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On cyclic $\alpha(\cdot)$-monotone multifunctions

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Abstract. Let (X,d) be a metric space. Let Φ be a linear family of real-valued functions defined on X. Let $\Gamma : X \to 2^{\Phi}$ be a maximal cyclic $\alpha(\cdot)$-monotone multifunction with non-empty values. We give a sufficient condition on $\alpha(\cdot)$ and Φ for the following generalization of the Rockafellar theorem to hold. There is a function f on X, weakly Φ-convex with modulus $\alpha(\cdot)$, such that Γ is the weak Φ-subdifferential of f with modulus $\alpha(\cdot)$, $\Gamma(x) = \partial_{\Phi}^{-\alpha} f|_x$.

Let (X, d_X) be a metric space. Let Φ be a family of continuous real-valued functions defined on X. Let f be a real-valued lower semicontinuous function on X. We say that f is *Φ-convex* if it is the majorant of some subset $\Phi_0 \subset \Phi$, $f(x) = \sup\{\phi(x) : \phi \in \Phi_0,\ \phi \le f\}$. We say that $\phi_0 \in \Phi$ is a *Φ-subgradient* of f at a point x_0 if

$$f(x) - f(x_0) \ge \phi_0(x) - \phi_0(x_0) \quad \text{for all } x \in X. \tag{1}$$

The set of all Φ-subgradients of f at x_0 is called the *Φ-subdifferential* of f at x_0, and is denoted by $\partial_{\Phi} f|_{x_0}$. Of course $\partial_{\Phi} f|_x$ is a multifunction mapping X into subsets of Φ, $\partial_{\Phi} f|_x : X \to 2^{\Phi}$.

Let $\alpha(\cdot)$ be a continuous non-decreasing function mapping $[0,\infty)$ into itself such that $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$.

We say that a function f is *weakly Φ-convex* at x_0 *with modulus* $\alpha(\cdot)$ if there is $\phi_0 \in \Phi$ such that

$$f(x) - f(x_0) \ge \phi_0(x) - \phi_0(x_0) - \alpha(d_X(x, x_0)) \quad \text{for all } x \in X. \tag{2}$$

The function ϕ_0 is then called a *weak Φ-subgradient* of f at x_0 *with modulus* $\alpha(\cdot)$.

The set of all Φ-subgradients of f at x_0 with modulus $\alpha(\cdot)$ is called the *weak Φ-subdifferential* of f at x_0 with modulus $\alpha(\cdot)$, and is denoted by $\partial_{\Phi}^{-\alpha} f|_{x_0}$. This yields a multifunction $\partial_{\Phi}^{-\alpha} f\big|_x : X \to 2^{\Phi}$. In the case when

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X is a normed space, $\Phi = X^*$ and $\alpha(t) = t^\gamma$ we obtain the definition of γ-subgradient and γ-subdifferential introduced by Jourani (1996).

If f is weakly Φ-convex at x_0 with the same modulus $\alpha(\cdot)$ for all $x_0 \in X$ we say that f is *uniformly weakly Φ-convex on X with modulus $\alpha(\cdot)$*.

In general, as in the case of Φ-subdifferentials, the knowledge of a weak Φ-subdifferential with modulus $\alpha(\cdot)$, $\partial_\Phi^{-\alpha} f|_x : X \to 2^\Phi$, does not permit one to determine the function f (up to a constant), as follows from

EXAMPLE 1. Let $X = [-1, 1]$. Let Φ be the class of functions

$$\Phi = \{\phi(x) = -|x - x_0| : -1 \le x_0 \le 1\}.$$

Suppose that

$$\lim_{t\to 0+0} \alpha(t)/t = 0.$$

Let f be an arbitrary Lipschitz function with constant less than 1. Then $\partial_\Phi^{-\alpha} f|_{x_0} = \{-|x - x_0|\}$.

In this example we can also construct two functions f and g such that $\partial f \subset \partial g$ and $\partial f \ne \partial g$. Indeed, let $0 \le a \le 1$ and let

$$f_a(x) = \begin{cases} |x| & \text{if } |x| \le a, \\ a & \text{if } a \le |x| \le 1. \end{cases}$$

By simple calculation we get

$$\partial_\Phi^{-\alpha} f_a|_{x_0} = \begin{cases} \{\phi(x) = -|x - x_0|\} & \text{for } a \le |x_0| \le 1, \\ \{\phi(x) = -|x - y| : -a \le y \le x_0 \le 0\} & \text{for } -a \le x_0 \le 0, \\ \{\phi(x) = -|x - y| : 0 \le x_0 \le y \le a\} & \text{for } 0 \le x_0 \le a, \\ \{\phi(x) = -|x - y| : |y| \le a\} & \text{for } x_0 = 0. \end{cases}$$

Thus $\partial_\Phi^{-\alpha} f_a|_{x_0} \subset \partial_\Phi^{-\alpha} f_b|_{x_0}$ and generally $\partial_\Phi^{-\alpha} f_a|_{x_0} \ne \partial_\Phi^{-\alpha} f_b|_{x_0}$ if $a < b \le 1$.

However in Banach spaces X and $\Phi = X^*$ we have

PROPOSITION 2. *Let X be a set in a Banach space E such that $X \subset \overline{\operatorname{Int} X}$ and $\operatorname{Int} X$ is arcwise connected. Let $\Phi = E^*|_X$ be the space of continuous linear functionals restricted to X. Suppose that*

$$\lim_{t\to 0+0} \alpha(t)/t = 0. \tag{3}$$

If for two locally Lipschitz functions f and g on X, $\partial_\Phi^{-\alpha} f|_x \subset \partial_\Phi^{-\alpha} g|_x$ for all $x \in X$, then the functions differ by a constant: $f(x) = g(x) + c$, $c \in \mathbb{R}$, and we have the equality $\partial_\Phi^{-\alpha} f|_x = \partial_\Phi^{-\alpha} g|_x$.

Proof (compare Rockafellar (1970), (1980)). Let $x, y \in X$ be such that the interval $[x, y] = \{tx + (1 - t)y : 0 \le t \le 1\}$ is contained in X. Now we consider two functions of a real variable: $\widetilde{f}(t) = f(ty + (1 - t)x)$ and $\widetilde{g}(t) = g(ty + (1 - t)x)$. Those functions are locally Lipschitz (even Lipschitz, since the interval $[x, y]$ is compact) and thus are differentiable almost everywhere.

Moreover we have

$$f(y) - f(x) = \widetilde{f}(1) - \widetilde{f}(0) = \int_0^1 \widetilde{f}'(t)\,dt,$$

$$g(y) - g(x) = \widetilde{g}(1) - \widetilde{g}(0) = \int_0^1 \widetilde{g}'(t)\,dt.$$

Since $\partial_\Phi^{-\alpha} f|_{x_0} \subset \partial_\Phi^{-\alpha} g|_{x_0}$ and (3) holds, we have $\widetilde{f}'(t) \le \widetilde{g}'(t)$ at each point of common differentiability of $\widetilde{f}$ and $\widetilde{g}$. Thus

$$f(y) - f(x) = \int_0^1 \widetilde{f}'(t)\,dt \le \int_0^1 \widetilde{g}'(t)\,dt = g(y) - g(x).$$

Interchanging the roles of x and y we obtain

$$f(y) - f(x) = g(y) - g(x). \tag{4}$$

Now take arbitrary two points $x, y \in \operatorname{Int} X$. Then there is a finite system of points $x = x_0, \ldots, x_n = y$ such that $[x_{i-1}, x_i] \subset \operatorname{Int} X$. By the previous considerations

$$f(x_{i-1}) - f(x_i) = g(x_{i-1}) - g(x_i), \quad i = 1, \ldots, n.$$

Adding all those equations we get (4).

Since f and g are continuous on X and $X \subset \overline{\operatorname{Int} X}$ we trivially deduce that (4) holds for all $x, y \in X$. ∎

REMARK 3. It is easy to observe that Proposition 2 holds if we replace the condition that f and g are locally Lipschitz by the condition

(L) *For any two x, y such that the interval $(x, y) = \{tx + (1 - t)y : 0 < t < 1\}$ is contained in X the functions f and g restricted to (x, y) are locally Lipschitz.*

We recall (see for example Pallaschke–Rolewicz (1997)) that a multifunction $\Gamma : X \to 2^\Phi$ is *monotone* if for all $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \ge 0. \tag{5}$$

In particular, when X is a linear space, and Φ is a linear space consisting of linear functionals $\phi(x) = \langle \phi, x \rangle$, we can rewrite (5) in the classical form

$$\langle \phi_x - \phi_y, x - y \rangle \ge 0.$$

A multifunction $\Gamma : X \to 2^\Phi$ is called *n-cyclic monotone* if, for all $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $i = 0, 1, \ldots, n$, we have

$$\sum_{i=1}^n [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \ge 0. \tag{6}$$

A multifunction $\Gamma : X \to 2^{\Phi}$ is called *cyclic monotone* if it is n-cyclic monotone for all $n = 2, 3, \ldots$ Of course, just from the definition, Γ is monotone if and only if it is 2-cyclic monotone.

A multifunction $\Gamma : X \to 2^{\Phi}$ is called *n-cyclic $\alpha(\cdot)$-monotone* if, for all $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$ $i = 0, 1, \ldots, n$, we have

$$\sum_{i=1}^{n}[\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] + \sum_{i=1}^{n}\alpha(d_X(x_i, x_{i-1})) \geq 0. \tag{7}$$

The 2-cyclic $\alpha(\cdot)$-monotone multifunctions are briefly called *$\alpha(\cdot)$-monotone* (Rolewicz (1999)). In the case when X is a normed space, $\Phi = X^*$ and $\alpha(t) = t^{\gamma}$ we obtain the definition of γ-monotone multifunctions introduced by Jourani (1996). A multifunction Γ is called *cyclic $\alpha(\cdot)$-monotone* if it is n-cyclic $\alpha(\cdot)$-monotone for all $n = 1, 2, \ldots$

Just from the definition we see that each monotone (resp. n-cyclic monotone, cyclic monotone) multifunction is $\alpha(\cdot)$-monotone (resp. n-cyclic $\alpha(\cdot)$-monotone, cyclic $\alpha(\cdot)$-monotone) for every $\alpha(\cdot)$. Moreover, if $\alpha_1(t) \geq \alpha(t)$ for all $0 \leq t < \infty$, then each $\alpha(\cdot)$-monotone (resp. n-cyclic $\alpha(\cdot)$-monotone, cyclic $\alpha(\cdot)$-monotone) multifunction is $\alpha_1(\cdot)$-monotone (resp. n-cyclic $\alpha_1(\cdot)$-monotone, cyclic $\alpha_1(\cdot)$-monotone).

Using the same method as in Section 1.1 of Pallaschke–Rolewicz (1997) we obtain

PROPOSITION 4. *Let f be a uniformly weakly Φ-convex function with modulus $\alpha(\cdot)$. Then the subdifferential $\partial_{\Phi}^{-\alpha} f|_x$, considered as a multifunction of x, is cyclic $\alpha(\cdot)$-monotone.*

Proof. Take $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \partial_{\Phi}^{-\alpha} f|_{x_i}$, $i = 0, 1, \ldots, n$. Since f is uniformly weakly Φ-convex with modulus $\alpha(\cdot)$, for $i = 1, \ldots, n$ we have

$$f(x_i) - f(x_{i-1}) \geq \phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}) - \alpha(d_X(x_i, x_{i-1})).$$

Adding all these inequalities we obtain

$$\begin{aligned} 0 &\geq \sum_{i=1}^{n}[\phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}) - \alpha(d_X(x_i, x_{i-1}))] \\ &= \sum_{i=1}^{n}[\phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1})] - \sum_{i=1}^{n}[\alpha(d_X(x_i, x_{i-1}))], \end{aligned}$$

which is (7). ■

An $\alpha(\cdot)$-monotone (resp. cyclic $\alpha(\cdot)$-monotone) multifunction Γ is called *maximal $\alpha(\cdot)$-monotone* (resp. *maximal cyclic $\alpha(\cdot)$-monotone*) if for each $\alpha(\cdot)$-monotone (resp. cyclic $\alpha(\cdot)$-monotone) multifunction Γ_1 such that $\Gamma(x) \subset \Gamma_1(x)$ for all x (in other words such that the graph of Γ, $G(\Gamma)$, is contained in $G(\Gamma_1)$), we have $\Gamma(x) = \Gamma_1(x)$ for all $x \in X$. It is easy to see that an $\alpha(\cdot)$-monotone multifunction Γ is maximal $\alpha(\cdot)$-monotone if and only if for all $x, y \in X$ and $\phi_x \in \Gamma(x)$, the inequality

$$\phi_x(x) + \psi(y) - \phi_x(y) - \psi(x) + 2\alpha(d_X(x, y)) \geq 0$$

implies that $\psi \in \Gamma(y)$. Observe that a maximal $\alpha(\cdot)$-monotone multifunction which is simultaneously cyclic $\alpha(\cdot)$-monotone is maximal cyclic $\alpha(\cdot)$-monotone. As follows from Example 1, in general the weak Φ-subdifferential with modulus $\alpha(\cdot)$ of a function f, $\partial_{\Phi}^{-\alpha} f|_x$, need not be a maximal $\alpha(\cdot)$-monotone multifunction.

Now we shall discuss the possibility of reversing Proposition 1.

Let (X, d_X) be a metric space. Let Φ be a family of continuous real-valued functions defined on X. Let $\Phi_{\alpha} = \{\phi(x) - \alpha(d_X(x, x_1)) : \phi \in \Phi,\ x_1 \in X\}$. Having this notation we can easily observe that if ϕ is a weak Φ-subgradient of a function f at a point x_0 with modulus $\alpha(\cdot)$ then $\phi(x) - \alpha(d_X(x, x_0))$ is a Φ_{α}-subgradient of f at x_0. However it may happen that $\phi(x) - \alpha(d_X(x, x_1))$ is a Φ_{α}-subgradient of a function g at x_0 and ϕ is not a weak Φ-subgradient of $g(x)$ at x_0 with modulus $\alpha(\cdot)$.

EXAMPLE 5. Let $X = [-1, 1]$, let Φ consist of the constant functions only and let $\alpha(t) = t^2$. Let $g(x) = 2x$. At the point 0 the function g has a Φ_{α}-subgradient $\psi(x) = 0 - (x - 1)^2$. On the other hand $\phi \equiv 0$ is not a weak Φ-subgradient of g at 0 with modulus $\alpha(\cdot)$.

It is essential to obtain conditions which guarantee that for all functions f and points x_0 the weak Φ-subdifferential with modulus $\alpha(\cdot)$ of f at x_0 with $\alpha(d_X(x, x_0))$ subtracted is equal to the Φ_{α}-subdifferential of f at x_0.

We shall show that such a condition is provided by the following property of $\alpha(\cdot)$ and the class Φ:

($\star$) *for every x_0 the function $\alpha(d(x, x_0))$ has at each $y \in X$ a subgradient $\phi_y \in \Phi$ such that for all $z \in X$,*

$$\alpha(d(z, x_0)) - \alpha(d(y, x_0)) + \phi_y(z) - \phi_y(y) \leq \alpha(d(z, y)). \tag{8}$$

It is interesting to know which $\alpha(\cdot)$ and Φ have property ($\star$).

PROPOSITION 6. *Let $X = \mathbb{R}$ and let Φ contain the class of linear functions. Let the function $\alpha(\cdot)$ be absolutely continuous. Assume that its derivative $\alpha'(t)$ exists for all $t > 0$ and moreover it satisfies the triangle inequality, $\alpha'(t + s) \leq \alpha'(t) + \alpha'(s)$. Then $\alpha(\cdot)$ and Φ have property ($\star$).*

Proof. Let $x_0, y, z \in \mathbb{R}$. Since $|x - y|$ is an invariant metric, without loss of generality we may assume that $x_0 = 0$. Thus (8) is equivalent to

$$\alpha(|z|) - \alpha(|y|) + \phi_y(z) - \phi_y(y) \leq \alpha(|z - y|). \tag{9}$$

We put $|z| = s+h$, $|y| = s$. Then

$$\alpha(s+h) - \alpha(s) - h\alpha'(s) = \int_s^{s+h} [\alpha'(t) - \alpha'(s)]\,dt.$$

If $h \ge 0$, then by the triangle inequality for $\alpha'(\cdot)$,

$$\int_s^{s+h} [\alpha'(t) - \alpha'(s)]\,dt \le \int_s^{s+h} \alpha'(t-s)\,dt = \int_0^h \alpha'(u)du = \alpha(u)|_0^h = \alpha(h),$$

i.e. (9) holds.

If $h < 0$, then again by the triangle inequality,

$$\int_s^{s+h} [\alpha'(t) - \alpha'(s)]\,dt = \int_{s-|h|}^{s} [\alpha'(s) - \alpha'(t)]\,dt \le \int_{s-|h|}^{s} \alpha'(s-t)\,dt$$

$$= -\int_{|h|}^{0} \alpha'(u)du = \int_0^{|h|} \alpha'(u)du = \alpha(|h|),$$

i.e. (9) also holds. ∎

Observe that the function $\alpha(t) = t^\gamma$, $1 < \gamma \le 2$, satisfies the assumption of Proposition 6. Indeed, in this case $\alpha'(t) = \gamma t^{\gamma-1}$ is a concave function, and thus it satisfies the triangle inequality.

If additionally $\alpha(\cdot)$ is convex (in particular if $\alpha(t) = t^\gamma$, $1 < \gamma \le 2$) we can extend Proposition 6 to normed spaces.

PROPOSITION 7. *Let $\alpha(\cdot)$ be convex. Assume that its upper derivative*

$$\alpha^+(t) = \lim_{h\downarrow 0} \frac{\alpha(t+h) - \alpha(t)}{h}$$

satisfies the triangle inequality, $\alpha^+(t+s) \le \alpha^+(t) + \alpha^+(s)$. Let $(X, \|\cdot\|)$ be a normed space and let Φ contain the conjugate space X^* of all continuous linear functionals. Then $\alpha(\cdot)$ and Φ have property $(\star)$.

Proof. Since $\alpha(\cdot)$ is convex it is absolutely continuous. As in the proof of Proposition 6, replacing $\alpha'(s)$ by the upper derivative $\alpha^+(s)$ we get

$$(10)\quad \alpha(\|z\|) - \alpha(\|y\|) + \alpha^+(\|y\|)(\|z\| - \|y\|) \le \alpha(\|z\| - \|y\|) \le \alpha(\|z-y\|).$$

Since $\alpha(\cdot)$ is convex we have $\alpha^+(\|y\|) \ge 0$. Let $y^* \in X^*$ be a functional of norm one such that $y^*(y) = \|y\|$. Of course $y^*(z) \le \|z\|$. Then by (10) we get

$$\alpha(\|z\|) - \alpha(\|y\|) + \alpha^+(\|y\|)y^*(z-y) \le \alpha(\|z-y\|),$$

i.e. (9) holds for $\phi_y = \alpha^+(\|y\|)y^*$. ∎

PROPOSITION 8. *Suppose that Φ is linear. Suppose that $\alpha(\cdot)$ and Φ satisfy condition $(\star)$. Then there is a weak Φ-subgradient $\phi \in \Phi$ of a function f at x_0*

with modulus $\alpha(\cdot)$ if and only if there is $\psi \in \Phi$ such that $\psi(x) - \alpha(d_X(x, x_0))$ is a Φ_α-subgradient of f at x_0, where $\Phi_\alpha = \{\phi(x) - \alpha(d_X(x, x_1)) : \phi \in \Phi, x_1 \in X\}$.

Proof. If ϕ is a weak Φ-subgradient of f at x_0 with modulus $\alpha(\cdot)$, then by definition

$$f(x) - f(x_0) \ge \phi(x) - \phi(x_0) - \alpha(d_X(x, x_0)).$$

This trivially implies that $\phi(x) - \alpha(d_X(x, x_0))$ is a Φ_α-subgradient of f at x_0.

Suppose now that $\phi(x) - \alpha(d_X(x, x_1))$ is a Φ_α-subgradient of f at x_0. Then by definition we have

$$(11)\quad f(x) - f(x_0) \ge \phi(x) - \phi(x_0) + \alpha(d_X(x_0, x_1)) - \alpha(d_X(x, x_1)).$$

By property $(\star)$ there is a $\phi_{x_0} \in \Phi$ such that for all $x \in X$,

$$\alpha(d_X(x, x_1)) - \alpha(d_X(x_0, x_1)) + \phi_{x_0}(x) - \phi_{x_0}(x_0) \le \alpha(d_X(x, x_0)),$$

i.e.

$$(12)\quad \alpha(d_X(x_0, x_1)) - \alpha(d_X(x, x_1)) \ge \phi_{x_0}(x) - \phi_{x_0}(x_0) - \alpha(d_X(x, x_0)).$$

Thus by (11) and (12) we get

$$f(x) - f(x_0) \ge \phi(x) - \phi(x_0) + \phi_{x_0}(x) - \phi_{x_0}(x_0) - \alpha(d_X(x, x_0)).$$

Therefore $\psi(\cdot) = \phi(\cdot) + \phi_{x_0}(\cdot) \in \Phi$ is a weak Φ-subgradient of f at x_0. ∎

Let Γ be a multifunction mapping X into 2^Φ. We denote by $(\Gamma - \alpha)$ the multifunction mapping X into 2^{Φ_α} defined in the following way:

$$(\Gamma - \alpha)(x) = \{\psi(\cdot) = \phi(\cdot) - \alpha(d_X(\cdot, x)) : \phi \in \Gamma(x)\}.$$

We call $(\Gamma - \alpha)$ the multifunction Γ with $\alpha(d_X(x, \cdot))$ subtracted.

From Proposition 8 we trivially obtain the following

COROLLARY 9. *Suppose that Φ is linear. Suppose that $\alpha(\cdot)$ and Φ satisfy condition $(\star)$. Then the weak Φ-subdifferential with modulus $\alpha(\cdot)$ of a function f at x with $\alpha(d_X(x, \cdot))$ subtracted is equal to the Φ_α-subdifferential of f at x,*

$$(\partial_\Phi^{-\alpha} f|_x - \alpha(d_X(x, \cdot))) = \partial_{\Phi_\alpha} f|_x.$$

By simple calculation we get

PROPOSITION 10. *Let Γ be a cyclic (resp. n-cyclic) $\alpha(\cdot)$-monotone multifunction mapping X into 2^Φ. Then $(\Gamma - \alpha)$ is cyclic (resp. n-cyclic) monotone.*

Proof. By definition for all $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $i = 0, 1, \ldots, n$, we have

$$\sum_{i=1}^n [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] + \sum_{i=1}^n \alpha(d_X(x_i, x_{i-1})) \ge 0.$$

Let $\psi_{x_i} \in (\Gamma - \alpha)(x_i)$. We put $\phi_{x_i}(\cdot) = \psi_{x_i}(\cdot) + \alpha(d_X(x_i, \cdot))$. Then $\phi_{x_i} \in \Gamma(x_i)$ and from the above we get

$$\sum_{i=1}^{n} [\psi_{x_{i-1}}(x_{i-1}) - \psi_{x_{i-1}}(x_i)] \geq 0,$$

which shows that $(\Gamma - \alpha)$ is cyclic (resp. n-cyclic) monotone. ■

COROLLARY 11. *Suppose that Φ is linear. Suppose that $\alpha(\cdot)$ and Φ satisfy condition $(\star)$. Then $\partial_{\Phi}^{-\alpha} f|_x$ is n-cyclic (resp. cyclic) $\alpha(\cdot)$-monotone if and only if $(\partial_{\Phi_\alpha} f|_x - \alpha(d_X(x, \cdot)))$ is n-cyclic (resp. cyclic) monotone.*

From Proposition 10, as in Section 1.1 of Pallaschke–Rolewicz (1997), we trivially obtain the following extension of the Rockafellar theorem (compare Rockafellar (1970)).

THEOREM 12. *Suppose that Φ is linear. Suppose that $\alpha(\cdot)$ and Φ satisfy condition $(\star)$. Let $\Gamma : X \to 2^{\Phi}$ be maximal cyclic $\alpha(\cdot)$-monotone. Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a function f weakly Φ-convex with modulus $\alpha(\cdot)$ such that Γ is the weak Φ-subdifferential of f with modulus $\alpha(\cdot)$, $\Gamma(x) = \partial_{\Phi}^{-\alpha} f|_x$.*

Proof. By Proposition 10 the multifunction $(\Gamma - \alpha)$ is cyclic monotone. We do not know if it is maximal or not. However, using the Kuratowski–Zorn Lemma we can find a maximal cyclic (resp. n-cyclic) monotone multifunction $(\Gamma - \alpha)_{\max}$ such that $(\Gamma - \alpha)(x) \subset (\Gamma - \alpha)_{\max}(x)$. Thus by Proposition 1.11 of Pallaschke–Rolewicz (1997) we can find a function f such that $\partial_{\Phi_\alpha} f|_x = (\Gamma - \alpha)_{\max}(x)$.

By $(\star)$ and Corollary 9 we get

$$(\partial_{\Phi}^{-\alpha} f|_x - \alpha(d_X(x, \cdot))) = (\Gamma - \alpha)_{\max}(x).$$

This implies

$$(\partial_{\Phi}^{-\alpha} f|_x - \alpha(d_X(x, \cdot))) \supset (\Gamma - \alpha)(x).$$

Therefore $\partial_{\Phi}^{-\alpha} f|_x \supset \Gamma(x)$, and by maximality of Γ we get $\partial_{\Phi}^{-\alpha} f|_x = \Gamma(x)$. ■

In general, the knowledge of a weak Φ-subdifferential with modulus $\alpha(\cdot)$, $\partial_{\Phi}^{-\alpha} f|_x : X \to 2^{\Phi}$, does not permit one to determine the function f (up to a constant) (see Example 1). But in Example 1, $\partial_{\Phi}^{-\alpha} f|_x$ is not a maximal cyclic $\alpha(\cdot)$-monotone multifunction, and we do not know if the equality $\partial_{\Phi}^{-\alpha} f|_x = \partial_{\Phi}^{-\alpha} g|_x$ together with the maximal cyclic $\alpha(\cdot)$-monotonicity of the multifunction $\partial_{\Phi}^{-\alpha} f|_x$ implies that $f(x) = g(x) + c$. In the case when X is a Banach space, $\Phi = X^*$ is the conjugate space and $\alpha(t) = t^\gamma$, $1 < \gamma \leq 2$, the answer is positive. More precisely we have

PROPOSITION 13. *Let X be a Banach space, let $\Phi = X^*$ be the conjugate space and let $1 < \gamma \leq 2$. Let $\Gamma : X \to 2^{\Phi}$ be maximal cyclic t^γ-monotone. Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a function f such that Γ is the γ-subdifferential of f, $\Gamma(x) = \partial_{\Phi}^{-t^\gamma} f|_x$, and the function f is uniquely determined up to a constant.*

Proof. By Theorem 12 there is a function f such that $\Gamma(x) = \partial_{\Phi}^{-t^\gamma} f|_x$. Using the result of Correa, Jofré and Thibault (1994) (see Jourani (1996), Theorem 7.1) we find that f is *γ-paraconvex*, i.e. there is $C > 0$ such that for all $x, y \in X$ and all $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C\|x - y\|^\gamma. \tag{13}$$

Observe that (13) immediately implies that f is bounded from above on $[x, y]$. We shall show that it is also bounded from below. Indeed, suppose that there is a sequence $\{z_n\} \subset [x, y]$ such that $\lim_{n\to\infty} f(z_n) = -\infty$. By compactness we can assume that $\{z_n = t_n x + (1 - t_n)y\}$ is convergent to $z = t_0 x + (1 - t_0)y \in [x, y]$. We can also assume that $\{t_n\}$ is either increasing or decreasing. In both cases we can choose $u \in [x, y]$ such that either $u \in \mathrm{Int}[x, z_n]$ or $u \in \mathrm{Int}[z_n, y]$ and replacing $[x, y]$ by $[x, z_n]$ and $[z_n, y]$ respectively we obtain a contradiction with (13).

Then by Jourani (1996), Remark 2.1, f is locally Lipschitz in the interval (x, y). Thus by Proposition 2 the equality $\partial_{\Phi}^{-\alpha} f|_x = \partial_{\Phi}^{-\alpha} g|_x$ implies that f and g differ by a constant, $f(x) = g(x) + c$, $c \in \mathbb{R}$. ■

COROLLARY 14. *Let X be a Banach space, let $\Phi = X^*$ be the conjugate space and let $1 < \gamma \leq 2$. Then the γ-subdifferential $\partial_{\Phi}^{-t^\gamma} f|_x$ of every function f, weakly Φ-convex with modulus t^γ, is a maximal cyclic t^γ-monotone multifunction.*

Proof. By Proposition 4, $\partial_{\Phi}^{-t^\gamma} f|_x$ is a cyclic t^γ-monotone multifunction. Of course we do not know if it is maximal or not. However, using the Kuratowski–Zorn Lemma we can find a maximal cyclic t^γ-monotone multifunction Γ such that

$$\partial_{\Phi}^{-t^\gamma} f|_x \subset \Gamma(x). \tag{14}$$

By Theorem 12 there is a function g, weakly Φ-convex with modulus t^γ, such that Γ is the weak Φ-subdifferential of g with modulus t^γ, $\Gamma(x) = \partial_{\Phi}^{-t^\gamma} g|_x$. Thus by (14), $\partial_{\Phi}^{-t^\gamma} f|_x \subset \partial_{\Phi}^{-t^\gamma} g|_x$. Therefore by Proposition 13, $f(x) = g(x) + c$ and we have the equality

$$\partial_{\Phi}^{-t^\gamma} f|_x = \partial_{\Phi}^{-t^\gamma} g|_x = \Gamma(x).$$

Since Γ is a maximal cyclic t^γ-monotone multifunction, so is $\partial_{\Phi}^{-t^\gamma} f|_x$. ■

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On the complemented subspaces of the Schreier spaces

by

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Abstract. It is shown that for every $1 \le \xi < \omega$, two subspaces of the Schreier space X^ξ generated by subsequences $(e^\xi_{l_n})$ and $(e^\xi_{m_n})$, respectively, of the natural Schauder basis (e^ξ_n) of X^ξ are isomorphic if and only if $(e^\xi_{l_n})$ and $(e^\xi_{m_n})$ are equivalent. Further, X^ξ admits a continuum of mutually incomparable complemented subspaces spanned by subsequences of (e^ξ_n). It is also shown that there exists a complemented subspace spanned by a block basis of (e^ξ_n), which is not isomorphic to a subspace generated by a subsequence of (e^ζ_n), for every $0 \le \zeta \le \xi$. Finally, an example is given of an uncomplemented subspace of X^ξ which is spanned by a block basis of (e^ξ_n).

1. Introduction. The Schreier families $\{S_\xi\}_{\xi<\omega_1}$ of finite subsets of positive integers (the precise definition is given in the next section), introduced in [1], have played a central role in the development of modern Banach space theory. We mention the use of Schreier families in the construction of mixed Tsirelson spaces which are asymptotic ℓ_1 and arbitrarily distortable [3]. The distortion of mixed Tsirelson spaces has been extensively studied in [2]. In that paper, as well as in [14], the moduli $(\delta_\alpha)_{\alpha<\omega_1}$ were introduced measuring the complexity of the asymptotic ℓ_1 structure of a Banach space. The definitions of those moduli also involve the Schreier families. Other applications can be found in [6] and [5] where the Schreier families form the main tool for determining the structure of those convex combinations of a weakly null sequence that tend to zero in norm, or are equivalent to the unit vector basis of c_0. For applications of the Schreier families in the construction of hereditarily indecomposable Banach spaces, we refer to [3] and [4].

A notion companion to the Schreier families is that of the Schreier spaces. These are Banach spaces whose norm is related to a corresponding Schreier family. More precisely, for every countable ordinal ξ, we define a norm $\|\cdot\|_\xi$ on c_{00}, the space of finitely supported real-valued sequences, in the following

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