EXAMPLE. Let X be the closed unit disc and $Y = T \cup \{0, 1/2, 1/3, \ldots\}$. Let A be the uniform algebra of all functions in $C(X)$ whose restriction to Y is in the restriction to Y of the disc algebra. It is easy to see that X is the Shilov boundary of A, and that the only non-R-points for A are the points of T and the point 0. Thus 0 is an isolated non-R-point for A. In fact, for $y \in Y$, $F_y = \{0, y\} \cup T$. All other points of X are points of continuity for A.

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Dirichlet series and uniform ergodic theorems for linear operators in Banach spaces

by

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Abstract. We study the convergence properties of Dirichlet series for a bounded linear operator T in a Banach space X. For an increasing sequence $\mu = \{\mu_n\}$ of positive numbers and a sequence $f = \{f_n\}$ of functions analytic in neighborhoods of the spectrum $\sigma(T)$, the Dirichlet series for $\{f_n(T)\}$ is defined by

$$D[f, \mu; z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T), \qquad z \in \mathbb{C}.$$

Moreover, we introduce a family of summation methods called Dirichlet methods and study the ergodic properties of Dirichlet averages for T in the uniform operator topology.

1. Introduction. In this paper we attempt to study the Dirichlet series in the ergodic theory setting for a bounded linear operator T in a Banach space X with a view to making up for a gap in the structural properties of the resolvent $R(\lambda; T)$ of T. In particular, the abscissa of uniform convergence of such Dirichlet series is investigated in an operator-theoretical sense. Moreover, we introduce a new summation method of what is called Dirichlet's type generalizing the Abel method and show that when $\|T^n\|/n \to 0$, the uniform $(C, 1)$ ergodicity of T is equivalent to the uniform ergodicity of Dirichlet's type.

Let X be a complex Banach space and let $B[X]$ denote the Banach algebra of bounded linear operators from X to itself. For a given $T \in B[X]$, the *resolvent set* of T, denoted by $\varrho(T)$, is the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ exists as an operator in $B[X]$ with domain X. The *spectrum* of T is the complement of $\varrho(T)$ and is denoted by $\sigma(T)$. $\varrho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a nonempty bounded closed subset of $\mathbb{C}$. So the spectral radius $\gamma(T)$ of T is well defined: in fact $\gamma(T) = \sup|\sigma(T)| = \lim_{n\to\infty} \|T^n\|^{1/n}$. The function $R(\lambda; T)$ defined by $R(\lambda; T) = (\lambda I - T)^{-1}$ for $\lambda \in \varrho(T)$ is called the *resolvent* of T. It is well known ([3], [10]) that $R(\lambda; T)$ is analytic in $\varrho(T)$

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and if $T \in B[X]$ and $|\lambda| > \gamma(T)$, then $\lambda \in \varrho(T)$ and

$$R(\lambda; T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n,$$

the series converging in the uniform operator topology. It is also known that if $d(\lambda)$ denotes the distance from $\lambda \in \mathbb{C}$ to $\sigma(T)$, then $\|R(\lambda; T)\| \geq 1/d(\lambda)$. If we take $\lambda = e^z$, $z = s + it$ $(s, t \in \mathbb{R})$, then the inequality $|\lambda| > \gamma(T)$ implies $s > \log \gamma(T)$ when $\gamma(T) > 0$. This characterization is of great interest in connection with the question of what is the abscissa of uniform convergence of $R(\lambda; T)$ as a series.

In this paper we consider a more general situation. Given $T \in B[X]$ let $\Phi(T)$ denote the class of all functions of a complex variable which are analytic in some open set containing $\sigma(T)$. We consider the Dirichlet series of the following type:

$$D[f, \mu; z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T),$$

where $z \in \mathbb{C}$, $f = \{f_n\}$ $(f_n \in \Phi(T))$ and $\mu = \{\mu_n\}$, $0 \leq \mu_0 < \mu_1 < \ldots < \mu_n \to \infty$.

2. Main results. We first discuss the uniform convergence of the series $D[f, \mu; z](T)$ and the abscissa of convergence. The first result is the following theorem which will play a fundamental role in dealing with Dirichlet averages for operators in $B[X]$.

THEOREM 1. *Let $T \in B[X]$ and $f_n \in \Phi(T)$, $n \geq 0$, and define*

$$a_\mu(f; T) = \begin{cases} \limsup_{n\to\infty} \dfrac{\log \|\sum_{k=0}^n f_k(T)\|}{\mu_n} & \text{if } \limsup_{n\to\infty} \Big\|\sum_{k=0}^n f_k(T)\Big\| > 0, \\ -\infty & \text{if } \limsup_{n\to\infty} \Big\|\sum_{k=0}^n f_k(T)\Big\| = 0, \end{cases}$$

where $f = \{f_n\}$ and $\mu = \{\mu_n\}$. Then the following statements hold:

(1) *If $s > 0$ and $D[f, \mu; s](T)$ converges in the uniform operator topology, then $s \geq a_\mu(f; T)$.*

(2) *When $a_\mu(f; T) < \infty$, the Dirichlet series $D[f, \mu; z](T)$ converges in the uniform operator topology for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \max(0, a_\mu(f; T))$.*

Proof. In order to prove (1), we assume $s > 0$ and that $D[f, \mu; s](T)$ converges in the uniform operator topology. Then there exists a constant $M > 0$, independent of n, such that

$$\Big\|\sum_{k=0}^{n} e^{-\mu_k s} f_k(T)\Big\| \leq M, \qquad n \geq 0.$$

For each $n \geq 0$, set

$$D_n[f, \mu; s](T) = \sum_{k=0}^{n} e^{-\mu_k s} f_k(T).$$

Making use of the partial summation formula of Abel (cf. [1], Theorem 8.27; [9], p. 2) we obtain

$$\begin{aligned} D_n[f, \mu; 0](T) &= \sum_{k=0}^{n} f_k(T) = \sum_{k=0}^{n} \{e^{-\mu_k s} f_k(T)\} e^{\mu_k s} \\ &= \sum_{k=0}^{n-1} \{e^{\mu_k s} - e^{\mu_{k+1} s}\} D_k[f, \mu; s](T) + e^{\mu_n s} D_n[f, \mu; s](T) \end{aligned}$$

and hence

$$\begin{aligned} \|D_n[f, \mu; 0](T)\| &\leq M \sum_{k=0}^{n-1} \{e^{\mu_{k+1} s} - e^{\mu_k s}\} + M e^{\mu_n s} \\ &= M\{2e^{\mu_n s} - e^{\mu_0 s}\} < 2M e^{\mu_n s}. \end{aligned}$$

Now for any given $\delta > 0$, choose an integer $N_1 = N_1(\mu, \delta)$ so large that

$$2M < e^{\mu_n \delta}, \qquad n > N_1,$$

which is possible since $\lim_{n\to\infty} \mu_n = \infty$ by assumption. Then we have

$$\Big\|\sum_{k=0}^{n} f_k(T)\Big\| = \|D_n[f, \mu; 0](T)\| < e^{\mu_n (s+\delta)}$$

for all $n > N_1$, which yields

$$\limsup_{n\to\infty} \frac{\log \|\sum_{k=0}^n f_k(T)\|}{\mu_n} \leq s + \delta$$

and we conclude that $s \geq a_\mu(f; T)$ as asserted.

Next we turn to the proof of (2). Since (2) holds trivially for the case $a_\mu(f; T) = -\infty$, we assume $a_\mu(f; T) > -\infty$. Fix $\delta > 0$ arbitrarily small such that $a_\mu(f; T) + \delta/2 > 0$. By assumption there is an integer $N_2 = N_2(\mu, a_\mu)$ $(a_\mu = a_\mu(f; T))$ so large that

$$\frac{\log \|D_n[f, \mu; 0](T)\|}{\mu_n} < a_\mu(f; T) + \frac{\delta}{2}, \qquad n > N_2,$$

so that

$$\|D_n[f, \mu; 0](T)\| < e^{\mu_n \{a_\mu(f; T) + \delta/2\}}, \qquad n > N_2.$$

Thus writing $a_\mu = a_\mu(f;T)$ for short and using the partial summation formula of Abel again, we have for $n > m+1 > N_2+1$,

$$\sum_{k=m+1}^{n} e^{-\mu_k(a_\mu+\delta)} f_k(T) = \sum_{k=m+1}^{n-1} \{e^{-\mu_k(a_\mu+\delta)} - e^{-\mu_{k+1}(a_\mu+\delta)}\} D_k[f,\mu;0](T) + e^{-\mu_n(a_\mu+\delta)} D_n[f,\mu;0](T) - e^{-\mu_{m+1}(a_\mu+\delta)} D_m[f,\mu;0](T),$$

so for such n and m,

$$\begin{aligned}\Big\|\sum_{k=m+1}^{n} e^{-\mu_k(a_\mu+\delta)} f_k(T)\Big\| &\le \sum_{k=m}^{n-1} e^{\mu_k(a_\mu+\delta/2)}\{e^{-\mu_k(a_\mu+\delta)} - e^{-\mu_{k+1}(a_\mu+\delta)}\} \\ &\quad + e^{\mu_n(a_\mu+\delta/2)-\mu_n(a_\mu+\delta)} + e^{\mu_m(a_\mu+\delta/2)-\mu_m(a_\mu+\delta)} \\ &= (a_\mu+\delta)\sum_{k=m}^{n-1} e^{\mu_k(a_\mu+\delta/2)} \int_{\mu_k}^{\mu_{k+1}} e^{-u(a_\mu+\delta)}\,du \\ &\quad + e^{-(\delta/2)\mu_n} + e^{-(\delta/2)\mu_m} \\ &\le (a_\mu+\delta)\sum_{k=m}^{n-1} \int_{\mu_k}^{\mu_{k+1}} e^{u(a_\mu+\delta/2)-u(a_\mu+\delta)}\,du \\ &\quad + e^{-(\delta/2)\mu_n} + e^{-(\delta/2)\mu_m} \\ &= \frac{2(a_\mu+\delta)}{\delta}\{e^{-(\delta/2)\mu_m} - e^{-(\delta/2)\mu_n}\} \\ &\quad + e^{-(\delta/2)\mu_n} + e^{-(\delta/2)\mu_m}.\end{aligned}$$

This gives

$$\lim_{n,m\to\infty}\Big\|\sum_{k=m+1}^{n} e^{-\mu_k(a_\mu+\delta)} f_k(T)\Big\| = 0,$$

implying that $D[f,\mu;a_\mu+\delta](T)$ converges in the uniform operator topology. Now let $z_0 = (a_\mu+\delta)+i0$ and $z = s+it$ $(s,t\in\mathbb{R})$, $s > a_\mu+\delta$. Since $D[f,\mu;z_0](T)$ converges in $B[X]$, there exists a constant $K>0$ such that

$$\sup_{\substack{n,m\\ 0\le m\le n}} \Big\|\sum_{k=m}^{n} e^{-\mu_k z_0} f_k(T)\Big\| \le K.$$

As before, for $n+1 > m \ge 0$ we obtain

$$\begin{aligned}\sum_{k=m+1}^{n} e^{-\mu_k z} f_k(T) &= \sum_{k=m+1}^{n} \{e^{-\mu_k z_0} f_k(T)\} e^{-\mu_k(z-z_0)} \\ &= \sum_{k=m+1}^{n-1} \{e^{-\mu_k(z-z_0)} - e^{-\mu_{k+1}(z-z_0)}\} D_k[f,\mu;z_0](T) \\ &\quad + e^{-\mu_n(z-z_0)} D_n[f,\mu;z_0](T) \\ &\quad - e^{-\mu_{m+1}(z-z_0)} D_m[f,\mu;z_0](T).\end{aligned}$$

Therefore, for such n and m,

$$\begin{aligned}\Big\|\sum_{k=m+1}^{n} e^{-\mu_k z} f_k(T)\Big\| &\le K \sum_{k=m+1}^{n-1} |e^{-\mu_k(z-z_0)} - e^{-\mu_{k+1}(z-z_0)}| \\ &\quad + K\{e^{-\mu_n \operatorname{Re}(z-z_0)} + e^{-\mu_{m+1}\operatorname{Re}(z-z_0)}\} \\ &\le K \sum_{k=m+1}^{n-1} |z-z_0| \int_{\mu_k}^{\mu_{k+1}} e^{-u\operatorname{Re}(z-z_0)}\,du \\ &\quad + K\{e^{-\mu_n \operatorname{Re}(z-z_0)} + e^{-\mu_m \operatorname{Re}(z-z_0)}\} \\ &\le \frac{K|z-z_0|}{\operatorname{Re}(z-z_0)}\{e^{-\mu_m \operatorname{Re}(z-z_0)} - e^{-\mu_n \operatorname{Re}(z-z_0)}\} \\ &\quad + K\{e^{-\mu_n \operatorname{Re}(z-z_0)} + e^{-\mu_m \operatorname{Re}(z-z_0)}\},\end{aligned}$$

which approaches zero as $n,m\to\infty$. Consequently, $D[f,\mu;z](T)$ converges in the uniform operator topology. The proof of Theorem 1 is complete.

When $0 \le a_\mu(f;T) < \infty$, we say that the number $a_\mu(f;T)$ is the *abscissa of uniform convergence* of the Dirichlet series $D[f,\mu;z](T)$.

EXAMPLE 2. If $T\in B[X]$ satisfies $\sup_{n\ge1}\|T^n\|/n^\omega = C < \infty$ for some real $\omega\ge0$, then $\gamma(T)\le1$, and this yields the uniform convergence of the series for the resolvent $R(\lambda;T)$ for $|\lambda|>1$ (i.e., $\log|\lambda|>0$). This fact can also be restated in terms of $a_\mu(f;T)$. Indeed, if $f=\{f_n\}$, $f_n(T)=T^n$, and $\mu=\{\mu_n\}$, $\mu_n=n+1$, then

$$\begin{aligned}a_\mu(f;T) &\le \limsup_{n\to\infty}\frac{\log\|\sum_{k=0}^n T^k\|}{n+1} \\ &\le \limsup_{n\to\infty}\frac{\log(n+1)+\log(\sup_{0\le k\le N}\|T^k\|+Cn^\omega)}{n+1} = 0,\end{aligned}$$

where N is a positive integer sufficiently large such that $\|T^n\|\le Cn^\omega$ for all $n>N$. Hence Theorem 1 is applicable to yield the uniform convergence of $\sum_{n=0}^\infty e^{-(n+1)s}T^n$.

EXAMPLE 3. If $T\in B[X]$ satisfies $\sup_{n\ge0}\|T^n\|/(n+1)^\omega = D < \infty$ for some real $\omega>0$, then for $z=s+it$ with $s>1+\omega$ we have, with

$\varepsilon = s - (1+\omega) > 0$,

$$\frac{\|T^n\|}{|(n+1)^z|} = \frac{\|T^n\|}{(n+1)^s} = \frac{\|T^n\|}{(n+1)^\omega} \cdot \frac{1}{(n+1)^{s-\omega}} \le \frac{D}{(n+1)^{1+\varepsilon}},$$

which yields the uniform convergence of $\sum_{n=0}^{\infty} T^n/(n+1)^z$. This fact can also be restated in terms of $a_\mu(f;T)$. Indeed, if $f = \{f_n\}$, $f_n(T) = T^n$, and $\mu = \{\mu_n\}$, $\mu_n = \log(n+1)$, then

$$\begin{aligned} a_\mu(f;T) &\le \limsup_{n\to\infty} \frac{\log \|\sum_{k=0}^{n} T^k\|}{\log(n+1)} \\ &\le \limsup_{n\to\infty} \frac{\log(n+1) + \log\{\sup_{0\le k\le N} \|T^k\| + D(n+1)^\omega\}}{\log(n+1)} \\ &= 1+\omega, \end{aligned}$$

where N is a positive integer so large that $\|T^n\| \le D(n+1)^\omega$ for all $n > N$. Hence Theorem 1 is applicable to yield the uniform convergence of $\sum_{n=0}^{\infty} e^{-\{\log(n+1)\}s} T^n$.

As mentioned in the introduction, we now introduce a summation method of Dirichlet's type with a view to relating the properties of $D[f,\mu;z](T)$ for an operator $T \in B[X]$ and the uniform ergodic theorem for T.

Let $\mu = \{\mu_n\}$ $(n \ge 0)$ be a sequence of real numbers satisfying the following conditions:

(i) $\mu_0 \ge 0$ and $\inf_{n\ge 0}\{\mu_{n+1} - \mu_n\} = \delta$ for some $\delta > 0$,

(ii) $\sup_{s>0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n\{e^{-\mu_n s} - e^{-\mu_{n+1} s}\} < \infty$,

where $g(s) = \sum_{n=0}^{\infty} e^{-\mu_n s}$ converges for $s > 0$. The basic assumption is (i) and it also implies the strict monotonicity of $\{\mu_n\}$ and $\mu_n \ge n\delta + \mu_0$. Moreover, it follows that $\lim_{s\to 0+} g(s) = \infty$, because

$$\lim_{s\to 0+} g(s) \ge \lim_{s\to 0+} \sum_{n=0}^{N-1} e^{-\mu_n s} = N$$

for every integer $N > 0$. Condition (ii) is needed whenever we deal with operators which satisfy $\|T^n\|/n^\omega \to 0$ for some $0 < \omega \le 1$. Such a sequence $\mu = \{\mu_n\}$ determines a strongly regular method of summability (Dirichlet summability) which will be called a (D,μ)-*method* in what follows. Then we can define the so-called *Dirichlet averages* $D_s^{(\mu)}[T]$ for T by the formula

$$D_s^{(\mu)}[T] = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n, \qquad s > 0,$$

where $a_\mu(f;T) \le 0$ with $f = \{f_n\}$, $f_n(T) = T^n$, $n \ge 0$.

For example, let $1 \le \alpha < \infty$ and define $\mu_n^{(\alpha)} = \{an+b\}^\alpha$ for some $a > 0$ and $b \ge 0$. Clearly

$$\mu_0^{(\alpha)} = b^\alpha \ge 0, \qquad \inf_{n\ge 0}\{\mu_{n+1}^{(\alpha)} - \mu_n^{(\alpha)}\} = (a+b)^\alpha - b^\alpha = \delta > 0$$

and

$$\sup_{s>0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n\{e^{-(an+b)^\alpha s} - e^{-(a(n+1)+b)^\alpha s}\} < \infty.$$

In particular, when $\mu_n = n+1$, we get the Abel averages $(1-r)\sum_{n=0}^{\infty} r^n T^n$, $0 < r < 1$ (or, equivalently, $(\lambda - 1)R(\lambda;T)$, $\lambda > 1$).

The study of Dirichlet methods is particularly natural, appropriate and interesting because they contain the Abel method as a special case. We are in particular interested in the connection between uniform convergence of Dirichlet averages and Cesàro averages of order α.

THEOREM 4 [1]. *Let $T \in B[X]$ satisfy $\|T^n\|/n \to 0$ as $n \to \infty$. Then the following are equivalent:*

(i) $n^{-1}\sum_{k=0}^{n-1} T^k$ *converges, as $n \to \infty$, in the uniform operator topology.*

(ii) $(1-r)\sum_{n=0}^{\infty} r^n T^n$ *converges, as $r \to 1-$, in the uniform operator topology.*

(iii) *For every (D,μ)-method, $D_s^{(\mu)}[T]$ converges, as $s \to 0+$, in the uniform operator topology.*

(iv) *For some (D,μ)-method, $D_s^{(\mu)}[T]$ converges, as $s \to 0+$, in the uniform operator topology.*

Proof. The proof starts with (iv). Assume that for some (D,μ)-method $\mu = \{\mu_n\}$, $D_s^{(\mu)}[T]$ converges, as $s \to 0+$, to some $E \in B[X]$ in the uniform operator topology. Given small $\varepsilon > 0$, choose a number $N = N(\varepsilon) \ge 1$ such that $\|T^n\| < \varepsilon n$ for all $n > N$. Then we have

$$\begin{aligned} \left\| \frac{1}{g(s)}(I-T)\sum_{n=0}^{\infty} e^{-\mu_n s} T^n \right\| &\le \frac{1}{g(s)}\Big[e^{-\mu_0 s} + \sum_{n=1}^{\infty}\{e^{-\mu_{n-1}s} - e^{-\mu_n s}\}\|T^n\|\Big] \\ &\le \frac{1}{g(s)}\Big[e^{-\mu_0 s} + \sum_{n=1}^{N}\{e^{-\mu_{n-1}s} - e^{-\mu_n s}\}\|T^n\| \\ &\quad + \varepsilon \sum_{n=N+1}^{\infty}\{e^{-\mu_{n-1}s} - e^{-\mu_n s}\}n\Big], \end{aligned}$$

[1] The statement of Theorem 4 is due to the referee's suggestion. This theorem remains valid even if X is a real Banach space.

which tends to zero by first letting $s \to 0+$ and then $\varepsilon \to 0$. (We use the fact that $\sup_{s>0} \|(1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s} T^n\| < \infty$.) That implies that $E = TE = ET$ and

$$E = (\mathrm{uo}) \lim_{s\to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n E = E^2.$$

Hence E is a projection operator and $EX = N(I-T)$. Now it follows that

$$\frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s}(I - T^n) = \frac{1}{g(s)}(I-T) \sum_{k=0}^{\infty} \Big(\sum_{n=k+1}^{\infty} e^{-\mu_n s} \Big) T^k.$$

Let $x \in X$ and $\overline{x} = x - Ex$. Clearly Ex is an element of $N(I-T)$. On the other hand,

$$\begin{aligned}\overline{x} &= (\mathrm{s}) \lim_{s\to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s}(I - T^n)x \\ &= (\mathrm{s}) \lim_{s\to 0+} \frac{1}{g(s)}(I-T) \sum_{k=0}^{\infty} \Big(\sum_{n=k+1}^{\infty} e^{-\mu_n s} \Big) T^k x \in \overline{R(I-T)}.\end{aligned}$$

We claim that $N(I-T) \cap \overline{R(I-T)} = \{0\}$. Let $\varepsilon > 0$ be given as before. If x is of the form $x = (I-T)y + y_0$, $y, y_0 \in X$, $\|y_0\| < \varepsilon$, then

$$\begin{aligned}&\Big\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n x \Big\| \\ &= \Big\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n (I-T)y + \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n y_0 \Big\| \\ &\le \frac{1}{g(s)} \Big\{ e^{-\mu_0 s}\|y\| + \sum_{n=1}^{\infty} (e^{-\mu_{n-1} s} - e^{-\mu_n s})\|T^n y\| + \varepsilon \sum_{n=0}^{\infty} e^{-\mu_n s}\|T^n\| \Big\} \\ &\le \frac{\|y\|}{g(s)} \Big\{ e^{-\mu_0 s} + \sum_{n=1}^{N} (e^{-\mu_{n-1} s} - e^{-\mu_n s})\|T^n\| + \varepsilon \sum_{n=N+1}^{\infty} (e^{-\mu_{n-1} s} - e^{-\mu_n s}) n \Big\} \\ &\quad + \frac{\varepsilon}{g(s)} \Big\{ \sum_{n=0}^{N} e^{-\mu_n s}\|T^n\| + \sum_{n=N+1}^{\infty} e^{-\mu_n s} n \Big\},\end{aligned}$$

so that $\|(1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s} T^n x\| \to 0$ as $s \to 0+$. This means that

$$(\mathrm{so}) \lim_{s\to 0+} \Big(\frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n \Big) \Big| \overline{R(I-T)} = \theta,$$

where θ denotes the zero operator in $B[X]$. Consequently, if $x \in N(I-T) \cap \overline{R(I-T)}$, we get $x = Ex = 0$ as asserted. Evidently $\overline{R(I-T)}$ is invariant

under T and we let $S = T|\overline{R(I-T)}$. Then on $\overline{R(I-T)}$,

$$(\mathrm{uo}) \lim_{s\to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n = \theta.$$

Thus for a fixed s sufficiently small, $I - (1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s} S^n$ is invertible on $\overline{R(I-T)}$. Hence, so is the operator $I - S$ and $R(I-T)$ must be closed because we have

$$I - \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n = \frac{1}{g(s)}(I-S) \sum_{k=0}^{\infty} \Big(\sum_{n=k+1}^{\infty} e^{-\mu_n s} \Big) S^k.$$

We have thus proved that

$$X = N(I-T) \oplus R(I-T), \qquad R(I-T) \text{ is closed.}$$

Hence we may apply Dunford's uniform ergodic theorem ([2], Theorem 3.16) to conclude that

$$(\mathrm{uo}) \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k = E,$$

and (i) follows.

(i) implies (ii) by Hille's theorem ([4], Theorem 6).

(ii) implies the ergodic decomposition (special case of (iv)). One gets

$$\begin{aligned}(I-T)R(I-T) &= (I-T)(I-T)X = (I-T)^2 X \\ &= (I-T)X = R(I-T).\end{aligned}$$

It turns out that $I - T$ is a bijection of $R(I-T)$ onto itself and $R(I-T)$ is invariant under T. Let $S = T|R(I-T)$. Then $I - S$ is invertible on $R(I-T)$. Since by assumption, $\mu_n \ge n\delta + \mu_0$ and $\|T^n\|/n \to 0$ as $n \to \infty$, it follows that $a_\mu(f;T) \le 0$, where $f = \{f_n\}$, $f_n(T) = T^n$. Taking into account that $[g(s)]^{-1} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n$ converges in $B[X]$ for $s > 0 \ge a_\mu(f;T)$ in virtue of Theorem 1, all that is to show is

$$\lim_{s\to 0+} \Big\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n \Big\| = 0.$$

Now for sufficiently small $\varepsilon > 0$, there exists by assumption a positive integer $N = N(\varepsilon)$ such that $\|S^n/n\| < \varepsilon$ for all $n > N$. Then

$$\begin{aligned}\Big\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n \Big\| &\le \frac{1}{g(s)} \|(I-S)^{-1}\| \Big\| \sum_{n=0}^{\infty} e^{-\mu_n s}(I-S)S^n \Big\| \\ &\le \frac{1}{g(s)} \|(I-S)^{-1}\| \Big[e^{-\mu_0 s} + \sum_{n=1}^{\infty} \{e^{-\mu_{n-1} s} - e^{-\mu_n s}\} \|S^n\| \Big]\end{aligned}$$

$$\leq \frac{1}{g(s)}\|(I-S)^{-1}\|\Big[e^{-\mu_0 s}+\sum_{n=1}^{N}\{e^{-\mu_{n-1}s}-e^{-\mu_n s}\}\|S^n\|$$
$$+\varepsilon\sum_{n=N+1}^{\infty}\{e^{-\mu_{n-1}s}-e^{-\mu_n s}\}n\Big],$$

whence the required convergence to 0 by first letting $s\to 0+$ and then $\varepsilon\to 0$ since $\lim_{s\to 0+} g(s)=\infty$ and $\sum_{n=1}^{\infty}\{e^{-\mu_{n-1}s}-e^{-\mu_n s}\}n$ converges uniformly for $s>0$. We have thus proved that (ii) implies (iii). This completes the proof of the theorem.

Next let $0<\alpha<\infty$ and let $A_n^{(\alpha)}$, $n\geq 0$, denote the (C,α) *coefficients* of order α, which means that

$$(1-r)^{-(\alpha+1)}=\sum_{n=0}^{\infty}A_n^{(\alpha)}r^n,\qquad 0<r<1.$$

Then the *Cesàro* (C,α) *averages* $C_n^{(\alpha)}[T]$, $n\geq 0$, of the sequence of powers T^n are defined by

$$C_n^{(\alpha)}[T]=\frac{1}{A_n^{(\alpha)}}\sum_{k=0}^{n}A_{n-k}^{(\alpha-1)}T^k,\qquad n\geq 0.$$

As early as 1945 E. Hille obtained, as applications of Abelian and Tauberian theorems to ergodic theorems, the uniform (strong) ergodic theorems for $T\in B[X]$ with a view to relating the uniform (strong) (C,α) ergodic theorems and the properties of $R(\lambda;T)$ (see [4], Theorems 6 and 7). In particular, the fact that the uniform (strong) convergence of $(\lambda-1)R(\lambda;T)$ as $\lambda\to 1+$ implies the uniform (strong) (C,α) convergence of $\{T^n\}$ has been established on supposing the power-boundedness of T. Hille's uniform ergodic theorem has recently been improved by the author [11], where the power-boundedness of T is replaced by the condition $\lim_{n\to\infty}\|T^n\|/n^{\omega}=0$ with $\omega=\min(1,\alpha)$. Using this fact, we have the following theorem which is a further extension of Theorem 4.

THEOREM 5. *Let $0<\alpha<\infty$ and let $T\in B[X]$ satisfy $\|T^n\|/n^{\omega}\to 0$ as $n\to\infty$, where $\omega=\min(1,\alpha)$. Then the following are equivalent*:

(i) *$C_n^{(\alpha)}[T]$ converges, as $n\to\infty$, in the uniform operator topology.*
(ii) *$(1-r)\sum_{n=0}^{\infty}r^nT^n$ converges, as $r\to 1-$, in the uniform operator topology.*
(iii) *For every (D,μ)-method, $D_s^{(\mu)}[T]$ converges, as $s\to 0+$, in the uniform operator topology.*
(iv) *For some (D,μ)-method, $D_s^{(\mu)}[T]$ converges, as $s\to 0+$, in the uniform operator topology.*

Proof. The equivalence of (i) and (ii) follows from the author's extension of Hille's uniform ergodic theorem ([4], Theorem 1). On the other hand, since $\|T^n\|/n\to 0$ as $n\to\infty$, it follows from Theorem 4 that (ii)–(iv) are equivalent. Hence the theorem follows.

THEOREM 6. *Let $0<\alpha<\infty$ and let $T\in B[X]$ satisfy $\|T^nx\|/n^{\omega}\to 0$ (as $n\to\infty$) for all $x\in X$, where $\omega=\min(1,\alpha)$. Suppose $\sup_{n\geq 0}\|C_n^{(\alpha)}[T]x\|<\infty$ for all $x\in\overline{(I-T)X}$. Then the following are equivalent*:

(i) *For all $x\in X$, $C_n^{(\alpha)}[T]x$ converges strongly as $n\to\infty$.*
(ii) *For all $x\in X$, $(1-r)\sum_{n=0}^{\infty}r^nT^nx$ converges strongly as $r\to 1-$.*
(iii) *For every (D,μ)-method and all $x\in X$, $D_s^{(\mu)}[T]x$ converges strongly as $s\to 0+$.*
(iv) *For some (D,μ)-method and all $x\in X$, $D_s^{(\mu)}[T]x$ converges strongly as $s\to 0+$.*

Proof. The equivalence of (i) and (ii) follows from the author's extension of Hille's strong ergodic theorem ([4], Theorem 2). Next, instead of $\{T^n\}$, we consider the sequence $\{T^nx\}$ for every $x\in X$. Then the proof of the equivalence of (ii)–(iv) follows exactly the same lines as the proof of Theorem 4. The theorem follows.

Following Laursen and Mbekhta [5], we say that $T\in B[X]$ is a *quasi-Fredholm operator* if there exist two closed T-invariant subspaces M and N of X such that

(1) $X=N\oplus M$;
(2) $T|N$ is nilpotent;
(3) $(T|M)(M)$ is closed;
(4) $(T|M)(M)$ contains all subspaces $N((T|M)^n)$, $n\geq 1$.

Using the uniform ergodic theorems in Dunford [2], Lin [6], Mbekhta and Zemánek [8] and Laursen and Mbekhta [5] together with our Theorem 4 we have the following theorem which shows that the uniform ergodic theorem of Dirichlet's type has a close connection with the usual uniform ergodic theorems and the spectral theory of bounded linear operators on X.

THEOREM 7. *Let $T\in B[X]$ and let $\mu=\{\mu_n\}$ be a (D,μ)-method. Then the following statements are equivalent*:

(1) *$n^{-1}\sum_{k=0}^{n-1}T^k$ converges, as $n\to\infty$, in the uniform operator topology.*
(2) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $D_s^{(\mu)}[T]$ converges, as $s\to 0+$, in the uniform operator topology.*
(3) *$\|T^n\|/n\to 0$ as $n\to\infty$ and the point $\lambda=1$ is either in $\varrho(T)$ or else a pole of $R(\lambda;T)$.*

(4) *T satisfies $\|n^{-1}(I-T)^k\sum_{i=0}^{n-1}T^i\| \to 0$ as $n\to\infty$ for some integer $k\ge 1$ and the point $\lambda=1$ is either in $\varrho(T)$ or else a simple pole of $R(\lambda;T)$.*

(5) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $I-T$ is a quasi-Fredholm operator.*

(6) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $\inf\{n\in\mathbb{N}\cup\{0\} : R((I-T)^n) = R((I-T)^{n+1})\}<\infty$.*

(7) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $R((I-T)^k)$ is closed for any integer $k\ge 1$.*

(8) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $X=N(I-T)\oplus R(I-T)$.*

(9) *$\|T^n\|/n\to 0$ as $n\to\infty$ and $N(I-T)+R(I-T)$ is closed.*

THEOREM 8 (2). *Let $T\in B[X]$ satisfy $a_\mu(f;T)\le 0$, where $f=\{f_n\}$, $f_n(T)=T^n$ and $\mu=\{\mu_n\}$, $\mu_n=an+b$ for some $a,b>0$. Let $x\in X$ and suppose that*

(1) *$\sup_{t>0}\|D_t^{(\mu)}[T]\|\le M$ for some constant $M>0$,*

(2) *there exists a sequence $\{t_k\}$, $t_k\to 0$ as $k\to\infty$, and an element $y\in X$ such that $(\mathrm{w})\lim_{k\to\infty}D_{t_k}^{(\mu)}[T]x=y$.*

Then $D_t^{(\mu)}[T]y=y$ for all $t>0$ and $(\mathrm{s})\lim_{t\to 0+}D_t^{(\mu)}[T]x=y$.

Proof. We first show that $D_t^{(\mu)}y=y$ for all $t>0$. For any $t>0$ we choose k so large that $0<t_k<t$. The resolvent equation yields

$$\begin{aligned}D_t^{(\mu)}[T]D_{t_k}^{(\mu)}[T] &= \frac{e^{(a-b)(t+t_k)}}{g(t)g(t_k)}R(e^{at};T)R(e^{at_k};T)\\ &= \frac{e^{(a-b)(t+t_k)}}{g(t)g(t_k)}\frac{1}{e^{at_k}-e^{at}}[R(e^{at};T)-R(e^{at_k};T)]\\ &= \frac{e^{(a-b)t}}{e^{at}-e^{at_k}}\frac{1}{g(t)}D_{t_k}^{(\mu)}[T]-\frac{e^{(a-b)t_k}}{e^{at}-e^{at_k}}\frac{1}{g(t_k)}D_t^{(\mu)}[T].\end{aligned}$$

So, if we let $k\to\infty$, the members of this equation converge in the weak operator topology and we obtain $D_t^{(\mu)}[T]y=y$ for all $t>0$ in the limit. Write $x=y+(x-y)$; then we have $D_t^{(\mu)}x=y+D_t^{(\mu)}[T](x-y)$. The assertion will therefore be proved if we can show that $D_t^{(\mu)}[T](x-y)$ converges strongly (as $t\to 0+$) to zero. Using the resolvent equation we have for $z\in X$ of the form $z=(I-D_1^{(\mu)}[T])u$, $u\in X$,

$$\begin{aligned}D_t^{(\mu)}[T]z &= D_t^{(\mu)}[T]u-D_t^{(\mu)}[T]D_1^{(\mu)}[T]u\\ &= \left\{1-\frac{1}{g(1)}\frac{e^{a-b}}{e^a-e^{at}}\right\}D_t^{(\mu)}[T]u-\frac{1}{g(t)}\frac{e^{(a-b)t}}{e^{at}-e^a}D_1^{(\mu)}[T]u,\end{aligned}$$

which approaches zero in norm as $t\to 0+$ since $\sup_{0<t<1}\|D_t^{(\mu)}[T]\|\le M$. Moreover, the same result is obviously true for all $z\in \overline{R(I-D_1^{(\mu)}[T])}$. Hence $D_t^{(\mu)}[T](x-y)\to 0$ in norm whenever $x-y\in\overline{R(I-D_1^{(\mu)}[T])}$. Now suppose on the contrary that $x-y$ does not belong to $\overline{R(I-D_1^{(\mu)}[T])}$. Then there exists an $x_0^*\in X^*$ such that $x_0^*(x-y)=1$ and $x_0^*(z)=0$ for all $z\in\overline{R(I-D_1^{(\mu)}[T])}$. Since $u-D_1^{(\mu)}[T]u\in R(I-D_1^{(\mu)}[T])$ for any $u\in X$, we have $x_0^*(u-D_1^{(\mu)}[T]u)=0$, i.e., $x_0^*(D_1^{(\mu)}[T]u)=x_0^*(u)$. It follows that

$$\begin{aligned}x_0^*(D_{t_k}^{(\mu)}[T]x) &= x_0^*(D_1^{(\mu)}[T]x)\\ &= \frac{e^{a-b}}{e^a-e^{at_k}}\frac{1}{g(1)}x_0^*(D_{t_k}^{(\mu)}[T]x)-\frac{e^{(a-b)t_k}}{e^a-e^{at_k}}\frac{1}{g(t_k)}x_0^*(x),\end{aligned}$$

so that since $g(t)=e^{(a-b)t}/(e^{at}-1)$, we have $x_0^*(D_{t_k}^{(\mu)}[T]x)=x_0^*(x)$ for all k. In the limit as $k\to\infty$ we obtain $x_0^*(y)=x_0^*(x)$ and this contradicts the assumption that $x_0^*(x-y)=1$. Hence $x-y\in\overline{R(I-D_1^{(\mu)}[T])}$ and $(\mathrm{s})\lim_{t\to 0+}D_t^{(\mu)}[T](x-y)=0$. This finishes the proof of the theorem.

THEOREM 9. *Let $T\in B[X]$ satisfy $\|T^n\|/n^\omega\to 0$ (as $n\to\infty$) for some $0<\omega\le 1$ and let $\mu=\{\mu_n\}$ be a (D,μ)-method. Suppose that for each $x\in X$, $\{D_s^{(\mu)}[T]x : s>0\}$ is weakly relatively compact. Then for each $x\in X$, $D_s^{(\mu)}[T]x$ converges strongly to Ex, where E is the projection of X onto the null space $N(I-T)$ of $I-T$.*

Proof. Let $x\in X$ be arbitrarily fixed. There exists a sequence $\{s_k\}$ with $s_k>0$, $s_k\to 0$ as $k\to\infty$, and an element $y\in X$ such that $(\mathrm{w})\lim_{k\to\infty}D_{s_k}^{(\mu)}[T]x=y$. Using the argument applied in the proof of Theorem 4, we see that $y\in N(I-T)$, $x-y\in\overline{(I-T)X}$, and

$$X=N(I-T)\oplus\overline{(I-T)X}.$$

All that remains is to show that $(\mathrm{so})\lim_{s\to 0+}D_s^{(\mu)}[T]=\theta$ on $\overline{(I-T)X}$. Assume $z\in\overline{(I-T)X}$; then for given $\varepsilon>0$ we can find $u\in X$ such that

(2) It should be noticed that $a_\mu(f;T)\le 0$ does not necessarily imply $\|T^n\|/n^\omega\to 0$ $(\omega\ge 0)$. However, if $T\in B[X]$ satisfies $\|T^n\|/n^\omega\to 0$ (as $n\to\infty$) for some $0<\omega\le 1$, we can use the proof of Theorem 4, which shows that $\|(I-T)D_s^{(\mu)}[T]\|\to 0$ as $s\to 0+$. This yields $(I-T)y=0$, and therefore $D_t^{(\mu)}[T]y=y$ for every $t>0$. Then some of the computations in the proof of Theorem 8 can be shortened.

$\|z-(I-T)u\|<\varepsilon$. Thus, writing $w=z-(I-T)u$, we get

$$D_s^{(\mu)}[T]z=\frac{1}{g(s)}\sum_{n=0}^{\infty}e^{-\mu_n s}T^n(u-Tu+w)$$
$$=\frac{1}{g(s)}\Big\{e^{-\mu_0 s}u+\sum_{n=0}^{\infty}(e^{-\mu_{n+1}s}-e^{-\mu_n s})T^{n+1}u\Big\}+D_s^{(\mu)}[T]w.$$

Since $\{D_s^{(\mu)}[T]x\}$ is assumed to be weakly relatively compact for each $x\in X$, it turns out that the operators $\{D_s^{(\mu)}[T]\}$ are uniformly bounded. Take an integer N so large that $\|T^n\|<\varepsilon n^{\omega}$ for all $n>N$. The uniform boundedness of $\{D_s^{(\mu)}[T]\}$ and the strong regularity of the (D,μ)-method give

$$\|D_s^{(\mu)}[T]z\|\le\frac{1}{g(s)}\Big\{e^{-\mu_0 s}\|u\|+\sum_{n=0}^{N}(e^{-\mu_n s}-e^{-\mu_{n+1}s})\|T^{n+1}u\|\Big\}+\varepsilon M,$$

where M is a positive constant independent of s and ε. Hence we have (s) $\lim_{s\to 0+}D_s^{(\mu)}[T]z=0$ by first letting $s\to 0+$ and then $\varepsilon\to 0$. The proof is complete.

EXAMPLE 10. Let $C_0[0,1]$ be the space of functions $f=f(t)$ continuous for $0\le t\le 1$ which vanish at 0, with $\|f\|=\max\|f(t)\|$. Let $\beta>0$ be any real number. Following Hille [4], we define $Q_\beta f=(I-J_\beta)f$ for $f\in C_0[0,1]$, where

$$(J_\beta f)(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-u)^{\beta-1}f(u)\,du,\qquad 0\le t\le 1.$$

Then for each $n\ge 1$, the iterate $Q_\beta^n f$ has the form

$$(Q_\beta^n f)(t)=f(t)-\int_0^t P_n(t-u,\beta)f(u)\,du,$$

where

$$P_n(w,\beta)=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{w^{k\beta-1}}{\Gamma(k\beta)}.$$

If $0<\beta\le 1$, then by Hille's theorems ([4], Theorems 7 and 11), Q_β is strongly (but not uniformly) (C,α)-ergodic for $\alpha>1/2$. Therefore the operators $\{C_n^{(\alpha)}[Q_\beta]\}$ are uniformly bounded and $\|Q_\beta^n\|/n^\alpha\to 0$ as $n\to\infty$. From this and Theorem 6 we see that for $0<\beta\le 1$, Q_β is strongly (but not uniformly) (D,μ)-ergodic.

Next we define $T_\beta=\Gamma(\beta+1)Q_1J_\beta$ for $\beta\ge 3/2$. Then $\|T_\beta^n\|=O(n^{1/4})$ and T_β is uniformly (C,α)-ergodic for $\alpha>1/4$ (see [11]). From this and Theorem 5 it now follows that T_β is uniformly (D,μ)-ergodic for $\beta\ge 3/2$.

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