bigcup_{j \in J \setminus \{\min J\}} F_{i_j k_j} \in S_\xi$ . Indeed,  $\min G_{j_1} < \min F_{i_{j_2} k_{j_2}}$  when  $j_1 < j_2$  in  $J$  and therefore, as  $\{\min G_j : j \leq q\} \in S_1$ ,  $\bigcup_{j \in J \setminus \{\min J\}} F_{i_j k_j}$  belongs to  $S_\xi$  by Lemma 3.8.

Next assume that  $I \subset \{1, \dots, p\}$  and  $J \subset \{1, \dots, q\}$  satisfy condition 2 of Lemma 4.1. Then  $J_i = \{j \in J : \text{supp } u_i \cap G_j \neq \emptyset\}$ ,  $i \in I$ . We set  $H_i = \{j \in J_i : G_j \subset \text{supp } u_i\}$ ,  $i \in I$ . Since  $\text{supp } u_i = F_i$  is an interval,  $|J_i| \leq |H_i| + 2$  for all  $i \in I$ . Moreover, since each  $G_j$  is a maximal  $S_{\xi-1}$  set and  $\tau_{\xi-1}(F_i) = i^2$ , we have  $|H_i| \leq i^2$  for all  $i \in I$ . To estimate  $\sum_{i \in I} \sum_{j \in J_i \setminus H_i} x(F_i) u_i(G_j)$ , choose  $j_i \in J_i \setminus H_i$  for every  $i \in I$  (we have assumed without loss of generality that  $J_i \setminus H_i \neq \emptyset$ ). Then the sets  $I$  and  $\{j_i : i \in I\}$  satisfy condition 1 of Lemma 4.1. We deduce from our preceding work that

$$\sum_{i \in I} \sum_{j \in J_i \setminus H_i} x(F_i) u_i(G_j) \leq 4\xi,$$

as  $|J_i \setminus H_i| \leq 2$  for every  $i \in I$ . We next choose, for every  $i \in I$ ,  $R_i \subset \{1, \dots, i^2\}$  with  $|R_i| = |H_i|$  and such that

$$\frac{1}{i^2} \sum_{k=1}^{i^2} x(F_{ik}) \leq \frac{1}{|H_i|} \sum_{k \in R_i} x(F_{ik}).$$

This choice is possible since  $|H_i| \leq i^2$ . (We make use of the following fact: Let  $(a_i)_{i=1}^n$  be scalars with  $a_i \leq a_j$ ,  $i \leq j$ , and let  $k < n$ . Then  $(1/n) \sum_{i=1}^n a_i \leq (1/(n-k)) \sum_{i=k+1}^n a_i$ .) We now have

$$\begin{aligned} \sum_{i \in I} \sum_{j \in H_i} x(F_i) u_i(G_j) &= \sum_{i \in I} \sum_{j \in H_i} \left[ \sum_{k=1}^{i^2} x(F_{ik}) \right] \left[ \frac{1}{i^2} \sum_{k=1}^{i^2} (\xi - 1)_k^{M_i}(G_j) \right] \\ &\leq \sum_{i \in I} \sum_{j \in H_i} \left[ \frac{1}{i^2} \sum_{k=1}^{i^2} x(F_{ik}) \right] \xi \quad \text{by Lemma 2.3} \\ &\leq \xi \sum_{i \in I} \sum_{j \in H_i} \frac{1}{|H_i|} \sum_{k \in R_i} x(F_{ik}) \leq \xi \sum_{i \in I} \sum_{k \in R_i} x(F_{ik}) \\ &\leq \xi x \left( \bigcup_{i \in I, k \in R_i} F_{ik} \right) \leq 2\xi. \end{aligned}$$

The last inequality follows since  $\|x\|_\xi \leq 1$  and  $\bigcup_{i \in I \setminus \{\min I\}, k \in R_i} F_{ik} \in S_\xi$ . Indeed, the cardinality of the set  $\{\min F_{ik} : k \in R_i, i \in I\}$  does not exceed that of  $J$  since  $|R_i| = |H_i|$  for all  $i \in I$ . It now follows, since  $|J| \leq \min G_1$ , that  $\{\min F_{ik} : k \in R_i, i \in I \setminus \{\min I\}\}$  belongs to  $S_1$  and thus  $\bigcup_{i \in I, k \in R_i} F_{ik}$

is the union of two members of  $S_\xi$ . Concluding,

$$\begin{aligned} \left( \sum_{i \in I} x(F_i) u_i \right) \left( \bigcup_{j \in J} G_j \right) &= \sum_{i \in I} \left[ \sum_{j \in J_i \setminus H_i} x(F_i) u_i(G_j) + \sum_{j \in H_i} x(F_i) u_i(G_j) \right] \\ &\leq 4\xi + 2\xi = 6\xi. \end{aligned}$$

Lemma 4.1 now implies that  $(\sum_{i=1}^p x(F_i) u_i)(G) \leq 18\xi$  for every  $p \in \mathbb{N}$  and  $x \in c_{00}$ ,  $\|x\|_\xi \leq 1$ , with  $x(\{i\}) \geq 0$ ,  $i \in \mathbb{N}$ . It follows that  $\|P\| \leq 18\xi$ . The proof of the proposition is now complete. ■

**PROPOSITION 4.7.** *Let  $1 \leq \xi < \omega$  and  $(u_n)$  be a block basis of  $(e_n^\xi)$  such that*

1.  $u_n = v_n + w_n$  with  $\text{supp } v_n \cap \text{supp } w_n = \emptyset$ ,  $n \in \mathbb{N}$ .
2.  $(w_n)$  is equivalent to the unit vector basis of  $c_0$ .
3.  $\lim_n \|v_n\|_\xi = 0$  yet  $\sup_n \|\sum_{i=1}^n v_i\|_\xi = \infty$ .

*Then there exists no projection from  $X^\xi$  onto the closed linear span of  $(u_n)$ .*

**Proof.** Let  $X$  denote the closed linear span of  $(u_n)$  in  $X^\xi$  and assume that  $P : X^\xi \rightarrow X$  is a bounded linear projection. Note that since  $(e_n^\xi)$  is unconditional our assumptions yield that  $(u_n)$  is semi-normalized in  $X^\xi$ . Lemma 2.a.11 of [11] now implies that  $(w_n)$  dominates  $(v_n)$ , contradicting 3 as  $\sup_n \|\sum_{i=1}^n w_i\|_\xi < \infty$ . ■

It is easy to construct a normalized convex block basis of  $(e_n^\xi)$ ,  $\xi \geq 1$ , satisfying conditions 1–3. Indeed, let  $M \in [\mathbb{N}]$ ,  $M = (m_n)$ , such that  $\sum_n (1/m_n) < \infty$ . Let  $q_n = \min F_n^\xi(M)$ ,  $n \in \mathbb{N}$  (recall that  $F_n^\xi(M) = \text{supp } \xi_n^M$ ). Because  $\xi \geq 1$ ,  $(e_{q_n}^\xi)$  is not dominated by the unit vector basis of  $c_0$ . It follows that there exists a sequence  $(a_n)$  of positive scalars such that  $\lim_n a_n = 0$  and  $\sup_n \|\sum_{i=1}^n a_i e_{q_i}^\xi\| = \infty$ . Set

$$v_n = a_n e_{q_n}^\xi \quad \text{and} \quad w_n = \frac{1 - a_n}{1 - \xi_n^M(q_n)} \sum_{i \in F_n^\xi(M) \setminus \{q_n\}} \xi_n^M(i) e_i^\xi, \quad n \in \mathbb{N}.$$

Finally, we let  $u_n = v_n + w_n$ ,  $n \in \mathbb{N}$ . Evidently,  $(u_n)$  is a normalized convex block basis of  $(e_n^\xi)$  satisfying 1 and 3. It remains to show that 2 holds. Since  $\sum_n 1/m_n < \infty$  and  $(\xi_n^M)$  is equivalent to the unit vector basis of  $c_0$ , letting  $x_n = \sum_{i \in F_n^\xi(M) \setminus \{q_n\}} \xi_n^M(i) e_i^\xi$ , we have  $\sup_n \|\sum_{i=1}^n x_i\|_\xi < \infty$ . It follows that  $(w_n)$  is equivalent to the unit vector basis of  $c_0$  as  $\lim_n a_n = 0$  and  $\lim_n \xi_n^M(q_n) = 0$ .

## References

- [1] D. E. Alspach and S. A. Argyros, *Complexity of weakly null sequences*, Dissertationes Math. 321 (1992).

- [2] G. Androulakis and E. Odell, *Distorting mixed Tsirelson spaces*, Israel J. Math. 109 (1999), 125–149.
- [3] S. A. Argyros and I. Deliyanni, *Examples of asymptotic  $l_1$  Banach spaces*, Trans. Amer. Math. Soc. 349 (1997), 973–995.
- [4] S. A. Argyros and V. Felouzis, *Interpolating hereditarily indecomposable Banach spaces*, J. Amer. Math. Soc. 13 (2000), 243–294.
- [5] S. A. Argyros and I. Gasparis, *Unconditional structures of weakly null sequences*, Trans. Amer. Math. Soc., to appear.
- [6] S. A. Argyros, S. Mercourakis and A. Tsarpalias, *Convex unconditionality and summability of weakly null sequences*, Israel J. Math. 107 (1998), 157–193.
- [7] P. Cembranos, *The hereditary Dunford–Pettis property on  $C(K, E)$* , Illinois J. Math. 31 (1987), 365–373.
- [8] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. Amer. Math. Soc., to appear.
- [9] A. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1994.
- [10] K. Kuratowski, *Applications of the Baire-category method to the problem of independent sets*, Fund. Math. 81 (1973), 65–72.
- [11] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, II*, Ergeb. Math. Grenzgeb. 92, 97, Springer, Berlin, 1977, 1979.
- [12] J. Mycielski, *Almost every function is independent*, Fund. Math. 81 (1973), 43–48.
- [13] E. Odell, *On quotients of Banach spaces having shrinking unconditional bases*, Illinois J. Math. 36 (1992), 681–695.
- [14] E. Odell, N. Tomczak-Jaegermann and R. Wagner, *Proximity to  $l_1$  and distortion in asymptotic  $l_1$  spaces*, J. Funct. Anal. 150 (1997), 101–145.
- [15] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. 2 (1930), 58–62.

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- [8] J. Kowalski, *Some remarks on  $J(X)$* , in: Algebra and Analysis (Edmonton, 1973), E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.
- [Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

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