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Operators with an ergodic power

by

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Abstract. We prove that if some power of an operator is ergodic, then the operator itself is ergodic. The converse is not true.

1. Introduction. Ergodic theory is concerned with the existence of the limit of the Cesàro means

$$M_n(T) := \frac{I + T + \ldots + T^{n-1}}{n}, \quad n \in \mathbb{N},$$

where T is a (bounded linear) operator on a Banach space. The operator T is said to be *uniformly*, *strongly* or *weakly ergodic* if the sequence $(M_n(T))$ is convergent in the uniform, strong or weak operator topology in the space $L(X)$ of all operators in X.

Dunford [7, Theorem 3.16] proved that, if 1 is a pole of the resolvent operator, then

$$M_n(T) \text{ converges in norm} \Leftrightarrow \lim_{n\to\infty} \frac{T^n}{n} = 0.$$

Wacker [14, Satz 4] extended the result of Dunford, proving that if 1 is a pole of the resolvent operator of order less than or equal to p, then

$$\frac{1}{n^p}\sum_{k=0}^{n-1} T^k \text{ converges in norm} \Leftrightarrow \lim_{n\to\infty} \frac{T^n}{n^p} = 0.$$

Note that the direct implications are trivial. Other generalizations and local versions of these results are given in [4, 6, 9, 10].

We say that an operator $T \in L(X)$ is *ergodic* if for every $x \in X$,

$$\lim_{n\to\infty} \frac{T^n x}{n} = 0 \Rightarrow M_n(T)x \text{ converges.} \tag{1}$$

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This concept is weaker than that of strongly ergodic, because the subspace $\{x \in X : \lim_{n\to\infty} T^n x/n = 0\}$ does not necessarily coincide with X or is closed. For example, take $T := 2B$, where B is the unilateral backward shift on $\ell^2(\mathbb{N})$. Then defining $c_{00} := \{x = (x(k)) \in \ell^2(\mathbb{N}) : \exists m,\ \forall k > m,\ x(k) = 0\}$, we have

$$c_{00} \subset \left\{x \in \ell^2(\mathbb{N}) : \lim_{n\to\infty} \frac{T^n x}{n} = 0\right\}.$$

Moreover, the set c_{00} is dense on $\ell^2(\mathbb{N})$, but $T^n y/n$ does not converge to 0 for $y := (1, 1/2, 1/3, \ldots)$.

In this paper, we study ergodic operators in the sense of the previous definition by analyzing the relations between some sets of vectors that satisfy the right side or the left side of (1).

Our main result is Theorem 3, where we prove that if some power of an operator T is ergodic, then T is ergodic. Our proof also works for the concepts of uniformly, strongly or weakly ergodic operator.

The converse of Theorem 3 is not true. We give an example of an operator T such that T^k is ergodic if and only if k is an odd integer.

We also give conditions on T and x so that the implication (1) holds, i.e., localizations of some results of ergodic theory. These conditions are related with the local spectral theory. Finally, for T a Riesz operator, we characterize the vectors $x \in X$ such that $(M_n(T)x)$ converges.

Some of these results were announced in [2].

2. Preliminaries and notation. Along the paper, X denotes a complex Banach space and $L(X)$ the Banach algebra of all bounded linear operators defined on X. If $T \in L(X)$, we denote the *kernel* and the *range* of T by $N(T)$ and $R(T)$, respectively. Moreover, a complex number λ belongs to the *resolvent set* $\varrho(T)$ of T if there exists $(\lambda - T)^{-1} \in L(X)$. We denote by $\sigma(T) := \mathbb{C} \setminus \varrho(T)$ the *spectrum* of T.

The *resolvent map* $\lambda \in \varrho(T) \mapsto (\lambda - T)^{-1} \in L(X)$ is analytic. Therefore, $w(\mu) = R(\mu, T)x$ for every $\mu \in \varrho(T)$ is an analytic solution of the equation

$$(\mu - T)w(\mu) = x. \tag{2}$$

The function $w(\mu)$ could have analytic extensions for certain $x \in X$. So, we say that a complex number λ belongs to the *local resolvent set* of T at x, denoted by $\varrho(x, T)$, if there exists an analytic function $w : U \to X$, defined on a neighbourhood U of λ, which satisfies (2) for every $\mu \in U$. The *local spectrum* of T at x is the complement $\sigma(x, T) := \mathbb{C} \setminus \varrho(x, T)$.

Since w is not necessarily unique, a complementary property is needed to prevent ambiguity. An operator $T \in L(X)$ has the *single-valued extension property* (hereafter referred to as the SVEP) if $(\lambda - T)h(\lambda) = 0$ has only trivial analytic solutions on any open subset of the plane. For example, every operator with empty interior of the point spectrum, $\operatorname{int}(\sigma_p(T)) = \emptyset$, has the SVEP.

If T satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_T$ on $\varrho(x, T)$ satisfying (2), which is called the *local resolvent function* of T at x.

For every subset $H \subset \mathbb{C}$, we define

$$X(T, H) := \{x \in X : \sigma(x, T) \subset H\}.$$

If $X(T, F)$ is closed for all closed sets F, we say that T has *property* (C). Radjabalipour [11, Theorem 2.3] proved that property (C) implies the SVEP.

The following characterization of the poles of the local resolvent function will be useful.

THEOREM 1 [1, Theorem 3.3]. *Assume that $T \in L(X)$ has the SVEP and let $x \in X$. Then α is a pole of $\widehat{x}_T$ of order n if and only if there exists a unique decomposition $x = y + z$ such that $y \in N((\alpha - T)^n) \setminus N((\alpha - T)^{n-1})$, and $\sigma(z, T) = \sigma(x, T) \setminus \{\alpha\}$.*

For $T \in L(X)$ we consider the following subsets of X:

$$\mathcal{E}_T := \left\{x \in X : \frac{T^n x}{n} \text{ converges}\right\},$$
$$E_T := \left\{x \in X : \lim_{n\to\infty} \frac{T^n x}{n} = 0\right\},$$
$$M_T := \{x \in X : M_n(T)x \text{ converges}\},$$
$$\mathcal{M}_T := \{x \in X : \lim_{n\to\infty} M_n(T)x = 0\}.$$

Clearly, these four sets are (not necessarily closed) subspaces of X which are invariant under any operator commuting with T. Moreover, $M_T \subset E_T$. Hence,

$$\mathcal{M}_T \subset M_T \subset E_T \subset \mathcal{E}_T.$$

DEFINITION 1. An operator $T \in L(X)$ is said to be *ergodic* if $E_T = M_T$.

REMARK 1. Note that $T \in L(X)$ is strongly ergodic if and only if T is ergodic and $E_T = X$.

The following result is well known as the *mean ergodic theorem* and is useful in finding examples of non-ergodic operators.

THEOREM 2 [8, Theorem 2.1.3]. *Let $T \in L(X)$ be an operator such that $E_T = X$ and $\sup_n \|M_n(T)\| < \infty$. Then $M_T = N(I - T) \oplus \overline{R(I - T)}$.*

REMARK 2. It follows from Theorem 2 that if T is *power bounded* (i.e., if there exists a constant $M > 0$ such that $\|T^n\| \le M$ for all $n \in \mathbb{N}$) and 1 is a pole of $(\lambda - T)^{-1}$ of order 1, then $E_T = M_T = X$; i.e., T is ergodic (and strongly ergodic).

See Corollary 1 below for related results.

3. Properties of ergodic operators. Now, we prove some basic results for the subsets $\mathcal{M}_T$, M_T, E_T, and $\mathcal{E}_T$ defined in Section 2 which will be useful later.

PROPOSITION 1. *Let $T \in L(X)$ and $n \in \mathbb{N}$. Then:*

(i) $(I-T)^{-1}M_T = \mathcal{E}_T$ *and* $(I-T)^{-1}\mathcal{M}_T = E_T$.

(ii) $M_T = \mathcal{M}_T \oplus N(I-T)$ *and* $\mathcal{E}_T = E_T + N((I-T)^2)$.

(iii) $\mathcal{E}_T \cap N((I-T)^{n+1}) = N(I-T)^2$ *and* $E_T \cap N((I-T)^{n+1}) = N(I-T)$.

Proof. Property (i) follows easily from

$$M_n(T)(I-T)x = (I-T^n)x/n. \tag{3}$$

For the first part of (ii), we fix $x \in M_T$. Since $M_T \subset E_T$, formula (3) implies that $z := \lim_{n\to\infty} M_n(T)x \in N(I-T)$. Thus we write $x = (x-z)+z$, where $x - z \in \mathcal{M}_T$ because $M_n(T)z = z$ for every n.

For the second part of (ii) we apply part (i), the first part of (ii) and $N(I-T)^2 = (I-T)^{-1}N(I-T)$ to get

$$\begin{aligned}\mathcal{E}_T &= (I-T)^{-1}M_T = (I-T)^{-1}(\mathcal{M}_T \oplus N(I-T))\\ &= (I-T)^{-1}\mathcal{M}_T + (I-T)^{-1}N(I-T) = E_T + N((I-T)^2).\end{aligned}$$

The case $n = 1$ of (iii) is proved by observing that $(I-T)\mathcal{E}_T \subset E_T$ and applying (i) and (ii).

For the case $n > 1$, assume that $x \in N((I-T)^{n+1}) \cap \mathcal{E}_T$. Then $(I-T)^{n-1}x \in E_T \cap N((I-T)^2) = N(I-T)$; hence $x \in \mathcal{E}_T \cap N((I-T)^n)$. Repeating this process $n-1$ times, we conclude $x \in \mathcal{E}_T \cap N((I-T)^2) = N((I-T)^2)$.

Moreover, $E_T \cap N((I-T)^{n+1}) = E_T \cap N((I-T)^2) = N(I-T)$. ■

COROLLARY 1. *Let $T \in L(X)$.*

(i) *If $1 \in \varrho(T)$, then $\mathcal{E}_T = E_T = (I-T)E_T = \mathcal{M}_T = M_T$. In particular, T is ergodic.*

(ii) *If 1 is a pole of $(\lambda - T)^{-1}$ of order $p > 1$, then $E_T \neq X$.*

Proof. The first part is an immediate consequence of the previous proposition.

For the second part, note that $p > 1$ implies $N(I-T) \neq N((I-T)^2)$ [13, Theorem V.6.2]. ■

Although $x \in M_T$ whenever $(T^n x)$ is convergent (convergence implies Cesàro convergence), the following example shows that the converse implication fails.

EXAMPLE 1. Let $T \in L(\ell^2(\mathbb{N}))$ be the unilateral weighted shift defined by

$$Te_n := \sqrt{\frac{n+1}{n}}\, e_{n+1}.$$

Taking $x := (I-T)e_1$, we deduce that $x \in M_T$ (since $e_1 \in E_T$) and $T^n x$ does not converge (since $T^n x = \sqrt{n+1}\, e_{n+1} - \sqrt{n+2}\, e_{n+2}$). ■

The following result will be useful to describe operators with an ergodic power. A related property was studied in [3, Proposition 2.4].

THEOREM 3. *Let $T \in L(X)$. Then $E_{T^k} = E_T$ and $M_{T^k} \subset M_T$ for every $k \in \mathbb{N}$.*

Proof. Clearly, $E_T \subset E_{T^k}$. Moreover, if $x \in E_{T^k}$, since $T^{nk}x/n \to 0$ as $n \to \infty$ the sequences

$$\left(\frac{T^{nk}x}{nk}\right), \left(\frac{T^{nk+1}x}{nk+1}\right), \ldots, \left(\frac{T^{nk+k-1}x}{nk+k-1}\right)$$

converge to zero. Hence $T^n x/n \to 0$ as $n \to \infty$, that is, $x \in E_T$.

On the other hand, if $x \in M_{T^k}$, the convergence of $M_n(T^k)x$ implies the convergence of

$$TM_n(T^k)x, \ldots, T^{k-1}M_n(T^k)x.$$

Consequently, $M_{nk}(T)x$ converges to y, hence

$$\frac{T^{nk+j}x}{nk+j} \to 0 \quad \text{as } n \to \infty$$

for $0 \le j \le k-1$. Taking into account the equality

$$\frac{T^{nk+j}}{nk+j} = \frac{nk+j+1}{nk+j} M_{nk+j+1}(T) - M_{nk+j}(T),$$

for $0 \le j \le k-1$, we find that $M_{nk+j}(T)x$ converges to the same limit y for $0 \le j \le k-1$. Hence $M_n(T)x$ converges, that is, $x \in M_T$. ■

A particular case of Theorem 3 can be found in [8, page 84]: If $T \in L(X)$ is a contraction and $M_n(T^2)x$ converges for all $x \in X$, then $M_n(T)x$ also converges for all $x \in X$.

The next result is a generalization of [12, Corollary].

COROLLARY 2. *Let $T \in L(X)$ and $k = 1, 2, \ldots$ If T^k is ergodic, then T is ergodic. Moreover, if T^k is uniformly, strongly or weakly ergodic, then T has the same property.*

The following example shows that the inclusion in Theorem 3 is not an equality and the converse of Corollary 2 is not valid, in general.

EXAMPLE 2. Let T be the multiplication operator on $C([-1,0])$ defined by $Tf(t) = tf(t)$ for $f \in C([-1,0])$. Then $\sigma(T) = [-1,0]$ and $\|T\| = 1$. Hence $E_T = E_{T^k} = C([-1,0])$ for every k. Since $1 \in \varrho(T)$, by Theorem 2 we get $M_T = C([-1,0])$, and T is ergodic.

On the other hand,

$$M_{T^2} = N(I-T^2) \oplus \overline{R(I-T^2)}.$$

Note that $N(I-T^2) = \{0\}$ and $R(I+T)$ is contained in the proper closed subspace $\{f \in C([-1,0]) : f(-1) = 0\}$, hence $\overline{R(I-T^2)} \neq C([-1,0])$. So, $M_{T^2} \neq C([-1,0]) = E_{T^2}$ and T^2 is not ergodic. ∎

REMARK 3. (a) Clearly, in the above example T^k is ergodic if and only if -1 is not a root of $z^k = 1$, or equivalently, if k is an odd integer.

In fact, for integers $1 < l < k$ such that l does not divide k, it is possible to give a multiplication operator similar to that in Example 2 such that T^k is ergodic, but T^l is not.

(b) In the case $E_T = X$, T is ergodic if and only if T is strongly ergodic. So the previous counterexample is also valid for strongly ergodic operators.

Next we prove a local ergodic result that gives conditions on a fixed vector $x \in X$ so that $x \in E_T$ implies $x \in M_T$.

PROPOSITION 2. *Suppose that $T \in L(X)$ has property* (C) *or* 1 *is an isolated point of $\sigma(T)$. Let $x \in E_T$. If $1 \in \varrho(x,T)$ or* 1 *is a pole of $\widehat{x}_T$ of order* 1, *then $x \in M_T$.*

Proof. As a consequence of Theorem 1, we can write $x = y + z$, where $y \in N(I-T)$ and $\sigma(z,T) = \sigma(x,T) \setminus \{1\}$. By part (ii) of Proposition 1, we have $y \in M_T$, hence $z \in E_T$.

Suppose that T has property (C) and define $F := \sigma(z,T)$. Using [5, Theorem I.3.8] we have $\sigma(T|X(T,F)) \subset F \cap \sigma(T)$, hence $1 \in \varrho(T|X(T,F))$. On the other hand, as $z \in X(T,F) \cap E_T$, by Corollary 1 we have $z \in M_T$. Hence $x \in M_T$.

Suppose that 1 is an isolated point of $\sigma(T)$. Then by [13, Theorem V.9.1] we obtain $X = Y \oplus Z$, where $\sigma(T|Y) = \{1\}$ and $\sigma(T|Z) = \sigma(T) \setminus \{1\}$. Hence $z \in Z \cap E_T$ and by Corollary 1 we have $z \in M_T$. Thus $x \in M_T$. ∎

COROLLARY 3. *Let $T \in L(X)$. If* 1 *is a pole of $(\lambda - T)^{-1}$, then T is ergodic.*

Proof. Suppose that $x \in E_T$. By [1, Theorem 4.2], either $1 \in \varrho(x,T)$ or the local resolvent function of T at x has a pole of order p at 1.

If $1 \in \varrho(x,T)$ or $p = 1$, then by the proof of Proposition 2 we see that $x \in M_T$.

Assume that $p > 1$. By Theorem 1, we can write $x = y + z$ where $y \in N((I-T)^p) \setminus N((I-T)^{p-1})$ and $\sigma(z,T) = \sigma(x,T) \setminus \{1\}$. Using part (iii) of Proposition 1 we get $y \notin E_T$; hence $z \notin E_T$. Now, as in the proof of Proposition 2, we have $X = Y \oplus Z$ with $z \in Z$ and $1 \in \varrho(T|Z)$. Then $(I-T)^p x = (I-T)^p z \in E_T$ and by Corollary 1 we obtain $z \in E_T$, which is a contradiction. ∎

Finally, we give several characterizations of the vectors in M_T for T a Riesz operator, similar to [10, Théorème 3] in the uniform case. Recall that $T \in L(X)$ is *Riesz* if every non-zero complex number is a pole of $(zI-T)^{-1}$ with finite multiplicity. We denote by $\mathbb{D}$ the open unit disc in the set of all complex numbers, and by Γ the boundary of $\mathbb{D}$.

THEOREM 4. *For a Riesz operator $T \in L(X)$ and $x \in X$, the following assertions are equivalent.*

(i) *$(T^n x)$ is bounded.*

(ii) *$x \in E_T$.*

(iii) *$x \in M_T$.*

(iv) *$\sigma(x,T) \subset \overline{\mathbb{D}}$ and $\sigma(x,T) \cap \Gamma$ consists of poles of $\widehat{x}_T$ of order* 1.

Proof. The implication (i)⇒(ii) is obvious and (ii)⇒(iii) follows from Corollary 3, since 1 is a pole of the resolvent $(\lambda - T)^{-1}$. Let us prove that (iii)⇒(iv). If $x \in M_T$ then $x \in E_T$, hence $\sigma(x,T) \subset \overline{\mathbb{D}}$. So, it is enough to prove that the poles of $\widehat{x}_T$ on the unit circle are of order 1. Since T is a Riesz operator, we can write $x = x_1 + \ldots + x_k + y$, where $x_i \in N(\lambda_i - T)^{n_i}$ for some n_i, $i = 1, \ldots, k$ and $\sigma(y,T) \subset \mathbb{D}$. Moreover, since each of the vectors $x_1, \ldots, x_k, y$ is the image of x under a projection commuting with T and $x \in M_T$, we have $x_1, \ldots, x_k, y \in M_T \subset E_T$. Note that $E_T = E_{\lambda T}$ when $|\lambda| = 1$. Thus, by part (iii) of Proposition 1, we obtain $x_i \in N(\lambda_i - T)$ for $i = 1, \ldots, k$. Hence, the poles of $\widehat{x}_T$ on the unit circle are of order 1.

For the implication (iv)⇒(i), assume that $\sigma(x,T) \subset \overline{\mathbb{D}}$ and $\sigma(x,T) \cap \Gamma$ consists of poles of $\widehat{x}_T$ of order 1. We write $x = x_1 + \ldots + x_k + y$, where $\sigma(x_i,T) = \{\lambda_i\}$ with $|\lambda_i| = 1$ and $\sigma(y,T) \subset \mathbb{D}$.

We have $x_i \in N(\lambda_i - T)$ by Theorem 1, hence $T^n x_i = \lambda_i^n x_i$. Moreover, $\sigma(y,T) \subset \mathbb{D}$ implies $\limsup_n \|T^n y\|^{1/n} < 1$; hence $\lim_n T^n y = 0$. Thus $(T^n x)$ is bounded. ∎

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Polynomial inequalities on algebraic sets

by

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Abstract. We give an estimate of Siciak's extremal function for compact subsets of algebraic varieties in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). As an application we obtain Bernstein–Walsh and tangential Markov type inequalities for (the traces of) polynomials on algebraic sets.

0. Preliminaries. The theory of the multivariate Markov inequality furnishing estimates of the derivatives of a polynomial in n variables in terms of its degree and its uniform norm on an n-dimensional compact subset of $\mathbb{C}^n$ or $\mathbb{R}^n$ was essentially developed in the last ten years. For an exhaustive survey on this subject we refer the reader to [Pl3]. In recent years, Markov and Bernstein type inequalities have been intensively investigated on algebraic subvarieties of $\mathbb{R}^n$ (see [BLT], [BLMT1], [BLMT2], [FeNa1], [FeNa2], [FeNa3], [Bru], [BaPl2], [BaPl3], [RoYo], [Gen]). In particular, in [BLT], [BLMT1], [BaPl2] and [BaPl3] the authors have characterized semialgebraic curves as well as semialgebraic manifolds in $\mathbb{R}^n$ in terms of tangential Markov or Bernstein and van der Corput–Schaake type inequalities.

The purpose of this paper is to establish Bernstein–Walsh or (tangential) Markov type inequalities on subsets N of an algebraic set in $\mathbb{R}^n$ that are images under non-degenerate analytic maps of non-pluripolar, compact sets in $\mathbb{R}^k$. Our results yield, as particular cases, some recent results of Bos–Levenberg–Milman–Taylor [BLMT1], [BLMT2] and Brudnyi [Bru].

Let us note that if N is a subset of an analytic variety then, in general, it need not admit a tangential Markov inequality with any finite exponent. A relevant example is due to Izumi [Iz] (see [BLMT1]).

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