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Stable inverse-limit sequences and automatic continuity

by

GRAHAM R. ALLAN (Cambridge)

Abstract. The elementary theory of stable inverse-limit sequences, introduced in [2], is used to extend the "stability lemma" of automatic continuity theory.

1. Introduction. In [2] we introduced a notion of *stable inverse-limit sequence* and gave a number of applications. In the present short paper we apply the idea of stability to extend a classical lemma in automatic continuity theory ([7], Lemma 1.6; coincidentally, this lemma is generally known as the "stability lemma"). The main tool is a new application of the familiar "sliding-hump" technique, to give a simple, general density lemma for an arbitrary stable sequence of subspaces of a vector space (Theorem 2.5).

We also prove and make use of a sufficient condition for stability (Theorem 3.1) that is derived by adapting an ingenious argument of Esterle [6].

A fuller account of the theory and applications of stable inverse-limit sequences is given in [4]. This contains substantial improvements of results in [2], [3], including characterizations of stable sequences in important contexts.

We briefly recall a few ideas from [2], §2. **ILG** is the category of IL-sequences of groups and homomorphisms; L is the inverse-limit functor on **ILG**. In the present paper we consider only *abelian* groups and use additive notation.

A sequence $\mathcal{G}$ of abelian groups and homomorphisms, say

$$G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \ldots \longleftarrow G_n \xleftarrow{g_n} G_{n+1} \longleftarrow \ldots,$$

is said to be *stable* if and only if, for every choice of $\gamma_n \in G_n$ $(n \geq 1)$, we may simultaneously solve the equations $x_n = g_n(x_{n+1}) + \gamma_n$, with $x_n \in G_n$ $(n \geq 1)$. Many examples and applications of stable sequences are given in [2] and [3]. We shall refer to [2] for simple properties of stable sequences.

As in [2], a *Mittag-Leffler sequence* is an IL-sequence $(G_n; g_n)$ of complete metrizable topological groups and continuous homomorphisms such that, for each n, $g_n(G_{n+1})$ is dense in G_n.

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THEOREM 1.1. *Every Mittag-Leffler sequence is stable.*

Proof. See [2], Theorem 4.

2. Stable sequences of subgroups

DEFINITION 2.1. Let G be a group (which is taken to be abelian). By a *stable sequence of subgroups of* G is meant a decreasing sequence,

$$G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq G_{n+1} \supseteq \ldots,$$

of subgroups which is stable as an IL-sequence,

$$G_1 \xleftarrow{j_1} G_2 \xleftarrow{j_2} \ldots \longleftarrow G_n \xleftarrow{j_n} G_{n+1} \longleftarrow \ldots,$$

in which each j_n is an inclusion mapping.

Thus, the decreasing sequence (G_n) is stable if and only if for every choice of $\gamma_n \in G_n$ $(n \geq 1)$, there is a simultaneous solution to the equations $x_n = x_{n+1} + \gamma_n$ with $x_n \in G_n$ $(n \geq 1)$.

EXAMPLES 2.2. (i) If E is any Fréchet space, let $(p_n)_{n\geq 1}$ be an increasing sequence of seminorms that define the topology of E. Then $(\ker p_n)_{n\geq 1}$ is a stable sequence of subspaces of E. (This is essentially the example given in [3], following Corollary 1.5 on page 69. The example there was of a Fréchet *algebra*, but it is quite clear that the same proof works for a Fréchet space.)

(ii) Let V be a vector space. Then V has the property that *every* decreasing sequence of subspaces is stable if and only if $\dim V < \infty$.

Of course, if $\dim V < \infty$, then every decreasing sequence of subspaces is eventually constant, so is certainly stable. Conversely, suppose that V is infinite-dimensional, and let $(e_n)_{n\geq 1}$ be an infinite, linearly independent sequence in V. Let $E_n = \operatorname{span}\{e_k : k \geq n\}$. In particular, $\{e_n : n \geq 1\}$ is a basis of E_1; let (e_n^*) in E_1^* be defined by $e_n^*(e_m) = \delta_{m,n}$ (for all m, n). Note that, if $x \in E_1$, then $e_n^*(x) = 0$ for all sufficiently large n.

Suppose that the decreasing sequence (E_n) is stable. Then, in particular, there exist $x_n \in E_n$ such that $x_n - x_{n+1} = e_n$ $(n \geq 1)$. Hence, for every $n \geq 1$, $x_1 - (e_1 + \ldots + e_n) \in E_{n+1}$, so that $\langle x_1, e_n^* \rangle = 1$ for all n, which is impossible. Hence the sequence (E_n) is not stable.

(iii) The idea of (ii) may be simply extended to show the following: if E is a normed space in which every decreasing sequence of *closed* subspaces is stable, then $\dim E < \infty$. The proof is left as a simple exercise.

(iv) Let $F = \mathbb{C}[[X]]$; then every decreasing sequence of ideals in F is stable.

In fact, the non-zero ideals of F are just $I_n = X^n F$ $(n \geq 1)$. Given any decreasing, infinite sequence of ideals in F, we may (by passing to a subsequence) suppose that the sequence is $I_{n(k)}$, where $(n(k))_{k\geq 1}$ is a strictly increasing sequence of positive integers. Then, for $f_k(X) \equiv X^{n(k)} g_k(X) \in I_{n(k)}$ $(k \geq 1)$, the formula $h_k(X) = \sum_{j=k}^{\infty} X^{n(j)} g_j(X)$ gives a well-defined element of $I_{n(k)}$. Clearly, $h_k(X) = h_{k+1}(X) + f_k(X)$ for all k, so the stability of the sequence $I_{n(k)}$ is proved.

Many other examples follow from the following theorem.

THEOREM 2.3. *Let $\mathcal{G} = (G_n\,; g_n)$ be a stable sequence in* **ILG**. *Then for every $m \geq 1$ the sequence*

$$\operatorname{im} g_m \supseteq \operatorname{im} g_m g_{m+1} \supseteq \operatorname{im} g_m g_{m+1} g_{m+2} \supseteq \ldots$$

is a stable sequence of subgroups of G_m.

Proof. Let the sequence

$$\mathcal{G}: \quad G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \ldots \longleftarrow G_n \xleftarrow{g_n} G_{n+1} \longleftarrow \ldots,$$

be a stable sequence in **ILG**. Define $h_n = g_1 \ldots g_n$ $(n \geq 1)$, and consider the diagram

$$\begin{array}{ccccccccccc}
G_1 & \xleftarrow{g_1} & G_2 & \xleftarrow{g_2} & \ldots & \longleftarrow & G_n & \xleftarrow{g_n} & G_{n+1} & \longleftarrow & \ldots \\
{\scriptstyle I}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & & & {\scriptstyle h_{n-1}}\big\downarrow & & {\scriptstyle h_n}\big\downarrow & & \\
G_1 & \xleftarrow{j_0} & \operatorname{im} h_1 & \xleftarrow{j_1} & \ldots & \longleftarrow & \operatorname{im} h_{n-1} & \xleftarrow{j_{n-1}} & \operatorname{im} h_n & \longleftarrow & \ldots
\end{array}$$

where I is the identity automorphism on G_1 and each j_n is an inclusion mapping. It is a triviality to see that the diagram commutes; it thus displays a surjective morphism in **ILG**. Hence, by [2], Proposition 1(i), the stability of the first sequence implies that of the second.

Finally, by a special case of [2], Lemma 3, for each $m \geq 1$, the truncated sequence $G_m \xleftarrow{g_m} G_{m+1} \xleftarrow{g_{m+1}} \ldots$ is stable. So, by the case already considered, the sequence of subgroups $\operatorname{im} g_m \supseteq \operatorname{im} g_m g_{m+1} \supseteq \operatorname{im} g_m g_{m+1} g_{m+2} \supseteq \ldots$ is stable.

DEFINITION 2.4. Let G be a group, and let τ be a topology on G. Then a decreasing sequence,

$$G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq G_{n+1} \supseteq \ldots,$$

of subgroups of G is said to have *finite closed descent* (or *FCD*) with respect to τ if and only if there is some $m \geq 1$ such that G_n is τ-dense in G_m for all $n \geq m$.

(This extends a notion of an "element of FCD" in a topological algebra A; an element $x \in A$ has (left-) FCD if and only if the sequence of (left) ideals $(x^n A)_{n\geq 1}$ has FCD in the sense just defined; see [1] and [3]. An improved version is given in [4].)

THEOREM 2.5. *Let E be a vector space (over $\mathbb{R}$ or $\mathbb{C}$), let $(E_n)_{n\geq 1}$ be a stable sequence of subspaces of E and let p be a seminorm on E. Then (E_n) has FCD with respect to p.*

Proof. Suppose that the result is false. Then, for each $m \geq 1$, there is some $m' > m$ such that $E_{m'}$ is not p-dense in E_m. In view of [2], Lemma 3, we may, by passing to a subsequence, suppose that, for each m, E_{m+1} is not p-dense in E_m.

For each $m \geq 1$ and every $x \in E$, define

$$p_m(x) = \inf\{p(x+y) : y \in E_m\}.$$

Then p_m is a seminorm on E and $p_m(x) \leq p(x)$ for all $x \in E$; in fact, $p_m(x)$ is just the "p-distance" of x from E_m. Hence, for each $m \geq 1$, there is some $x_m \in E_m$ with $p_{m+1}(x_m) > 0$.

For any positive numbers $\lambda_1, \ldots, \lambda_m$ $(m \geq 2)$,

$$\begin{aligned} p_{m+1}(\lambda_1 x_1 + \ldots + \lambda_m x_m) & \\ &\geq \lambda_m p_{m+1}(x_m) - p_{m+1}(\lambda_1 x_1 + \ldots + \lambda_{m-1}x_{m-1}) \\ &\geq \lambda_m p_{m+1}(x_m) - p(\lambda_1 x_1 + \ldots + \lambda_{m-1}x_{m-1}). \end{aligned}$$

It is thus an easy inductive argument to define a sequence $(\lambda_m)_{m\geq 1}$, with $\lambda_m > 0$, so that

$$p_{m+1}(\lambda_1 x_1 + \ldots + \lambda_m x_m) > m$$

for all $m \geq 1$.

By the hypothesis of stability, there is some $y_m \in E_m$ $(m \geq 1)$ such that

$$y_m = y_{m+1} + \lambda_m x_m \quad (m \geq 1).$$

But then, for every m,

$$y_1 = y_2 + \lambda_1 x_1 = y_3 + \lambda_1 x_1 + \lambda_2 x_2 = \ldots = y_{m+1} + (\lambda_1 x_1 + \ldots + \lambda_m x_m).$$

Thus

$$p(y_1) \geq p_{m+1}(y_1) = p_{m+1}(\lambda_1 x_1 + \ldots + \lambda_m x_m) > m$$

for all m, which is a contradiction.

The result is therefore proved.

This last result may be compared with [5], Lemme 2.4, which may be deduced from 2.5.

COROLLARY 2.6. *Let (E_n) be a decreasing sequence of subspaces of a normed space. If (E_n) is stable, then there is some $m \geq 1$ such that $\overline{E}_n = \overline{E}_m$ for all $n \geq m$.*

Proof. The deduction from Theorem 2.5 is clear.

3. A generalized stability lemma. We first adapt an ingenious argument of Esterle (proof of [6], Theorem 6.5) to give a useful condition for stability.

THEOREM 3.1. *Let $0 \to \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\beta} \mathcal{Z} \to 0$ be a short exact sequence in* **ILG**. *Suppose that $\mathcal{Y} = (Y_n\,; g_n)$ is a sequence of complete metrizable, abelian groups and continuous homomorphisms, and that, for every n, $\alpha_n(X_n)$ is dense in Y_n. Then the sequence $\mathcal{Z}$ is stable.*

Proof. Write $0 \to \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\beta} \mathcal{Z} \to 0$ in extended form:

$$\begin{array}{ccccccccccc}
0 & & 0 & & & & 0 & & 0 & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & \ldots & \longleftarrow & X_n & \xleftarrow{f_n} & X_{n+1} & \longleftarrow & \ldots \\
\alpha_1\downarrow & & \alpha_2\downarrow & & & & \alpha_n\downarrow & & \alpha_{n+1}\downarrow & & \\
Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & \ldots & \longleftarrow & Y_n & \xleftarrow{g_n} & Y_{n+1} & \longleftarrow & \ldots \\
\beta_1\downarrow & & \beta_2\downarrow & & & & \beta_n\downarrow & & \beta_{n+1}\downarrow & & \\
Z_1 & \xleftarrow{h_1} & Z_2 & \xleftarrow{h_2} & \ldots & \longleftarrow & Z_n & \xleftarrow{h_n} & Z_{n+1} & \longleftarrow & \ldots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

Of course, since each Y_n is abelian, and each column is exact, it is clear that *every* group in the diagram is abelian.

Define a sequence of groups (K_n) by setting

$$K_1 = Y_1, \quad K_n = Y_n \times X_1 \times \ldots \times X_{n-1} \quad (n \geq 2).$$

Each group Y_n has its given complete metric; each X_k is given the discrete metric, and then K_n is given the corresponding product topology; so K_n is a complete metrizable, abelian group.

For each n define $T_n : K_{n+1} \to K_n$ by setting, for every $y \in Y_{n+1}$ and $x_k \in X_k$ $(k = 1, \ldots, n)$,

$$T_n(y; x_1, \ldots, x_{n-1}, x_n) = (g_n(y) + \alpha_n(x_n); x_1, \ldots, x_{n-1}).$$

Then T_n is a continuous homomorphism and, because of the density of $\alpha_n(X_n)$ in Y_n, $T_n(K_{n+1})^- = K_n$. The sequence $\mathcal{K} = (K_n\,; T_n)$ is thus a Mittag-Leffler sequence, and so is stable.

We now define a surjective morphism $\varrho = (\varrho_n)_{n\geq 1}$ from $\mathcal{K}$ onto $\mathcal{Z}$ by defining, for each n and each $(y; x_1, \ldots, x_{n-1}) \in K_n$,

$$\varrho_n(y; x_1, \ldots, x_{n-1}) = \beta_n(y).$$

It is clear that each ϱ_n is a surjective homomorphism of K_n onto Z_n. To see that ϱ is a morphism in **ILG**, we must check the commutativity relations, $h_n \varrho_{n+1} = \varrho_n T_n$ $(n \geq 1)$. But for $(y; x_1, \ldots, x_{n-1}, x_n) \in K_{n+1}$,

$$\begin{aligned} \varrho_n T_n(y; x_1, \ldots, x_{n-1}, x_n) &= \varrho_n(g_n(y) + \alpha_n(x_n); x_1, \ldots, x_{n-1}) \\ &= \beta_n g_n(y) = h_n \beta_{n+1}(y) \\ &= h_n \varrho_{n+1}(y; x_1, \ldots, x_{n-1}, x_n). \end{aligned}$$

Thus ϱ is a surjective morphism of $\mathcal{K}$ onto $\mathcal{Z}$, so the stability of $\mathcal{Z}$ follows from [2], Proposition 1(i).

COROLLARY 3.2. *Let $\mathcal{X} = (X_n; f_n)$ be an IL-sequence in which each X_n is a metrizable, abelian topological group with translation-invariant metric and each f_n is a continuous homomorphism. Let $\widetilde{\mathcal{X}} = (\widetilde{X}_n; \widetilde{f}_n)$ be the sequence of completions (where $\widetilde{f}_n : \widetilde{X}_{n+1} \to \widetilde{X}_n$ is the unique continuous extension of f_n). Suppose that $\mathcal{X}$ is stable; then $\widetilde{\mathcal{X}}$ is also stable.*

Proof. With a hopefully obvious notation, consider the short exact sequence $0 \to \mathcal{X} \to \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}/\mathcal{X} \to 0$. Then $\mathcal{X}$ is stable by hypothesis, $\widetilde{\mathcal{X}}/\mathcal{X}$ is stable by Theorem 3.1, so $\widetilde{\mathcal{X}}$ is stable by [2], Proposition 1(ii).

COROLLARY 3.3. *Let E be a complete metrizable abelian group and let (E_n) be a stable sequence of subgroups of E. Then the sequence of closures $(\overline{E}_n)$ is also stable.*

Proof. This follows from (3.2), by taking $X_n = E_n$, $\widetilde{X}_n = \overline{E}_n$ and $f_n : E_{n+1} \to E_n$ as inclusion.

The last result will be a general form of the classical Stability Lemma of automatic continuity theory ([7], Lemma 1.6). The only property of the separating subspace that will be used is its definition. To make this point quite clear, we shall, with no extra difficulty, set the result in the context of complete metrizable abelian groups—where the closed graph theorem does, anyway, not generally apply.

Let X, Y be metrizable, abelian topological groups, and let $S : X \to Y$ be a homomorphism. Then the *graph* of S, denoted by $\Gamma(S)$, is defined to be

$$\Gamma(S) = \{(x, Sx) : x \in X\}.$$

Thus $\Gamma(S)$ is a subgroup of $X \times Y$; the latter group is given the product metric.

The *separating subgroup* of S is

$$\mathcal{S}(S) = \{y \in Y : \text{there is a sequence } (x_n) \text{ in } X,\ x_n \to 0,\ S(x_n) \to y\}.$$

The following lemma is a very simple (and essentially well known) exercise.

LEMMA 3.4. *Let X, Y be metrizable, abelian topological groups, and let $S : X \to Y$ be a homomorphism. Then:*

(i) *$\Gamma(S)$ is a subgroup of $X \times Y$;*
(ii) *$\mathcal{S}(S)$ is a closed subgroup of Y;*
(iii) *if S is continuous, then $\Gamma(S)$ is closed in $X \times Y$ and $\mathcal{S}(S) = (0)$.*

A simple, but important, connection between the graph and the separating subgroup is given in the following lemma.

LEMMA 3.5 (Exact sequence of the separating subgroup). *Let X, Y be metrizable, abelian topological groups, and let $S : X \to Y$ be a homomorphism. Let $\Gamma = \Gamma(S)$, $\mathcal{S} = \mathcal{S}(S)$ and let $\overline{\Gamma}$ be the closure of Γ in $X \times Y$. Then there is a short (split) exact sequence of groups and homomorphisms,*

$$0 \longrightarrow X \xrightarrow{\gamma} \overline{\Gamma} \xrightarrow{\sigma} \mathcal{S} \longrightarrow 0,$$

where

$$\gamma(x) = (x, Sx) \quad (x \in X), \qquad \sigma(x, y) = y - Sx \quad ((x, y) \in \overline{\Gamma}).$$

Also, $\operatorname{im} \gamma = \Gamma$.

Proof. Evidently, γ is injective, and $\operatorname{im} \gamma = \Gamma$. It is also clear that $\sigma(x, y) = 0$ if and only if $(x, y) \in \Gamma = \operatorname{im} \gamma$.

Now let $(x, y) \in \overline{\Gamma}$; then there is a sequence (x_n) in X with $x_n \to x$ and $Sx_n \to y$. But then $x_n - x \to 0$ and $S(x_n - x) \to y - Sx = \sigma(x, y)$, so $\sigma(x, y) \in \mathcal{S}$.

It remains to prove that σ is surjective. Let $y \in \mathcal{S}$; then there is a sequence (x_n) in X with $x_n \to 0$ and $Sx_n \to y$, so that $(0, y) \in \overline{\Gamma}$ and evidently $\sigma((0, y)) = y$. This completes the proof of exactness. (Moreover, the mapping $y \mapsto (0, y)$ $(y \in \mathcal{S})$ is a right-inverse of σ, so that the sequence is even split exact—but we do not use this stronger fact.)

THEOREM 3.6 (Generalized Stability Lemma). *Let $\mathcal{X} = (X_n; T_n)$, $\mathcal{Y} = (Y_n; R_n)$ be IL-sequences of complete metrizable, abelian topological groups and continuous homomorphisms. For each n, let $S_n : X_n \to Y_n$ be a homomorphism and suppose that, for every n, $S_n T_n = R_n S_{n+1}$. For each n let $\mathcal{S}_n = \mathcal{S}(S_n)$. Then, for every n, $R_n(\mathcal{S}_{n+1}) \subseteq \mathcal{S}_n$ and (writing $\overline{R}_n = R_n \mid \mathcal{S}_{n+1} : \mathcal{S}_{n+1} \to \mathcal{S}_n$), the sequence $(\mathcal{S}_n; \overline{R}_n)$ is stable.*

Proof. We first check that $R_n(\mathcal{S}_{n+1}) \subseteq \mathcal{S}_n$. Let $y \in \mathcal{S}_{n+1}$; then there is a sequence $x_k \to 0$ in X_{n+1}, with $S_{n+1}(x_k) \to y$ as $k \to \infty$. But R_n is continuous, so $R_n S_{n+1} x_k \to R_n y$, i.e. $S_n T_n x_k \to R_n y$. But again T_n is continuous, so $T_n x_k \to 0$ and therefore $R_n y \in \mathcal{S}_n$. Hence $R_n(\mathcal{S}_{n+1}) \subseteq \mathcal{S}_n$.

For each n let $\Gamma_n = \Gamma(S_n)$ and let $\overline{\Gamma}_n$ be the closure of Γ_n in $X_n \times Y_n$.

Now consider the diagram

$$\begin{array}{ccccccccccc}
0 & & 0 & & & & 0 & & 0 & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
X_1 & \xleftarrow{T_1} & X_2 & \xleftarrow{T_2} & \cdots & \longleftarrow & X_n & \xleftarrow{T_n} & X_{n+1} & \longleftarrow & \cdots \\
{\scriptstyle\gamma_1}\downarrow & & {\scriptstyle\gamma_2}\downarrow & & & & {\scriptstyle\gamma_n}\downarrow & & {\scriptstyle\gamma_{n+1}}\downarrow & & \\
\overline{\Gamma}_1 & \xleftarrow{(T_1,R_1)} & \overline{\Gamma}_2 & \xleftarrow{(T_2,R_2)} & \cdots & \longleftarrow & \overline{\Gamma}_n & \xleftarrow{(T_n,R_n)} & \overline{\Gamma}_{n+1} & \longleftarrow & \cdots \\
{\scriptstyle\sigma_1}\downarrow & & {\scriptstyle\sigma_2}\downarrow & & & & {\scriptstyle\sigma_n}\downarrow & & {\scriptstyle\sigma_{n+1}}\downarrow & & \\
\mathcal{S}_1 & \xleftarrow{\overline{R}_1} & \mathcal{S}_2 & \xleftarrow{\overline{R}_2} & \cdots & \longleftarrow & \mathcal{S}_n & \xleftarrow{\overline{R}_n} & \mathcal{S}_{n+1} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

Note that, in the middle row, $(T_n, R_n)(u, v) \equiv (T_n u, R_n v)$. The nth column is the exact sequence of the separating subgroup of S_n, given by Lemma 3.5. Commutativity of the diagram follows easily from the relations $S_n T_n = R_n S_{n+1}$.

For each n, $\operatorname{im}\gamma_n = \Gamma_n$ is dense in $\overline{\Gamma}_n$, and the latter is complete. The stability of the third row thus follows from Theorem 3.1.

COROLLARY 3.7. *In the notation of Theorem* 3.6, *let* $\mathcal{X} = (X_n\,; T_n)$, $\mathcal{Y} = (Y_n\,; R_n)$ *now be IL-sequences of complete, metrizable, topological vector spaces and continuous linear mappings. Also, let each* $S_n : X_n \to Y_n$ *be linear, such that, for every* n, $S_n T_n = R_n S_{n+1}$. *Then, for every* $n \geq 1$ *and every seminorm* p *on* Y_n, *there exists* $m(n) \geq n$ *such that* $R_n \ldots R_m(\mathcal{S}_{m+1})$ *is* p*-dense in* $R_n \ldots R_{m(n)}(\mathcal{S}_{m(n)+1})$ *for every* $m \geq m(n)$.

Proof. By the main theorem, the sequence $(\mathcal{S}_n\,; \overline{R}_n)$ is stable. The result now follows at once from 2.3 and 2.5.

References

[1] G. R. Allan, *Elements of finite closed descent in a Banach algebra*, J. London Math. Soc. (2) 7 (1973), 462–466.

[2] —, *Stable inverse-limit sequences, with application to Fréchet algebras*, Studia Math. 121 (1996), 277–308.

[3] —, *Stable elements of Banach and Fréchet algebras*, ibid. 129 (1998), 67–96.

[4] —, *Inverse-limit sequences in functional analysis*, to appear.

[5] J. Esterle, *Semi-normes sur* $C(K)$, Proc. London Math. Soc. (3) 36 (1978), 27–45.

[6] J. Esterle, *Mittag-Leffler methods in the theory of Banach algebras and a new approach to Michael's problem*, in: Contemp. Math 32, Amer. Math. Soc., 1984, 107–129.

[7] A. M. Sinclair, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge Univ. Press, Cambridge, 1976.

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WB, U.K.
E-mail: G.R.Allan@dpmms.cam.ac.uk

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