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Raising bounded groups and splitting of radical extensions of commutative Banach algebras

by

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Abstract. Let $\mathcal{A}$ be a commutative unital Banach algebra and let $\mathcal{A}/\mathcal{R}$ be the quotient algebra of $\mathcal{A}$ modulo its radical $\mathcal{R}$. This paper is concerned with raising bounded groups in $\mathcal{A}/\mathcal{R}$ to bounded groups in the algebra $\mathcal{A}$. The results will be applied to the problem of splitting radical extensions of certain Banach algebras.

1. Introduction. Let A be a commutative, semisimple, unital Banach algebra. A unital Banach algebra $\mathcal{A}$ with Jacobson radical $\mathcal{R}$ is an *extension* of A if $\mathcal{A}/\mathcal{R}$ is topologically isomorphic to A. Thus we have a short exact sequence

$$0 \to \mathcal{R} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} A \to 0.$$

The sequence *splits topologically* if there is a continuous algebra homomorphism $\theta : A \to \mathcal{A}$ such that $\pi \circ \theta = \mathrm{id}_A$. The topological splitting of an extension is equivalent to $\mathcal{A}$ having a Wedderburn decomposition $\mathcal{A} = \mathcal{B} \oplus \mathcal{R}$, where $\mathcal{B} = \theta(A)$ is a closed subalgebra of $\mathcal{A}$. An extension is *commutative* if $\mathcal{A}$ is commutative, *nilpotent* if $\mathcal{R}^n = 0$ for some $n \in \mathbb{N}$, and *singular* if $\mathcal{R}^2 = 0$. A treatment of splittings of extensions of Banach algebras may be found in [BDL].

The question of the topological splitting of commutative extensions $\mathcal{A}$ of the algebra $C(X)$, X a compact Hausdorff space, has been explored under a variety of conditions on X and on the radical $\mathcal{R}$ of $\mathcal{A}$. In 1960 Bade and Curtis [BC2] proved the splitting in the case where $\mathcal{R}$ is nilpotent and X is totally disconnected. They proved that one may take $\mathcal{B} = \overline{\mathrm{sp}}\{p \in \mathcal{A} : p^2 = p\}$. A crucial observation was the fact that under these two conditions the set of idempotents in $\mathcal{A}$ is uniformly bounded in norm. In 1962 Kamowitz [Ka] established splitting in the case where $\mathcal{R}$ is nilpotent and $\mathcal{R}$ and the closure of all even powers of $\mathcal{R}$ have a Banach space complement in $\mathcal{A}$. He did not require that X be totally disconnected. Returning to the hypothesis that X

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is totally disconnected, Gorin and Lin [GL] in 1967 proved the boundedness of idempotents, and hence splitting of extensions, when the radical $\mathcal{R}$ satisfies the following condition weaker than nilpotence, which will concern us later.

DEFINITION. A commutative radical algebra $\mathcal{R}$ is *uniformly topologically nilpotent* (UTN) if there is a positive sequence (μ_n) with $\mu_n \to 0$ such that

$$\|r^n\|^{1/n} \le \|r\|\mu_n \quad (n \in \mathbb{N},\ r \in \mathcal{R}).$$

REMARK. A radical Banach algebra $\mathcal{R}$ is called *topologically nilpotent* if

$$\limsup_{n\to\infty}\{\|r_1 \dots r_n\|^{1/n} : r_i \in \mathcal{R},\ \|r_i\| = 1\ (i = 1, \dots, n)\} = 0.$$

It is called *uniformly topologically nil* if

$$\limsup_{n\to\infty}\{\|r^n\|^{1/n} : r \in \mathcal{R},\ \|r\| = 1\} = 0.$$

When $\mathcal{R}$ is commutative, these definitions are equivalent, and equivalent to *uniformly topologically nilpotent* as we have defined it. We prefer "nilpotent" to "nil" in this context. See [Di] and [P, page 515] for excellent discussions.

Let $\mathbb{I} = [0,1]$ and let C_* denote the Banach space $C(\mathbb{I})$ with the uniform norm and convolution multiplication. The subalgebra $C_{*,0} = \{f \in C_* : f(0) = 0\}$ is a simple example of a radical Banach algebra $\mathcal{R}$ which is UTN. In this case $\|r^n\|^{1/n} \le \|r\|\mu_n$ $(r \in \mathcal{R},\ n \in \mathbb{N})$, where

$$\mu_n = [(n-1)!]^{-1/n} \quad (n \in \mathbb{N}).$$

After [GL], all subsequent results required that $\mathcal{R}$ be nilpotent, but not that X be totally disconnected. The first progress toward the general case of nilpotent extensions came in 1993 when M. Solovej ([So1] and [So2]) proved the existence of a topological splitting in the case where $X = \mathbb{I}$ and $C(\mathbb{I})$ is isometrically isomorphic to $\mathcal{A}/\mathcal{R}$. Recently, E. Albrecht and O. Ermert [AlEr], building on the ideas of Solovej, proved splitting for all nilpotent radicals. Supposing $\mathcal{R}^2 = 0$, the strategy in [So1] and [AlEr] is to resolve the problem for X a compact subset of $\mathbb{R}$ by passing to the second duals $\mathcal{A}^{**}$ and $C(X)^{**}$, which are Banach algebras under the first Arens product. We have the diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}^{**} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi^{**}} \\ C(X) & \longrightarrow & C(X)^{**} \end{array}$$

Fortunately, $C(X)^{**} = C(\Omega)$ for Ω a certain extremely disconnected compact Hausdorff space. The problem is that $\mathcal{A}^{**}$ need not be commutative. However, by deep arguments they show there exists a short exact sequence

$$0 \to \mathcal{R}_1 \to \mathcal{A}_1 \to C(\Omega_1) \to 0,$$

where Ω_1 is totally disconnected, $\mathcal{A}_1$ is a commutative subalgebra of $\mathcal{A}^{**}$, $\pi_1 = \pi^{**}|_{\mathcal{A}_1}$ and $\mathcal{R} \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_1^{**}$, where $\mathcal{R}_1^{**^2} = 0$. This sequence splits by the results of [BC2]. From this splitting one obtains the splitting of the sequence

$$0 \to \mathcal{R} \to \mathcal{A} \to C(X) \to 0.$$

Albrecht and Ermert then show how, knowing the result for X a compact subset of $\mathbb{R}$, one obtains the result for all nilpotent extensions and arbitrary compact spaces.

In contrast to these results there is an example in [BC2] of a radical extension of the algebra c of convergent sequences which does not split even algebraically.

In this paper we give a direct proof of the following theorem which applies directly to all UTN extensions and all compact spaces and avoids the complications associated with the second duals.

1. THEOREM. *Let X be a compact Hausdorff space and*

$$0 \to \mathcal{R} \to \mathcal{A} \to \mathcal{A}/\mathcal{R} \to 0$$

be a commutative extension, where $\mathcal{A}/\mathcal{R}$ is topologically isomorphic to $C(X)$. If the radical $\mathcal{R}$ of $\mathcal{A}$ is uniformly topologically nilpotent, then the extension splits topologically and this topological splitting is unique.

2. Raising bounded groups. Let $\mathcal{A}$ be a commutative unital Banach algebra with radical $\mathcal{R}$. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{R}$ the canonical quotient map. Our aim is to show that if G is a bounded commutative group of invertible elements in $\mathcal{A}/\mathcal{R}$ and $\mathcal{R}$ is uniformly topologically nilpotent, then there exists a unique bounded group $\widetilde{G}$ in $\mathcal{A}$ such that $\pi(\widetilde{G}) = G$ and the map $\tau : G \to \widetilde{G}$, $g \to \widetilde{g}$ is a continuous isomorphism.

We first prove this result for the simplest type of group. If h is an invertible element of a Banach algebra $\mathcal{A}$, we call the set

$$\mathrm{orb}(h) = \{\dots, h^{-2}, h^{-1}, e, h, h^2, \dots\}$$

the *orbit* of h. The element h is *doubly power bounded* (DPB) if $\mathrm{orb}(h)$ is a bounded subset of $\mathcal{A}$. Clearly, as $\mathcal{A}$ is commutative, the product of DPB elements is DPB. We are going to prove that if $g \in \mathcal{A}$, $[g + \mathcal{R}]$ is DPB in $\mathcal{A}/\mathcal{R}$ and $\mathcal{R}$ is UTN, then in the coset $[g + \mathcal{R}]$ there exists a unique element $\widetilde{g}$ such that $\widetilde{g}$ is DPB in $\mathcal{A}$. Moreover, $\widetilde{g}$ has the form $\widetilde{g} = ge^r$, where $r = r_g$ is a unique element of $\mathcal{R}$. We can formulate the result as follows.

2. THEOREM. *Let $\mathcal{A}$ be a commutative unital Banach algebra with radical $\mathcal{R}$ which is* UTN. *Let g be an invertible element of $\mathcal{A}$ and K be a constant with $K > 1$ such that*

$$\|[g + \mathcal{R}]^n\|_{\mathcal{A}/\mathcal{R}} \le K \quad (n \in \mathbb{Z}).$$

Then there exists a unique $r \in \mathcal{R}$ and a constant M which depends only on K such that

$$\|e^{nr}g^n\|_{\mathcal{A}} \le M \quad (n \in \mathbb{Z}).$$

The proof proceeds by several steps. The first is to exploit the hypothesis that $\mathcal{R}$ is UTN.

3. LEMMA. *Let C be a positive constant. If $b \in \mathcal{R}$ with $\|b\| \le C$, then $\|\log(1+b)\| \le L$, where L is a constant depending only on C and the sequence (μ_n). Moreover,*

$$1 + b = \exp(\log(1+b)).$$

Proof. Since $\|b^n\|^{1/n} \to 0$ for $b \in \mathcal{R}$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}b^n/n$ converges absolutely. Since $\mathcal{R}$ is uniformly topologically nilpotent, $\|b^n\| \le C^n\mu_n^n$, where $\mu_n \to 0$, and hence

$$\|\log(1+b)\| \le \sum_{n=1}^{\infty}\frac{C^n}{n}\mu_n^n \equiv L.$$

That $1 + b = \exp(\log(1+b))$ follows by standard analytic functional arguments (see [DS], VII.3.12). ∎

Returning to the proof of Theorem 2, suppose $g \in \mathcal{A}$ is such that $[g+\mathcal{R}]$ is DPB, say, $\|[g+\mathcal{R}]^n\| \le K$ $(n \in \mathbb{Z})$. We seek $r \in \mathcal{R}$ such that $g \exp r$ is DPB in $\mathcal{A}$. By definition of the quotient norm, for each $n \in \mathbb{Z}$ there exists $s_n \in \mathcal{R}$ such that

$$\|g^n + s_n\| < K + 1 \quad (n \in \mathbb{Z}).$$

Let $y_n = g^{-n}s_n$ for each $n \in \mathbb{Z}$. Then

$$\|g^n(1+y_n)\| = \|g^n(1+g^{-n}s_n)\| = \|g^n + s_n\| < K + 1. \tag{2.1}$$

Let $w(n) = \log(1+y_n)$, so $w(n) \in \mathcal{R}$. For $m, n \in \mathbb{Z}$,

$$\begin{aligned}
&\|\exp[w(m+n) + w(-m) + w(-n)]\| \\
&\quad = \|g^{m+n}\exp w(m+n) \cdot g^{-m}\exp w(-m) \cdot g^{-n}\exp w(-n)\| \\
&\quad = \|g^{m+n}(1+y_{m+n}) \cdot g^{-m}(1+y_{-m}) \cdot g^{-n}(1+y_{-n})\| \\
&\quad < (K+1)^3.
\end{aligned}$$

Then

$$\|w(m+n) + w(-m) + w(-n)\| \le L \quad (m, n \in \mathbb{Z})$$

where L is the constant in Lemma 3 corresponding to $\|b\| \le (K+1)^3$.

4. PROPOSITION. *Let X be a Banach space and $w : \mathbb{Z} \to X$ be a map such that*

$$\|w(m+n) + w(-m) + w(-n)\| < L \quad (m, n \in \mathbb{Z}).$$

Then there exists $r \in X$ such that

$$\|w(n) - nr\| < L \quad (n \in \mathbb{Z}).$$

Proof. Replacing m and n by $2n$ gives

$$\|w(4n) + 2w(-2n)\| < L,$$

and replacing m and n by $-n$,

$$\|w(-2n) + 2w(n)\| < L.$$

Hence

$$\begin{aligned}
\|w(4n) - 4w(n)\| &= \|w(4n) + 2w(-2n) - 2w(-2n) - 4w(n)\| \\
&\le \|w(4n) + 2w(-2n)\| + 2\|w(-2n) + 2w(n)\| \\
&< L + 2L = 3L.
\end{aligned}$$

Replacing n by $4n$ successively, we get

$$\left\|\frac{w(4n)}{4} - w(n)\right\| < \frac{3L}{4},$$

$$\left\|\frac{w(4^2n)}{4^2} - \frac{w(4n)}{4}\right\| < \frac{3L}{4^2},$$

$$\left\|\frac{w(4^3n)}{4^3} - \frac{w(4^2n)}{4^2}\right\| < \frac{3L}{4^3},$$

$$\vdots$$

For each $p \in \mathbb{N}$, set

$$u_p(n) = \frac{w(4^{p+1}n)}{4^{p+1}} - \frac{w(4^pn)}{4^p}.$$

Then $\|u_p(n)\| < 3L/4^{p+1}$ $(n \in \mathbb{Z})$, so $(w(4^pn)/4^p)$ is a Cauchy sequence in X uniformly in n. For

$$\begin{aligned}
\left\|\frac{w(4^{p+k}n)}{4^{p+k}} - \frac{w(4^pn)}{4^p}\right\| &< \|u_{p+k-1}(n) + u_{p+k-2}(n) + \ldots + u_p(n)\| \\
&< \frac{3L}{4^{p+1}}\sum_{k=0}^{\infty}\frac{1}{4^k} = \frac{L}{4^p}.
\end{aligned}$$

Set $V(n) = \lim_{p\to\infty} w(4^pn)/4^p$ $(n \in \mathbb{Z})$. The limit exists uniformly for $n \in \mathbb{Z}$. Now

$$\left\|\frac{w(4^p(m+n))}{4^p} + \frac{w(4^p(-m))}{4^p} + \frac{w(4^p(-n))}{4^p}\right\| < \frac{L}{4^p},$$

so

$$V(m+n) + V(-m) + V(-n) = 0 \quad (m, n \in \mathbb{Z}).$$

Letting $m = -n$ yields

$$V(0) + V(n) + V(-n) = 0 \quad (n \in \mathbb{Z}),$$

so (taking $n = 0$) we get $V(0) = 0$. Thus $V(-n) = -V(n)$ $(n \in \mathbb{Z})$. Consequently,

$$V(m+n) = V(m) + V(n) \quad (m, n \in \mathbb{Z}).$$

It follows that

$$V(n) = nV(1) \quad (n \in \mathbb{Z}).$$

Now for each $n \in \mathbb{Z}$, we have

$$\begin{aligned}\|w(n) - V(n)\| &\le \left\| w(n) - \frac{w(4n)}{4} \right\| + \left\| \frac{w(4n)}{4} - \frac{w(4^2 n)}{4^2} \right\| + \dots \\ &\quad + \left\| \frac{w(4^{k-1})}{4^{k-1}} - \frac{w(4^k n)}{4^k} \right\| + \left\| \frac{w(4^k n)}{4^k} - V(n) \right\| \\ &< \sum_{k=0}^{\infty} \left\| \frac{w(4^k n)}{4^k} - \frac{w(4^{k+1} n)}{4^{k+1}} \right\| \\ &\le \sum_{k=0}^{\infty} \frac{3L}{4^{k+1}} = \frac{3L}{4} \left(\frac{1}{1 - 1/4} \right) = L.\end{aligned}$$

Taking $r = V(1)$, we have

$$\|w(n) - nr\| < L \quad (n \in \mathbb{Z}). \ \blacksquare$$

Now consider the element ge^r. Then

$$\begin{aligned}\|(ge^r)^n\| &\le \|g^n e^{w(n)}\| \cdot \|e^{nr - w(n)}\| \\ &\le (K+1)\|e^{nr - w(n)}\| \le (K+1)e^L = M \quad (n \in \mathbb{Z}).\end{aligned}$$

Thus ge^r is DPB.

We now prove that ge^r is the unique DPB element in the coset $[g + \mathcal{R}]$. Indeed, a coset can contain at most one DPB element. This result does not require that $\mathcal{R}$ be UTN.

5. PROPOSITION. *Let h, k be* DPB *elements in $\mathcal{A}$. If $h - k = r \in \mathcal{R}$, then $h = k$.*

Proof. Suppose $\|h^n\|$ and $\|k^n\|$ are bounded by M, N for $n \in \mathbb{Z}$. Then $k^{-1}h = 1 + k^{-1}r$ is DPB with bound MN. By Gelfand's Theorem [HP, Theorem 4.10.1], $k^{-1}r = 0$, so $r = 0$, proving $h = k$. ■

6. THEOREM. *Let $\mathcal{A}$ be a commutative unital Banach algebra whose radical $\mathcal{R}$ is uniformly topologically nilpotent. Let G be a bounded subgroup of the group of invertibles in $\mathcal{A}/\mathcal{R}$. Then there exists a unique bounded group $\widetilde{G}$ in $\mathcal{A}$ such that $\pi(\widetilde{G}) = G$ and $\widetilde{G}$ is isomorphic to G.*

Proof. The elements of G are cosets $[g + \mathcal{R}]$. We form ge^r, which is the unique DPB element in $[g + \mathcal{R}]$. (We remind the reader that we could have chosen any other element $h \in [g + \mathcal{R}]$ and $s \in \mathcal{R}$ so that he^s is DPB. But then $ge^r = he^s$ by Proposition 5.) For $[g + \mathcal{R}] \in G$, let $\tau([g + \mathcal{R}]) = ge^r$ and set

$$\widetilde{G} = \{\tau([g + \mathcal{R}]) : [g + \mathcal{R}] \in G\}.$$

Let $[g + \mathcal{R}]$, $[h + \mathcal{R}]$ be in G and let $a = \tau([g + \mathcal{R}])$, $b = \tau([h + \mathcal{R}])$ and $c = \tau([g + \mathcal{R}][h + \mathcal{R}]) = \tau[gh + \mathcal{R}]$ be in $\mathcal{A}$. Then ab and c are DPB in $\mathcal{A}$ and $\pi(ab) = \pi(c)$, so by the uniqueness of Proposition 5, $ab = c$. This shows τ is a homomorphism. The product is well defined on $\widetilde{G}$. Moreover,

$$\tau([g + \mathcal{R}][g + \mathcal{R}]^{-1}) = \tau([g + \mathcal{R}][g^{-1} + \mathcal{R}]) = \tau([1 + \mathcal{R}]) = e^0 = 1,$$

the identity in $\mathcal{A}$. Thus $\widetilde{G}$ is a group and τ is an isomorphism of G onto $\widetilde{G}$. If K is a bound for the norms of elements of G, then the constant M of Theorem 2 is a bound for the norms of elements of $\widetilde{G}$. ■

3. The splitting of topologically nilpotent extensions of $l^1(G)$.

As a first application of Theorem 6, let G be an abelian group and let A be the Banach algebra $l^1(G)$ under convolution multiplication. For each $g \in G$, the unit mass δ_g is DPB in A and has norm one. Suppose that $\mathcal{A}$ is a commutative extension of A. Let $\mathcal{R} = \operatorname{rad} \mathcal{A}$ and suppose $\mathcal{R}$ is uniformly topologically nilpotent.

Consider the short exact sequence

$$0 \to \mathcal{R} \to \mathcal{A} \xrightarrow{\pi} A \to 0.$$

We show that this sequence splits topologically. Thus we will show there exists a continuous homomorphism $\theta : A \to \mathcal{A}$ such that $\pi \circ \theta = \mathrm{id}_A$. Then $\mathcal{B} = \theta(A)$ is a closed subalgebra of $\mathcal{A}$ such that

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{R}.$$

7. THEOREM. *Every commutative uniformly topologically nilpotent extension of $l^1(G)$ splits topologically and this topological splitting is unique.*

Proof. For each $g \in G$, let δ_g denote the unit point mass at g. Given $g_1, g_2 \in G$ we have

$$\delta_{g_1 + g_2} = \delta_{g_1} \delta_{g_2}.$$

By Theorem 6 there exists a unique bounded group $\{h_g : g \in G\}$ in $\mathcal{A}$ and $M \ge 1$ such that

$$\pi(h_g) = \delta_g, \quad \|h_g\| < M \quad (g \in G).$$

The map $\theta : \delta_g \to h_g$ $(g \in G)$ defines an isomorphism of G onto a bounded multiplicative group of elements of $\mathcal{A}$. We extend θ to a bounded isomorphism of $l^1(G)$ into $\mathcal{A}$ by

$$\theta\Big(\sum_{g\in G}\alpha_g\delta_g\Big)=\sum_{g\in G}\alpha_g h_g.$$

It is clear that $(\pi\circ\theta)(a)=a$ $(a\in l^1(G))$. Since the map θ is uniquely defined, the splitting is unique. ∎

4. Raising Hermitian equivalent elements. Let A be a unital commutative Banach algebra. An element $a\in A$ is *Hermitian* if

$$\|\exp iat\|=1 \quad (t\in\mathbb{R}).$$

It is *Hermitian equivalent* if there exists a constant $M\ge 1$ such that

$$\|\exp iat\|\le M \quad (t\in\mathbb{R}).$$

More generally, let $\mathfrak{F}\subseteq A$ and let $S(\mathfrak{F})$ be the real linear span of the elements of $\mathfrak{F}$. The *exponential group* of $\mathfrak{F}$ is the set

$$G(\mathfrak{F})=\{\exp ih: h\in S(\mathfrak{F})\}.$$

We say $\mathfrak{F}$ is *Hermitian equivalent* if $G(\mathfrak{F})$ is a bounded set in A. The theory of Hermitian elements may be found in [BD2] and [Lu].

We record the following theorem which will be important to us. (See [Lu], [Gr].)

8. THEOREM. *If $\mathfrak{F}\subseteq A$ is Hermitian equivalent, then there exists an equivalent norm $|\cdot|$ on A such that, with respect to $|\cdot|$, the elements of $\mathfrak{F}$ are all Hermitian.*

Let $[a+\mathcal{R}]$ be Hermitian equivalent in $\mathcal{A}/\mathcal{R}$. We can raise the bounded group $\{\exp it[a+\mathcal{R}]: t\in\mathbb{R}\}$ to $\mathcal{A}$.

9. THEOREM. *Let $\mathcal{A}$ be a commutative unital Banach algebra with uniformly topologically nilpotent radical $\mathcal{R}$, and let $a\in\mathcal{A}$ satisfy*

$$\|\exp it([a+\mathcal{R}])\|<K \quad (t\in\mathbb{R})$$

where $K>1$ is a constant. Then there is a constant $M>1$ depending only on K, and a unique $r\in\mathcal{R}$ such that

$$\|\exp it(a+r)\|<M \quad (t\in\mathbb{R}).$$

Proof. Clearly,

$$\|\exp in[a+\mathcal{R}]\|=\|[(\exp ia)+\mathcal{R}]^n\|<K \quad (n\in\mathbb{Z}),$$

so $\exp i[a+\mathcal{R}]$ is DPB in $\mathcal{A}/\mathcal{R}$. It follows from Theorem 2 that there exists $r\in\mathcal{R}$ so that $\exp i(a+r)$ is DPB in $\mathcal{A}$. Now for $t\in\mathbb{R}$,

$$\begin{aligned}\|\exp it(a+r)\|&\le\|\exp i[t](a+r)\|\cdot\|\exp i(t-[t])(a+r)\|\\&\le M\sup_{u\in[0,1]}\|\exp iu(a+r)\|.\end{aligned}$$

It follows that $a+r$ is Hermitian equivalent in $\mathcal{A}$. To show r is unique, suppose $s\in\mathcal{R}$ and $\|\exp it(a+s)\|$ is bounded for $t\in\mathbb{R}$. Then

$$\exp i(a+r)\cdot\exp(-i(a+s))=\exp i(r-s)$$

is DPB and hence $r=s$ as in the proof of Proposition 5. ∎

5. Splitting of extensions of $C(X)$. Let X be a compact Hausdorff space and let

$$0\to\mathcal{R}\to\mathcal{A}\xrightarrow{\pi}C(X)\to 0$$

be a radical extension of $C(X)$, where $C(X)$ is topologically isomorphic to $\mathcal{A}/\mathcal{R}$ and $\mathcal{R}$ is uniformly topologically nilpotent. As we remarked in the introduction, the splitting is known to hold when $\mathcal{R}$ is nilpotent. Our results on raising bounded groups give us a new method for proving splitting theorems which is applicable to the larger class of algebras with uniformly topologically nilpotent radicals.

The proof that the extension splits has two parts. In the first part we use the raising of Hermitian elements to prove there always exists a splitting

$$\mathcal{A}=\mathcal{Y}\oplus\mathcal{R},$$

where $\mathcal{Y}$ is a closed *subspace* of $\mathcal{A}$. We use this fact in the second part to show that $\mathcal{A}$ has a splitting $\mathcal{A}=\mathcal{B}\oplus\mathcal{R}$, where $\mathcal{B}$ is a closed *subalgebra*.

10. THEOREM. *Let X be a compact Hausdorff space and*

$$0\to\mathcal{R}\to\mathcal{A}\to C(X)\to 0$$

be a commutative extension of $C(X)$, where $\mathcal{R}$ is uniformly topologically nilpotent and $C(X)$ is topologically isomorphic to $\mathcal{A}/\mathcal{R}$. Then $\mathcal{R}$ has a closed Banach space complement in $\mathcal{A}$.

Proof. Let $\Psi: C(X)\to\mathcal{A}/\mathcal{R}$ be a continuous isomorphism, and let K be a constant which satisfies $K>\|\Psi\|$. Note that if $f\in C_{\mathbb{R}}(X)$, then

$$\|\exp(it\Psi(f))\|=\|\Psi(\exp itf)\|<K$$

for all $t\in\mathbb{R}$. The set $\Psi(C_{\mathbb{R}}(X))$ is Hermitian equivalent in $\mathcal{A}/\mathcal{R}$. If $[a+\mathcal{R}]\in\Psi(C_{\mathbb{R}}(X))$, let $S([a+\mathcal{R}])=a+r$ be the unique Hermitian equivalent element of $[a+\mathcal{R}]$ selected by Theorem 9, and define

$$\varrho=S\circ\Psi: C_{\mathbb{R}}(X)\to\mathcal{A}.$$

Given $f_1,f_2\in C_{\mathbb{R}}(X)$ and $\alpha_1,\alpha_2\in\mathbb{R}$, we see that

$$\alpha_1\varrho(f_1)+\alpha_2\varrho(f_2)-\varrho(\alpha_1 f_1+\alpha_2 f_2)$$

is a Hermitian equivalent element which belongs to the radical $\mathcal{R}$ and hence is zero. It follows that ϱ is a real linear map. Since the product of Hermitian equivalent elements need not be Hermitian equivalent (see [Lu]), it is not clear whether ϱ is a real homomorphism. However, ϱ is continuous. To see this let $|\cdot|$ be an equivalent norm on $\mathcal{A}$ such that with respect to $|\cdot|$, the set $\mathcal{Y}_{\mathbb{R}} = \varrho(C_{\mathbb{R}}(X))$ is Hermitian. Since the norm of a Hermitian element is its spectral radius, we see that if $f \in C_{\mathbb{R}}(X)$, then $|\varrho(f)| = \nu(\varrho(f)) = \|f\|_\infty$. Hence ϱ is isometric for $|\cdot|$. Consequently, $\mathcal{Y}_{\mathbb{R}}$ is closed for $|\cdot|$ and hence for $\|\cdot\|$.

Now consider $\mathcal{Y} = \mathcal{Y}_{\mathbb{R}} + i\mathcal{Y}_{\mathbb{R}}$ and adopt the norm $|\cdot|$. If $b, c \in \mathcal{Y}_{\mathbb{R}}$, then

$$\frac{|b|+|c|}{2} = \frac{\nu(b)+\nu(c)}{2} \le \nu(b+ic) \le |b+ic| \le |b|+|c|.$$

Consequently, $\mathcal{Y}$ is closed for $|\cdot|$ and $\|\cdot\|$. We now extend ϱ from $C_{\mathbb{R}}(X)$ to $C(X)$ by defining

$$\varrho(f+ig) = \varrho(f) + i\varrho(g).$$

A calculation shows that the extension is complex linear. For each $a \in \mathcal{A}$, there is $f \in C(X)$, $f = f_1 + if_2$, where $f_1, f_2 \in C_{\mathbb{R}}(X)$, and $a_1, a_2 \in \mathcal{A}$ so that

$$a = a_1 + ia_2, \qquad \pi(a_i) = f_i.$$

Let $b_i = \varrho(f_i)$, $i = 1, 2$. Then $a_i = b_i + r_i$, where $b_i \in \mathcal{Y}_{\mathbb{R}}$, $r_i \in \mathcal{R}$. Thus $a \in \mathcal{Y} + \mathcal{R}$. Since clearly $\mathcal{Y} \cap \mathcal{R} = (0)$, we have proved that $\mathcal{Y}$ is a closed complement to $\mathcal{R}$ in $\mathcal{A}$. ■

11. THEOREM. *Under the hypotheses of Theorem 9, every short exact sequence*

$$0 \to \mathcal{R} \to \mathcal{A} \to C(X) \to 0$$

splits topologically and this topological splitting is unique.

Proof. As the first step of this proof we construct a closed amenable subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that we have the short exact sequence

$$0 \to \operatorname{rad}(\mathcal{B}) \to \mathcal{B} \xrightarrow{\widetilde{\pi}} \mathcal{B}/\operatorname{rad}(\mathcal{B}) \to 0,$$

where $\widetilde{\pi} = \pi|_{\mathcal{B}}$ and $\mathcal{B}/\operatorname{rad}\mathcal{B}$ is isomorphic to $C(X)$.

To prove this let G be the unitary group of $C(X)$. Thus

$$G = \{f \in C(X) : |f(x)| = 1 \ (x \in X)\}$$

and let $\Psi : C(X) \to \mathcal{A}/\mathcal{R}$ be the continuous isomorphism of $C(X)$ onto $\mathcal{A}/\mathcal{R}$. By Theorem 6 we can raise $\Psi(G)$ to a bounded group in $\mathcal{A}$. Recall that for $g \in \mathcal{A}$ such that $[g+\mathcal{R}] \in \Psi(G)$, there exists a unique $r = r_g \in \mathcal{R}$ such that ge^r is DPB in $\mathcal{A}$. Let $\psi : G \to \widetilde{G} \subseteq \mathcal{A}$ be the continuous group isomorphism defined by $\psi([g+\mathcal{R}]) = ge^r$. We extend ψ linearly and continuously as a map of all of $\ell^1(G)$ into $\mathcal{A}$ as in the proof of Theorem 7. Define

$$\mathcal{B} = \overline{\psi(\ell^1(G))},$$

where the closure is in $\mathcal{A}$. Since $\ell^1(G)$ is amenable, so is $\mathcal{B}$. (For facts about amenable Banach algebras see the references [CuLo], [Jo], or [He].)

We next note that $\widetilde{\pi}$ maps $\mathcal{B}$ onto $C(X)$. This follows from an observation of Kamowitz [Ka, p. 366] that each $f \in C(X)$ has a representation as a linear combination $f = \sum_{i=1}^4 \lambda_i g_i$, where $g_i \in G$ and $\lambda_i \in \mathbb{C}$. Since $\widetilde{\pi}(\mathcal{B})$ contains G, it follows that $\widetilde{\pi}(\mathcal{B}) = C(X)$. Hence $\mathcal{B}/\operatorname{rad}(\mathcal{B})$ is isomorphic to $C(X)$. We have $\mathcal{B} + \mathcal{R} = \mathcal{A}$ and $\operatorname{rad}(\mathcal{B}) = \mathcal{B} \cap \mathcal{R}$. We next show that $\operatorname{rad}(\mathcal{B}) = 0$. Since $\operatorname{rad}(\mathcal{B}) \subseteq \mathcal{R}$, it is uniformly topologically nilpotent. Thus by Theorem 9,

$$\mathcal{B} = \mathcal{Y} \oplus \operatorname{rad}(\mathcal{B}),$$

where $\mathcal{Y}$ is a closed *subspace* of $\mathcal{B}$, complementary to $\operatorname{rad}(\mathcal{B})$. Now $\mathcal{B}$ is amenable, and if $\operatorname{rad}(\mathcal{B})$ is different from zero, $\operatorname{rad}(\mathcal{B})^\perp$ is complemented in $\mathcal{B}^*$. Therefore by [CuLo, Theorem 3.7] or [He, II, Proposition 32] we have $\operatorname{rad}(\mathcal{B}) = (\operatorname{rad}(\mathcal{B}))^2$. However, by Dixon [Di], a UTN radical algebra never has such a factorization. Consequently, $\operatorname{rad}(\mathcal{B}) = 0$. It follows that

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{R}.$$

Since the unitary group G can be raised uniquely to the bounded group $\widetilde{G}$ of Theorem 6, it follows that $\mathcal{B}$ is the unique closed subalgebra satisfying $\mathcal{A} = \mathcal{B} \oplus \mathcal{R}$. ■

12. COROLLARY. *Let X be a locally compact Hausdorff space and*

$$0 \to \mathcal{R} \to \mathcal{A} \to C_0(X) \to 0$$

be a commutative topologically nilpotent extension where $C_0(X)$ is topologically isomorphic to $\mathcal{A}/\mathcal{R}$. Then the extension splits topologically and this splitting is unique.

Proof. Adjoin an identity e to $\mathcal{A}$ to form $\mathcal{A}^\#$. Then $\mathcal{A}^\#$ has radical $\mathcal{R}$ and $\mathcal{A}^\#/\mathcal{R} \simeq C(X^\#)$ where $X^\#$ is the one-point compactification of X. But $\mathcal{A}^\# = \mathcal{B} \oplus \mathcal{R} = \{\lambda e\} \oplus \mathcal{B}_0 \oplus \mathcal{R}$, where $\mathcal{B}_0 \simeq C_0(X)$ and $\mathcal{A} \simeq \mathcal{B}_0 \oplus \mathcal{R}$. Since $\mathcal{B}$ is unique in $\mathcal{A}^\#$, $\mathcal{B}_0$ is unique in $\mathcal{A}$. ■

13. COROLLARY. *The subspace $\mathcal{Y}$ coincides with the subalgebra $\mathcal{B}$.*

Proof. Let $\theta : C(X) \to \mathcal{B}$ be the continuous homomorphism such that $\pi\theta = \operatorname{id}_{C(X)}$ and let $\varrho : C(X) \to \mathcal{Y}$ be the continuous linear map constructed in the proof of Theorem 10. It suffices to show that for f real we have $\theta(f) = \varrho(f)$. Since $\varrho(f)$ is Hermitian equivalent, $\exp i\varrho(f)$ is DPB and $\exp if$ is DPB in $C(X)$. Since θ is continuous, $\exp i\theta(f)$ is DPB in $\mathcal{A}$. But $\pi\circ\theta = \pi\circ\varrho$ and hence $\theta(f) = \varrho(f) + r$. But then by Theorem 6, $\theta(f) = \varrho(f)$. ■

REMARK. In the case of a nilpotent radical $\mathcal{R}$, the splitting of the short exact sequence $0 \to \mathcal{R} \to \mathcal{A} \to C(X) \to 0$ is necessarily algebraically unique by [BC1, Theorem 4.5]. However, if $\mathcal{R}$ is only uniformly topologically nilpotent, the splitting is unique topologically, but not necessarily algebraically. For, assuming the continuum hypothesis, H. G. Dales has shown how, for any infinite compact Hausdorff space X, one can construct an extension $\mathcal{A}$

of $C(X)$ for which the radical $\mathcal{R}$ is the UTN algebra $C_{*,0}$ and for which there are both a strong splitting $\mathcal{A} = \mathcal{B} \oplus \mathcal{R}$ and a splitting $\mathcal{A} = \mathcal{B}_0 \oplus \mathcal{R}$, where $\mathcal{B}_0$ is not closed in $\mathcal{A}$. See the construction in the proof of [Da, Theorem 7.8]. As shown there, a similar construction can be carried out for certain other radicals.

6. Applications. For X locally compact, situations in which the short exact sequence

$$(*) \qquad 0 \to \mathcal{R} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} C_0(X) \to 0$$

arises most naturally occur in the study of problems of spectral synthesis for locally compact Abelian groups. For a locally compact Abelian group G, if we regard $\mathcal{A} = L^1(G) \simeq A(\Gamma)$ via the Gelfand representation of $L^1(G)$, where Γ is the dual group of G, then the short exact sequence $(*)$ describes the existence of a Helson set X of non-spectral synthesis for the group Γ. (For a proof that every non-discrete locally compact group contains such a set see [Sa].) If

$$I(X) = \{f \in A(\Gamma) : f(X) = 0\}$$

and

$$J(X) = \{f \in A(\Gamma) : \operatorname{supp} f \text{ is compact and } (\operatorname{supp} f) \cap X = \emptyset\},$$

then $I(X) \neq \overline{J(X)}$ and $\mathcal{R} = I(X)/\overline{J(X)}$ is the radical of $\mathcal{A} = A(\Gamma)/\overline{J(X)}$. Always $\overline{J(X)^2} = \overline{J(X)}$ and our next result shows that for Helson sets of non-spectral synthesis this relationship holds for both $I(X)$ and the radical $I(X)/\overline{J(X)}$.

14. THEOREM. *Let X be a Helson set of non-spectral synthesis in the locally compact non-discrete Abelian group Γ. Then $\overline{I(X)^2} = I(X)$ and $\overline{\mathcal{R}^2} = \mathcal{R}$.*

Proof. Assume $K \equiv \overline{I(X)^2} \neq I(X)$. Then $\mathcal{B} \equiv A(\Gamma)/K$ has radical $\mathcal{R}_0 = I(X)/K$, which is nilpotent of order 2, and furthermore

$$\mathcal{B}/\mathcal{R}_0 \simeq C_0(X).$$

By Corollary 12 the short exact sequence

$$0 \to \mathcal{R}_0 \to \mathcal{B} \to C_0(X) \to 0$$

splits topologically. Consequently, $\mathcal{B} \simeq C_0(X) \oplus \mathcal{R}_0$ and $\mathcal{R}_0$ is a complemented subspace of $\mathcal{B}$.

Since Γ is an amenable group, $A(\Gamma)$ and hence $\mathcal{B}$ are amenable as Banach algebras. Since $\mathcal{R}_0$ is complemented in $\mathcal{B}$, $\mathcal{R}_0^{\perp}$ is complemented in $\mathcal{B}^*$. Therefore by [CuLo, Theorem 3.7] or [He, II, Proposition 32], $\mathcal{R}_0$ has a bounded approximate identity. This is clearly impossible since $\mathcal{R}_0^2 = 0$. The same argument shows that $\overline{\mathcal{R}^2} = \mathcal{R}$. Indeed it shows that if $K(X)$ is any closed ideal of $A(\Gamma)$ satisfying $J(X) \subset K(X) \subset I(X)$, then

$$\overline{(I(X)/K(X))^2} = I(X)/K(X).$$

An interesting question in this context is whether for a Helson set X of non-synthesis the radical $\mathcal{R} = I(X)/\overline{J(X)}$ can ever be weakly complemented in $\mathcal{A}$, i.e. $\mathcal{R}^{\perp}$ complemented in $\mathcal{A}^*$. This would imply that $\mathcal{R}$ has a bounded approximate identity and consequently $\mathcal{R}$ would be amenable. ■

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Edward Marczewski (Szpilrajn) (1907–1976), one of the most distinguished Polish mathematicians, was a disciple and an active member of the Warsaw School of Mathematics between the two World Wars. His life and work after the Second World War were connected with Wrocław, where he was among the creators of the Polish scientific centre.

Marczewski's main fields of interest were measure theory, descriptive set theory, general topology, probability theory and universal algebra. He also published papers on real and complex analysis, applied mathematics and mathematical logic.

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