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Received May 6, 1998
Revised version March 6, 2000 (4096)

STUDIA MATHEMATICA 141 (3) (2000)

Polynomial inequalities on algebraic sets

by

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Abstract. We give an estimate of Siciak's extremal function for compact subsets of algebraic varieties in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). As an application we obtain Bernstein–Walsh and tangential Markov type inequalities for (the traces of) polynomials on algebraic sets.

0. Preliminaries. The theory of the multivariate Markov inequality furnishing estimates of the derivatives of a polynomial in n variables in terms of its degree and its uniform norm on an n-dimensional compact subset of $\mathbb{C}^n$ or $\mathbb{R}^n$ was essentially developed in the last ten years. For an exhaustive survey on this subject we refer the reader to [Pl3]. In recent years, Markov and Bernstein type inequalities have been intensively investigated on algebraic subvarieties of $\mathbb{R}^n$ (see [BLT], [BLMT1], [BLMT2], [FeNa1], [FeNa2], [FeNa3], [Bru], [BaPl2], [BaPl3], [RoYo], [Gen]). In particular, in [BLT], [BLMT1], [BaPl2] and [BaPl3] the authors have characterized semialgebraic curves as well as semialgebraic manifolds in $\mathbb{R}^n$ in terms of tangential Markov or Bernstein and van der Corput–Schaake type inequalities.

The purpose of this paper is to establish Bernstein–Walsh or (tangential) Markov type inequalities on subsets N of an algebraic set in $\mathbb{R}^n$ that are images under non-degenerate analytic maps of non-pluripolar, compact sets in $\mathbb{R}^k$. Our results yield, as particular cases, some recent results of Bos–Levenberg–Milman–Taylor [BLMT1], [BLMT2] and Brudnyi [Bru].

Let us note that if N is a subset of an analytic variety then, in general, it need not admit a tangential Markov inequality with any finite exponent. A relevant example is due to Izumi [Iz] (see [BLMT1]).

2000 *Mathematics Subject Classification*: 41A17, 41A25, 14P10, 14P05.

Key words and phrases: traces of polynomials on algebraic sets, Siciak's extremal function, pluricomplex Green function, Bernstein–Walsh and tangential Markov type inequalities on algebraic sets.

Research partially supported by KBN grant No 2 P03A 045 14. The second-named author wishes to thank the Laboratory of Geometry, Analysis and Topology of Université des Sciences et Technologies, Lille, where a part of the paper was written, and personally Professor Anne-Marie Chollet, for the invitation and hospitality.

Let E be a subset of the space $\mathbb{C}^k$. We set

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^k),\ u \le 0 \text{ on } E\},$$

where $\mathcal{L}(\mathbb{C}^k) = \{u \in \mathrm{PSH}(\mathbb{C}^k) : \sup_{z\in\mathbb{C}^k}[u(z) - \log(1+|z|)] < \infty\}$ is the *Lelong class* of plurisubharmonic functions with minimal growth. The function V_E is called the (*plurisubharmonic*) *extremal function* associated with E (see [Si2]). By the pluripotential theory due to E. Bedford and B. A. Taylor (see [K]), if E is non-pluripolar in $\mathbb{C}^k$ then the upper semicontinuous regularization V_E^* of V_E belongs to $\mathcal{L}(\mathbb{C}^k)$ and is a solution (in $\mathbb{C}^k \setminus \widehat{E}$, where $\widehat{E}$ denotes the polynomial hull of E) of the homogeneous complex *Monge–Ampère equation*, which reduces in the one-dimensional case to the *Laplace equation*. Therefore V_E^* is a multidimensional counterpart of the classical *Green function* for $\mathbb{C} \setminus \widehat{E}$. It is a result of Siciak [Si2] that if E is compact then

$$(0.1)\qquad V_E(z) = \sup\left\{\frac{1}{\deg p}\log|p(z)| : p \text{ is a polynomial with } \deg p \ge 1 \text{ and } \|p\|_E \le 1\right\} = \log \Phi_E(z),$$

where Φ_E is the (*polynomial*) *extremal function* of E introduced by Siciak [Si1]. In what follows, we shall be working with both the plurisubharmonic and the polynomial extremal functions.

We recall that a subset E of $\mathbb{C}^k$ is said to be *pluripolar* if there is a plurisubharmonic function u on $\mathbb{C}^k$ such that $E \subset \{u = -\infty\}$. By Josefson [Jos], E is pluripolar if and only if it is *locally pluripolar*, i.e. if for each point $a \in E$ there exist an open neighbourhood U of a and a plurisubharmonic function u on U such that $E \cap U \subset \{u = -\infty\}$.

Let now $\mathbb{M}$ be a locally analytic subset of $\mathbb{C}^n$ such that the set $\mathbb{M}_{\mathrm{reg}}$ of regular points of $\mathbb{M}$ is a complex submanifold of $\mathbb{C}^n$ of pure dimension k, where $k \le n$. A function u defined on $\mathbb{M}$ is said to be plurisubharmonic on $\mathbb{M}$ if it is plurisubharmonic on $\mathbb{M}_{\mathrm{reg}}$ and locally bounded above on $\mathbb{M}$. Let N be a subset of $\mathbb{M}$. Then N is said to be *pluripolar* in $\mathbb{M}$ if there exists a plurisubharmonic function u on $\mathbb{M}$ such that $N \cap \mathbb{M}_{\mathrm{reg}} \subset \{u = -\infty\}$.

In our paper, a crucial role is played by the following

LEMMA 0.1. *Let E be a non-pluripolar compact subset of $\mathbb{C}^k$ and let f be an analytic map defined in an open neighbourhood U of E, with values in a locally analytic subset $\mathbb{M}$ of $\mathbb{C}^n$ of pure dimension* $\min(k,n)$, *where we set $\mathbb{M} = \mathbb{C}^n$ if $k > n$. If* $\mathrm{rank}_V f := \sup_{z\in V} \mathrm{rank}_z f = \min(k,n)$ *for a connected component V of U such that $V \cap E$ is non-pluripolar, then $f(E)$ is a non-pluripolar subset of $\mathbb{M}$.*

Proof. First assume that $k \le n$. Let $\mathbb{M}_{\mathrm{sing}}$ be the set of singular points of $\mathbb{M}$. Then $\mathbb{M}_{\mathrm{sing}}$ is an analytic subset of $\mathbb{M}$ with $\dim \mathbb{M}_{\mathrm{sing}} < k$ (see e.g. [Ł, Chapt. V.4]). Consequently, the set $A_1 := V \cap f^{-1}(\mathbb{M}_{\mathrm{sing}})$ is an analytic subset of V. Since $\mathrm{rank}_V f = k$, we have $A_1 \ne V$, as otherwise we would have $\dim f(V) < k$, which is impossible. In particular, by Josefson's theorem, the set A_1 is pluripolar (in $\mathbb{C}^k$).

Let now A_2 be the set $\{z \in V : \mathrm{rank}_z f < k\}$. Then A_2 is also pluripolar, whence $F := E \setminus (A_1 \cup A_2)$ is a non-pluripolar subset of $\mathbb{C}^k$. Again by Josefson's theorem, there is a point $a \in F$ such that for each open neighbourhood V_a of a the set $F \cap V_a$ is non-pluripolar. Since $\mathrm{rank}_a f = k$, and since the sets A_1 and A_2 are closed (in the induced topology of V), we may choose V_a so that $V_a \subset V \setminus (A_1 \cup A_2)$ and the restriction of f to V_a is a biholomorphism of V_a onto $f(V_a) \subset \mathbb{M}_{\mathrm{reg}}$.

Suppose now that $f(F)$ is (locally) pluripolar. Then there exist an open neighbourhood Ω of $f(a)$ with $\Omega \subset f(V_a) \subset \mathbb{M}_{\mathrm{reg}}$ and a plurisubharmonic function u on Ω such that $\Omega \cap f(F) \subset \{u = -\infty\}$. Thus, for the plurisubharmonic function $u \circ f$ we would have $f^{-1}(\Omega) \cap F \subset \{u \circ f = -\infty\}$. On the other hand, since $f^{-1}(\Omega)$ is a neighbourhood of a, the set $f^{-1}(\Omega) \cap F$ cannot be pluripolar. This gives a contradiction. Consequently, $f(F)$ as well as $f(E)$ must be non-pluripolar.

The case where $k > n$ is now obvious.

In what follows, we will be assuming that $\mathbb{R}^n$ is the real part of the space $\mathbb{C}^n$, i.e. $\mathbb{R}^n = \mathbb{R}^n + i0 \subset \mathbb{C}^n$. In Section 2, we shall deal with real algebraic subsets of $\mathbb{R}^n$. In such a setting Lemma 0.1 yields the following

COROLLARY 0.2. *Let E be a non-pluripolar compact subset of $\mathbb{R}^k$ and let f be a real-analytic map defined in an open neighbourhood of E, with values in a real algebraic subset $\mathbb{M}$ of $\mathbb{R}^n$ of dimension k with $1 \le k \le n$, where we put $\mathbb{M} = \mathbb{R}^n$ if $k = n$. If $\mathrm{rank}_E f = k$ then $f(E)$ is a non-pluripolar subset of (the complexification $\widetilde{\mathbb{M}}$ of) $\mathbb{M}$.*

1. Estimates for Siciak's extremal function on algebraic sets.

We shall need the following beautiful result of Sadullaev [Sa].

SADULLAEV'S CRITERION. *An analytic subset $\mathbb{M}$ of $\mathbb{C}^n$ is algebraic if and only if Siciak's extremal function Φ_N is locally bounded in $\mathbb{M}$ for some (and hence for each) non-pluripolar compact subset N of $\mathbb{M}$.*

The above criterion together with Lemma 0.1 permits one to prove numerous versions of Bernstein–Walsh type inequalities on algebraic varieties.

EXAMPLE 1.1. Let S^{n-1} denote the unit sphere in $\mathbb{C}^n$ and let Ω be an open subset of S^{n-1}. We claim that there exists a positive constant A depending only on Ω such that for each polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\sup_{z\in S^{n-1}} |P(z)| \le A^{\deg P} \sup_{z\in\Omega} |P(z)|.$$

Indeed, S^{n-1} can be considered as a compact, $(2n-1)$-dimensional real algebraic subset N of $\mathbb{R}^{2n}$,

$$N = \{x_1^2 + y_1^2 + \ldots + x_n^2 + y_n^2 = 1\},$$

whose complexification $\mathbb{M}$ is a complex algebraic subset of $\mathbb{C}^{2n}$ of complex dimension $2n-1$. Let $T : I \to N$ be the standard spherical parametrization of N, where I is a compact cube in $\mathbb{R}^{2n-1}$ (see e.g. [K, Section 2.1]). If Ω is an open subset of S^{n-1}, one can find a closed subcube J of I such that $T(J) \subset \Omega$. Hence by Lemma 0.1, Ω is a non-pluripolar subset of $\mathbb{M}$. By Sadullaev's Criterion, $A := \sup \Phi_{T(J)}(N) < \infty$. Hence by the definition of the extremal function, for any polynomial P in n complex variables $z_1, \ldots, z_n$ we get for $u = (x_1, y_1, \ldots, x_n, y_n) \in N$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$,

$$|P(u)| \le A^{\deg P} \sup_{u \in T(J)} |P(u)|,$$

which proves our claim.

By Sadullaev's Criterion, it is also clear that the same holds true if we replace Ω by a subset Ω' of S^{n-1} that is non-pluripolar in $\mathbb{M}$.

EXAMPLE 1.2. Let $\mathbb{M}$ be an algebraic subset of $\mathbb{R}^n$ of pure dimension k, where $1 \le k \le n-1$, and let x be a regular point of $\mathbb{M}$. Then one can find an open neighbourhood Ω of x and an analytic homeomorphism ϕ of the open unit ball $B = B(0,1)$ in $\mathbb{R}^k$ onto Ω. If $S \subset \Omega$, let

$$\sigma(S) = \int_{\phi^{-1}(S)} |d_t\phi| \, d\lambda(t)$$

be the measure on Ω induced by the Lebesgue measure λ in $\mathbb{R}^k$. We claim that for every ball $B(\delta) = B(x,\delta) \cap \Omega$ on $\mathbb{M}$ with $0 < \delta \le \delta_0$, and for every set $S \subset B(\delta)$ with $\sigma(S) > 0$, there exists a positive constant $A \ge 1$ that depends only on $B(\delta)$ and S, such that for every polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d we have

$$\sup_{B(\delta)} |P(x)| \le A^d \sup_{x \in S} |P(x)|.$$

To see this, choose a compact subset E of $\phi^{-1}(S)$ with $\lambda(E) > 0$. Then the set E is non-pluripolar (in $\mathbb{C}^k$) and by Corollary 0.2 the set $\phi(E)$ is a non-pluripolar, compact subset of $\mathbb{M}$. Hence applying Sadullaev's Criterion to the extremal Siciak function $\Phi_{\phi(E)}$ we prove our claim.

Let us mention that the above example yields an inequality that is close to the main result of Brudnyi [Bru, Theorem 1.2].

In Section 2 of this paper, we shall need more refined information about Siciak's extremal function than that furnished by Sadullaev's Criterion. It is provided by the following

PROPOSITION 1.3. *Let E be a compact, non-pluripolar subset of $\mathbb{C}^k$ and let f be an analytic map defined in an open neighbourhood U of $\widehat{E}$, the polynomial hull of E, with values in a $\min(k,n)$-dimensional algebraic set $\mathbb{M}$ in $\mathbb{C}^n$ (where $\mathbb{M} = \mathbb{C}^n$ if $k \ge n$). Assume that $\operatorname{rank}_E f = \min(k,n)$. Then there exist constants $M > 0$ and $\delta_0 > 0$ such that*

$$V_{f(E)}(f(z)) \le M\, V_E(z) \quad \text{as } \operatorname{dist}(z,E) \le \delta \le \delta_0.$$

Proof. We have to prove that there exist constants $M > 0$ and $\delta_0 > 0$ such that for any d, and any polynomial $p \in \mathbb{C}[w_1, \ldots, w_n]$ of degree d,

$$|p(f(z))| \le \|p\|_{f(E)} \Phi_E^{Md}(z) \quad \text{as } \operatorname{dist}(z,E) \le \delta \le \delta_0.$$

We may assume that f is bounded in U. Let $\delta_0 > 0$ be so small that the polynomial hull F of the set $E(\delta_0) := \{z \in \mathbb{C}^k : \operatorname{dist}(z,E) \le \delta_0\}$ is contained in U. By a uniform version of the Bernstein–Walsh–Siciak theorem (see [Pl1, Lemma 2.1]), there exist constants M_1 and $a \in (0,1)$ such that for any polynomial $p \in \mathbb{C}[w_1, \ldots, w_n]$ of degree d one can find polynomials $q_l \in \mathbb{C}[z_1, \ldots, z_k]$, $l = 1, 2, \ldots$, of degree l satisfying

$$\begin{aligned} \|p \circ f - q_l\|_F &\le M_1 \|p \circ f\|_U a^l = M_1 \|p\|_{f(U)} a^l \\ &\le M_1 \|p\|_{f(E)} (\sup \Phi_{f(E)}(f(U)))^d a^l. \end{aligned}$$

Since $\operatorname{rank}_E f = \min(k,n)$, Lemma 0.1 shows that $f(E)$ is a non-pluripolar subset of $\mathbb{M}$, and by Sadullaev's Criterion, $A := \sup \Phi_{f(E)}(f(U)) < \infty$. Choose $l = Md$, where M is a positive integer so large that $Aa^M \le a$. Then we have, for $z \in E(\delta)$ with $0 < \delta \le \delta_0$,

$$|p(f(z))| \le \|p \circ f - q_{Md}\|_{E(\delta)} + |q_{Md}(z)| \le M_1 \|p\|_{f(E)} a^d + \|q_{Md}\|_E \Phi_E^{Md}(z).$$

Observe that

$$\|q_{Md}\|_E \le \|p \circ f - q_{Md}\|_E + \|p \circ f\|_E \le (M_1 a^d + 1) \|p \circ f\|_E.$$

Hence, for $z \in E(\delta)$,

$$\begin{aligned} |p \circ f(z)| &\le M_1 \|p\|_{f(E)} a^d + (M_1 a^d + 1) \|p\|_{f(E)} \Phi_E^{Md}(z) \\ &\le \|p\|_{f(E)} (2M_1 a^d + 1) \Phi_E^{Md}(z). \end{aligned}$$

Now, applying the above inequality to the polynomials p^r, $r = 1, 2, \ldots$, then taking the rdth root of both sides and letting r tend to infinity gives

$$\Phi_{f(E)}(f(z)) \le \Phi_E^M(z) \quad \text{for } z \in E(\delta).$$

In particular,

$$V_{f(E)}(f(z)) \le M\, V_E(z) \quad \text{for } z \in E(\delta) \text{ with } 0 < \delta \le \delta_0.$$

REMARK. Define the modulus of continuity of V_E by

$$\omega(V_E; \delta) := \sup\{V_E(z) : \operatorname{dist}(z,E) \le \delta\} \quad \text{for } 0 < \delta \le \delta_0.$$

By a remark due to Z. Błocki (unpublished), if the extremal function V_E is continuous on E, it is uniformly continuous on the whole space $\mathbb{C}^k$ with the same modulus of continuity $\omega(V_E;\delta)$. For the sake of completeness we give Błocki's reasoning.

Let E be a non-pluripolar subset of $\mathbb{C}^k$ and let $u \in \mathcal{L}(\mathbb{C}^k)$, $u \le 0$ on E. If $\zeta \in \mathbb{C}^k$ and $|\zeta| \le \delta_0$, we set

$$v_\zeta(z) := u(z+\zeta) - \omega(V_E;|\zeta|).$$

Then $v_\zeta \in \mathcal{L}(\mathbb{C}^k)$. If $z \in E$ then $\operatorname{dist}(z+\zeta, E) \le |z+\zeta-z| = |\zeta|$. Hence $u(z+\zeta)-\omega(V_E;|\zeta|) \le 0$, whence $v_\zeta(z) \le 0$ for $z \in E$. Thus, by the definition of V_E, we get

$$V_E(z+\zeta) - V_E(z) \le \omega(V_E;|\zeta|) \quad \text{for any } z \in \mathbb{C}^k.$$

By the same argument we show that for any $z \in \mathbb{C}^k$ and $|\zeta| \le \delta_0$,

$$V_E(z-\zeta) - V_E(z) \le \omega(V_E;|\zeta|).$$

Consequently, for any point $z \in \mathbb{C}^k$ and $0 < \delta \le \delta_0$, we get

$$\sup\{|V_E(z+\zeta) - V_E(z)| : |\zeta| \le \delta\} \le \omega(V_E;\delta),$$

as claimed.

REMARK. Proposition 1.3 does not assert that any non-degenerate analytic map preserves the modulus of continuity of V_E. (Consider e.g. $E = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x\}$ and $f(x,y) = (x,y^2)$.) However, if $\operatorname{rank}_t f = \min(k,n)$ for each $t \in E$ and $f(E) \subset \mathbb{M}_{\text{reg}}$, then by using Merrien–Tougeron's version of the implicit function theorem (see [Tou, Chap. I, Proposition 5.1]) one can show that f preserves the modulus of continuity of V_E (see [Pl2], the case where $\min(k,n) = n$).

2. Tangential Markov inequality on algebraic sets. Let now E be a non-pluripolar compact set in the space $\mathbb{C}^k$ and let f be an analytic map defined in an open neighbourhood U of the polynomial hull $\widehat{E}$ of E, with values in a k-dimensional algebraic subset $\mathbb{M}$ of $\mathbb{C}^n$, where $\mathbb{M} = \mathbb{C}^n$ if $k \ge n$. Assume that f is non-degenerate. Then by Lemma 0.1 the set $N = f(E)$ is a non-pluripolar compact subset of $\mathbb{M}$.

Let $Q \in \mathbb{C}[z_1,\ldots,z_n]$ be a polynomial of degree d. For any vector $v \in S^{k-1}$ and for any fixed $t \in E$, consider the function $g(s) = Q(f(t+sv))$ defined in a sufficiently small neighbourhood of $0 \in \mathbb{C}$. By Cauchy's Integral Formula and by Proposition 1.3 we get

$$|g'(0)| \le \delta^{-1} \sup_{|s|\le\delta} |Q(f(t+sv))| \le \delta^{-1}\|Q\|_N \sup_{|s|=\delta} \Phi_E^{Md}(t+sv),$$

for $\delta > 0$ sufficiently small, with an appropriate constant $M > 0$, where $\|Q\|_N := \sup|Q|(N)$. On the other hand, $g'(0) = D_{\mathcal{T}(t,v)}Q$, where $\mathcal{T}(t,v) = D_v f(t)$, the derivative at the point t of the map f in direction v. Hence by (0.1) we get the following formula:

$$(2.1)\qquad |D_{\mathcal{T}(t,v)}Q(x)| \le \delta^{-1}(M_1\omega(V_E;\delta)d+1)\|Q\|_N \quad \text{as } 0<\delta\le\delta_0,$$

with positive constants M_1 and δ_0 that depend only on E and f.

The above formula can be specified in case E is a compact subset of $\mathbb{C}^k$ with the *Hölder Continuity Property* (of V_E), which means that the extremal function V_E associated with E satisfies

$$(\text{HCP})\qquad \omega(V_E;\delta) \le M\delta^{1/r} \quad \text{for } 0<\delta\le\delta_0,$$

where the constants $M > 0$ and $r \ge 1$ do not depend on δ. Indeed, by setting $\delta = 1/d^r$ in (2.1) we get

THEOREM 2.1. *With the above assumptions on f, if E is an HCP compact subset of $\mathbb{C}^k$ with parameter r, then there exists a constant $C_1 > 0$ such that for any polynomial $Q \in \mathbb{C}[z_1,\ldots,z_n]$ of degree d one has*

$$|D_{\mathcal{T}(t,v)}Q(z)| \le C_1 d^r \|Q\|_{f(E)},$$

where $z = f(t)$ with $t \in E$.

If $z = f(t)$ is a regular point of $\mathbb{M}$ then for every $v \in S^{k-1}$ the vector $\mathcal{T}(t,v)$ is a vector of $T_z\mathbb{M}$, the tangent space of $\mathbb{M}$ at z. Hence by Theorem 2.1 we get

COROLLARY 2.2. *Assume that $k \le n$, E is an HCP compact subset of $\mathbb{C}^k$ with parameter r, $N = f(E) \subset \mathbb{M}_{\text{reg}}$, and for each $t \in E$, $\operatorname{rank}_t f = k$. Then there exists a positive constant C_2 such that for any polynomial $Q \in \mathbb{C}[z_1,\ldots,z_n]$ of degree d we have*

$$|D_{\mathcal{T}_z}Q(z)| \le C_2 d^r \|Q\|_N \quad \text{for } z \in N,$$

where $\mathcal{T}_z \in \{\mathcal{T}(t,v)/\|\mathcal{T}(t,v)\| : v \in S^{n-1},\ f(t) = z\}$ is any unit vector of the tangent space $T_z\mathbb{M}$.

REMARK. In the special case where $k = 1$ and $E = [0,1]$ (then $r = 2$), Corollary 2.2 yields Proposition 6.1 of [BLMT1].

An important family of sets that are HCP is the family of UPC sets. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Following [PaPl], let us recall that E is *uniformly polynomially cuspidal* (briefly, UPC) with parameters $M > 0$, $m \ge 1$ and $d \in \mathbb{Z}_+$ if there exists a mapping $\phi : E \times [0,1] \to E$ such that for every $t \in E$, $\phi(t,\cdot)$ is a polynomial map from $\mathbb{R}$ to $\mathbb{K}^k$ of degree d, $\phi(t,1) = t$ and

$$\operatorname{dist}(\phi(t,s), \mathbb{K}^k \setminus E) \ge M(1-s)^m \quad \text{for } (t,s) \in E \times [0,1].$$

The family of UPC sets is large enough. For example, if E is a compact subanalytic subset of $\mathbb{R}^p$ with int E dense in E then by Hironaka's Rectilinearization Theorem E is UPC (see [PaPl, Corollary 6.6]). Moreover, by

[PaPl, Theorem 4.1] for any compact UPC subset E of $\mathbb{K}^k$ with parameters M, m and d,

$$\Phi_E(t) \le 1 + C\delta^{1/(2[m])} \quad \text{if } \operatorname{dist}(x,E) \le \delta \le 1,$$

where C is a positive constant that depends only on M, m and d, and $[m] := l$ if $l-1 < m \le l$ with $l \in \mathbb{Z}$. Hence in particular, every UPC compact subset of $\mathbb{K}^k$ admits Markov's inequality of Theorem 2.1 with exponent $2[m]$.

REMARK. A more subtle technique (due to Baran), based on properties of the Joukowski function $g(z) = \frac{1}{2}(z + 1/z)$, permits one to show that (in the case of E being UPC with parameter m) the exponent $2[m]$ of the Markov inequality can be replaced by $2m$ (see [Ba]).

In what follows, we shall assume that $\mathbb{K} = \mathbb{R}$. Thus f is an analytic map with values in an algebraic set $\mathcal{M} \subset \mathbb{R}^n$, defined in an open neighbourhood U (in $\mathbb{R}^k$) of a UPC compact subset E of $\mathbb{R}^k$ with parameter m. We let S^{k-1} denote the unit sphere in $\mathbb{R}^k$. To prove further corollaries to Theorem 2.1 we shall need the following two lemmas.

LEMMA 2.3. *Let $\mathbf{A} = [a_{ij}]$ be a symmetric, positive semi-definite matrix of dimension k. Let $Q_{\mathbf{A}}(v) = \sum_{i,j=1}^k a_{ij}v_iv_j$ be the quadratic form associated with $\mathbf{A}$. Then for all $v \in S^{k-1}$,*

$$Q_{\mathbf{A}}(v) \ge (1 + \operatorname{tr}\mathbf{A})^{1-k}\det\mathbf{A}.$$

Proof. Let $0 \le \lambda_1 \le \ldots \le \lambda_k$ be the eigenvalues of the matrix $\mathbf{A}$. Then it is well known that

$$\lambda_1 = \min_{v\in S^{k-1}} Q_{\mathbf{A}}(v).$$

Hence for any $v \in S^{k-1}$ we have

$$Q_{\mathbf{A}}(v) \ge \lambda_1 \ge \lambda_1 \frac{\lambda_2}{1+\lambda_2}\cdots\frac{\lambda_k}{1+\lambda_k} \ge (1+\operatorname{tr}\mathbf{A})^{1-k}\det\mathbf{A}.$$

LEMMA 2.4. *Let $Z = \{t \in U : \operatorname{rank}_t f < k\}$. Then there exist constants $A > 0$ and $\alpha \ge 0$ such that for any $t \in E$ and $v \in S^{k-1}$,*

$$\|D_v f(t)\| \ge A\,(\mathit{dist}(t,Z))^\alpha.$$

Proof. Let $\mathbf{A}_f(t)$ be the (n,k)-matrix of d_tf (in the canonical basis). Then $\mathbf{B}_f(t) = \mathbf{A}_f^*(t)\mathbf{A}_f(t)$ is a quadratic matrix of dimension k. It is known that if $\mathbf{A}$ is an (n,k)-matrix with real entries then $\operatorname{rank}\mathbf{A} = \operatorname{rank}(\mathbf{A}^*\mathbf{A})$. Hence

$$Z = \{t \in U : \det\mathbf{B}_f(t) = 0\}.$$

By Lemma 2.3,

$$\|D_vf(t)\|^2 = Q_{\mathbf{B}_f(t)}(v) \ge B_1\det\mathbf{B}_f(t),$$

where $B_1 = (1+\sup_{t\in E}\operatorname{tr}\mathbf{B}_f(t))^{1-k}$. By Łojasiewicz's Inequality there exist constants $B_2 > 0$ and $\beta \ge 0$ such that for any $t \in E$,

$$\det\mathbf{B}_f(t) \ge B_2(\operatorname{dist}(t,Z))^\beta.$$

This completes the proof of the lemma.

COROLLARY 2.5. *There exist constants $C_2 > 0$ and $\alpha \ge 0$ such that for any polynomial $Q \in \mathbb{R}[x_1,\ldots,x_n]$ of degree d and $x \in f(E\setminus Z)$,*

$$|D_{\mathcal{T}_x}Q(x)| \le C_2d^{2m}(\operatorname{dist}(f^{-1}(x)\cap E, Z))^{-\alpha}\|Q\|_{f(E)}.$$

In particular, if $Z \cap \operatorname{int} E = \emptyset$ then we get

$$|D_{\mathcal{T}_{f(t)}}Q(f(t))| \le C_2d^{2m}(\operatorname{dist}(t,\partial E))^{-\alpha}\|Q\|_{f(E)}.$$

REMARK. If $Z \cap \partial E \ne \emptyset$, one cannot expect a Markov inequality of Corollary 2.5 with exponent $2m$ (see [BaPl1, Example 2.9]).

We end this paper by considering the following special case.

PROPOSITION 2.6. *Let f be a polynomial map from $\mathbb{R}$ to $\mathbb{R}^n$ with*

$$f'(t) = (1-t)^{s_1}(1+t)^{s_2}Q(t),$$

where Q is a polynomial map from $\mathbb{R}$ to $\mathbb{R}^n$, $Q(t) \ne 0$ on $[-1,1]$. Let $\alpha = \max(s_1,s_2)$. Then there exists a constant $A > 0$ such that for any polynomial $P \in \mathbb{R}[x_1,\ldots,x_n]$ of degree d we have

$$|D_{\mathcal{T}_x}P(x)| \le Ad^{2+2\alpha}\|P\|_N \quad \textit{for } x \in N = f([-1,1]),$$

with $D_{\mathcal{T}_x}P(x) = D_{Q(t)/\|Q(t)\|}P(x)$, $x = f(t)$.

Proof. By Theorem 2.1 there exists a constant $C_1 > 0$ such that

$$|D_{f'(t)}P(f(t))| \le C_1d^2\|P\|_N.$$

Hence we get

$$|D_{Q(t)}P(f(t))| \le C_1d^2(1-t^2)^{-\alpha}\|P\|_N \quad \text{for } t \in (-1,1),$$

whence by a generalized Schur Inequality (see [Ba, Lemma 2.4]) we obtain

$$|D_{Q(t)}P(f(t))| \le C_1(d\deg f)^{2\alpha}d^2\|P\|_N.$$

Therefore we get the required inequality with

$$A = C_1(\deg f)^{2\alpha}\big(\min_{t\in[-1,1]}|Q(t)|\big)^{-1}.$$

EXAMPLE 2.7. Let m and l, where $m > l \ge 2$, be two relatively prime natural numbers. Let

$$f(t) = \left(\left(\frac{1+t}{2}\right)^l, \left(\frac{1+t}{2}\right)^m\right).$$

Then $N = f([-1,1]) = \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1 \text{ and } x^m = y^l\}$. Since in this case $\alpha = l - 1$, by Proposition 2.6 we get

$$|D_{T_{(x,y)}} P(x,y)| \le A d^{2l} \|P\|_N$$

for any polynomial $P \in \mathbb{R}[x,y]$ of degree d.

REMARK. A result related to Example 2.7 has recently been announced in [BLMT2]. (See also [Gen].)

REMARK. Let $f : [-1,1] \to \mathbb{R}^n$ be a continuous spline function (i.e. $f : [-1,1] \to \mathbb{R}^n$ is continuous and there are points $-1 = s_0 < s_1 < \ldots < s_m = 1$ such that $f_{|[s_i,s_{i+1}]}$ is the restriction of a polynomial map with $f'(t) \ne 0$ for $t \in (s_i, s_{i+1})$, $i = 0, \ldots, m-1$. Then by Proposition 2.6 there exist constants $A > 0$ and $\beta_i > 0$ such that for each $i = 0, \ldots, m-1$, and for each polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d,

$$|D_{T_x} P(x)| \le A d^{\beta_i} \|P\|_{f([-1,1])}$$

for $x \in f((s_i, s_{i+1}))$, $i = 0, \ldots, m-1$, where $D_{T_x} P$ is a tangential derivative of P. Moreover, if f is an arc of class C^1 then the tangential Markov inequality holds for any $x \in f([-1,1])$.

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Received June 24, 1998
Revised version February 29, 2000 (4132)