Example. Let $X$ be the closed unit disc and $Y = \mathbb{T} \cup \{0, 1/2, 1/3, \ldots\}$. Let $A$ be the uniform algebra of all functions in $C(X)$ whose restriction to $Y$ is in the restriction to $Y$ of the disc algebra. It is easy to see that $X$ is the Shilov boundary of $A$, and that the only non-$R$-points for $A$ are the points of $T$ and the point 0. Thus $0$ is an isolated non-$R$-point for $A$. In fact, for $y \in Y$, $F_y = \{0, y\} \cup \mathbb{T}$. All other points of $X$ are points of continuity for $A$.

References


Dirichlet series and uniform ergodic theorems for linear operators in Banach spaces

by

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Abstract. We study the convergence properties of Dirichlet series for a bounded linear operator $T$ in a Banach space $X$. For an increasing sequence $\mu = \{\mu_n\}$ of positive numbers and a sequence $f = \{f_n\}_n$ of functions analytic in neighborhoods of the spectrum $\sigma(T)$, the Dirichlet series for $\{f_n(T)\}$ is defined by $D[f, \mu, z](T) = \sum_{n=0}^{\infty} e^{-i\mu_n z} f_n(T)$, $z \in \mathbb{C}$. Moreover, we introduce a family of summation methods called Dirichlet methods and study the ergodic properties of Dirichlet averages for $T$ in the uniform operator topology.

1. Introduction. In this paper we attempt to study the Dirichlet series in the ergodic theory setting for a bounded linear operator $T$ in a Banach space $X$ with a view to making up for a gap in the structural properties of the resolvent $R(\lambda; T)$ of $T$. In particular, the abscissa of uniform convergence of such Dirichlet series is investigated in an operator-theoretical sense. Moreover, we introduce a new summation method of what is called Dirichlet's type generalizing the Abel method and show that when $|T^n|/n \to 0$, the uniform $(C, 1)$ ergodicity of $T$ is equivalent to the uniform ergodicity of Dirichlet's type.

Let $X$ be a complex Banach space and let $B[X]$ denote the Banach algebra of bounded linear operators from $X$ to itself. For a given $T \in B[X]$, the resolvent set of $T$, denoted by $\rho(T)$, is the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ exists as an operator in $B[X]$ with domain $X$. The spectrum of $T$ is the complement of $\rho(T)$ and is denoted by $\sigma(T)$. $\rho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a nonempty bounded closed subset of $\mathbb{C}$. So the spectral radius $\gamma(T)$ of $T$ is well defined: in fact $\gamma(T) = \sup \{\sigma(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$. The function $R(\lambda; T)$ defined by $R(\lambda; T) = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$ is called the resolvent of $T$. It is well known ([3], [10]) that $R(\lambda; T)$ is analytic in $\rho(T)$.

and if $T \in B[X]$ and $|\lambda| > \gamma(T)$, then $\lambda \in \sigma(T)$ and
\[
R(\lambda; T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n,
\]
the series converging in the uniform operator topology. It is also known that
if $d(\lambda)$ denotes the distance from $\lambda \in \mathbb{C}$ to $\sigma(T)$, then $\|R(\lambda; T)\| \geq 1/d(\lambda)$. If we take $\lambda = e^{i\theta}$, $\theta = s + it$ ($s, t \in \mathbb{R}$), then the inequality $|\lambda| > \gamma(T)$ implies $s > \log \gamma(T)$ when $\gamma(T) > 0$. This characterization is of great interest in connection with the question of what is the abscissa of uniform convergence of $R(\lambda; T)$ as a series.

In this paper we consider a more general situation. Given $T \in B[X]$ let $\Phi(T)$ denote the class of all functions of a complex variable which are analytic in some open set containing $\sigma(T)$. We consider the Dirichlet series of the following type:
\[
D[f; \mu; \gamma](T) = \sum_{n=0}^{\infty} e^{-\mu_n \gamma} f_n(T),
\]
where $\gamma \in \mathbb{C}$, $f = \{f_n\}$ ($f_n \in \Phi(T)$) and $\mu = \{\mu_n\}$, $0 \leq \mu_0 < \mu_1 < \ldots < \mu_n \to \infty$.

2. Main results. We first discuss the uniform convergence of the series $D[f; \mu; \gamma](T)$ and the abscissa of convergence. The first result is the following theorem which will play a fundamental role in dealing with Dirichlet averages for operators in $B[X]$.

**Theorem 1.** Let $T \in B[X]$ and $f_n \in \Phi(T)$, $n \geq 0$, and define
\[
a_{\mu}(f; T) = \left\{ \begin{array}{ll}
\limsup_{n \to \infty} \frac{\log \| \sum_{k=0}^{n} f_k(T) \|}{\mu_n} & \text{if } \limsup_{n \to \infty} \frac{\sum_{k=0}^{n} f_k(T)}{\mu_n} > 0, \\
-\infty & \text{if } \limsup_{n \to \infty} \frac{\sum_{k=0}^{n} f_k(T)}{\mu_n} = 0,
\end{array} \right.
\]
where $f = \{f_n\}$ and $\mu = \{\mu_n\}$. Then the following statements hold:

1. If $s > 0$ and $D[f; \mu; \gamma](T)$ converges in the uniform operator topology, then $s \geq a_{\mu}(f; T)$.

2. When $a_{\mu}(f; T) < \infty$, the Dirichlet series $D[f; \mu; \gamma](T)$ converges in the uniform operator topology for any $\gamma \in \mathbb{C}$ with $\text{Re}(\gamma) > \max(0, a_{\mu}(f; T))$.

**Proof.** In order to prove (1), we assume $s > 0$ and that $D[f; \mu; \gamma](T)$ converges in the uniform operator topology. Then there exists a constant $M > 0$, independent of $n$, such that
\[
\left\| \sum_{k=0}^{n} e^{-\mu_k \gamma} f_k(T) \right\| \leq M, \quad n \geq 0.
\]
For each $n \geq 0$, set
\[
D_n[f; \mu; \gamma](T) = \sum_{k=0}^{n} e^{-\mu_k \gamma} f_k(T).
\]
Making use of the partial summation formula of Abel (cf. [1], Theorem 8.27; [9], p. 2) we obtain
\[
D_n[f; \mu; 0](T) = \sum_{k=0}^{n} f_k(T) = \sum_{k=0}^{n} \left( e^{-\mu_k \gamma} f_k(T) e^{\mu_k \gamma} \right) = \sum_{k=0}^{n} \left( e^{\mu_k \gamma} - e^{\mu_{k+1} \gamma} \right) D_k[f; \mu; \gamma](T) + e^{\mu_n \gamma} D_n[f; \mu; \gamma](T),
\]
and hence
\[
\left\| D_n[f; \mu; 0](T) \right\| \leq M \sum_{k=0}^{n-1} \left( e^{\mu_{k+1} \gamma} - e^{\mu_k \gamma} \right) + M e^{\mu_n \gamma} = M \left( 2 e^{\mu_n \gamma} - e^{\mu_0 \gamma} \right) < 2 M e^{\mu_n \gamma}.
\]
Now for any given $\delta > 0$, choose an integer $N_1 = N_1(\mu, \delta)$ so large that
\[
2 M < e^{\mu_1 \gamma}, \quad n > N_1,
\]
which is possible since $\lim_{n \to \infty} \mu_n = \infty$ by assumption. Then we have
\[
\left\| \sum_{k=0}^{n} f_k(T) \right\| = \left\| D_n[f; \mu; 0](T) \right\| < e^{\mu_n(\gamma+\delta)},
\]
for all $n > N_1$, which yields
\[
\limsup_{n \to \infty} \frac{\log \| \sum_{k=0}^{n} f_k(T) \|}{\mu_n} \leq s + \delta
\]
and we conclude that $s \geq a_{\mu}(f; T)$ as asserted.

Next we turn to the proof of (2). Since (2) holds trivially for the case $a_{\mu}(f; T) = -\infty$, we assume $a_{\mu}(f; T) > -\infty$. Fix $\delta > 0$ arbitrarily small such that $a_{\mu}(f; T) + \delta/2 > 0$. By assumption there is an integer $N_2 = N_2(\mu, a_{\mu})(a_{\mu} = a_{\mu}(f; T))$ so large that
\[
\frac{\log \| D_n[f; \mu; 0](T) \|}{\mu_n} < a_{\mu}(f; T) + \frac{\delta}{2}, \quad n > N_2,
\]
so that
\[
\left\| D_n[f; \mu; 0](T) \right\| < e^{\mu_n(\alpha_{\mu}(f; T)+\delta/2)}, \quad n > N_2.
\]
Thus writing \(a_\mu = a_\mu(f; T)\) for short and using the partial summation formula of Abel again, we have for \(n > m + 1 > N_2 + 1\),

\[
\sum_{k=m+1}^{n} e^{-\mu_k(z-x_0)} f_k(T) = \sum_{k=m+1}^{n-1} \left\{ e^{-\mu_k(a_\mu+\delta)} - e^{-\mu_{k+1}(a_\mu+\delta)} \right\} D_k[f; \mu; z_0](T) + e^{-\mu_m(a_\mu+\delta)} D_m[f; \mu; z_0](T) - e^{-\mu_{m+1}(a_\mu+\delta)} D_m[f; \mu; z_0](T),
\]

so for such \(n\) and \(m\),

\[
\left\| \sum_{k=m+1}^{n} e^{-\mu_k(z-x_0)} f_k(T) \right\| \leq \sum_{k=m+1}^{n-1} e^{-\mu_k(a_\mu+\delta)} \left\{ e^{-\mu_k(a_\mu+\delta)} - e^{-\mu_{k+1}(a_\mu+\delta)} \right\} + e^{-\mu_m(a_\mu+\delta)} - e^{-\mu_{m+1}(a_\mu+\delta)} + e^{-\mu_m(a_\mu+\delta)} - e^{-\mu_{m+1}(a_\mu+\delta)}
\]

\[
= (a_\mu + \delta) \sum_{k=m}^{n-1} \sum_{\mu_k+1}^{\mu_{k+1}} e^{-\mu_k(u(a_\mu+\delta)/\mu_k)} du + e^{-((\delta/2)\mu_k)} + e^{-((\delta/2)\mu_{k+1})}
\]

\[
\leq (a_\mu + \delta) \sum_{k=m}^{n-1} \sum_{\mu_k+1}^{\mu_{k+1}} e^{-u(a_\mu+\delta)/\mu_k} du + e^{-((\delta/2)\mu_k)} + e^{-((\delta/2)\mu_{k+1})}
\]

\[
= \frac{2(a_\mu + \delta)}{\delta} \left( -e^{-((\delta/2)\mu_k)} - e^{-((\delta/2)\mu_{k+1})} \right) + e^{-((\delta/2)\mu_k)} + e^{-((\delta/2)\mu_{k+1})}.
\]

This gives

\[
\lim_{n,m \to \infty} \left\| \sum_{k=m+1}^{n} e^{-\mu_k(z-x_0)} f_k(T) \right\| = 0,
\]

implying that \(D[f; \mu; z](T)\) converges in the uniform operator topology.

Now let \(z_0 = (a_\mu + \delta) + 0\) and \(z = s + it\) \((s, t \in \mathbb{R}, s > a_\mu + \delta)\). Since \(D[f; \mu; z_0](T)\) converges in \(B[X]\), there exists a constant \(K > 0\) such that

\[
\sup_{n \leq m \leq n} \left\| \sum_{k=m}^{n} e^{-\mu_k x_0} f_k(T) \right\| \leq K.
\]

As before, for \(n+1 > m \geq 0\) we obtain

\[
\sum_{k=m+1}^{n} e^{-\mu_k x_0} f_k(T) = \sum_{k=m+1}^{n} \left\{ e^{-\mu_k x_0} f_k(T) \right\} e^{-\mu_k(z-x_0)}
\]

\[
= \sum_{k=m+1}^{n} \left\{ e^{-\mu_k(z-x_0)} - e^{-\mu_{k+1}(z-x_0)} \right\} D_k[f; \mu; z_0](T) + e^{-\mu_m(z-x_0)} D_m[f; \mu; z_0](T) - e^{-\mu_{m+1}(z-x_0)} D_m[f; \mu; z_0](T).
\]

Therefore, for such \(n\) and \(m\),

\[
\left\| \sum_{k=m+1}^{n} e^{-\mu_k x_0} f_k(T) \right\| \leq K \sum_{k=m+1}^{n-1} \left| e^{-\mu_k(z-x_0)} - e^{-\mu_{k+1}(z-x_0)} \right| + K \left\{ e^{-\mu_m \text{Re}(z-x_0)} + e^{-\mu_{m+1} \text{Re}(z-x_0)} \right\}
\]

\[
\leq K \sum_{k=m+1}^{n-1} \left| z - z_0 \right| \left\{ e^{-\mu_k \text{Re}(z-x_0)} + e^{-\mu_{k+1} \text{Re}(z-x_0)} \right\}
\]

\[
\leq K \left\{ \frac{\left| z - z_0 \right|}{\text{Re}(z-x_0)} \left\{ e^{-\mu_m \text{Re}(z-x_0)} + e^{-\mu_{m+1} \text{Re}(z-x_0)} \right\} + K \left\{ e^{-\mu_m \text{Re}(z-x_0)} + e^{-\mu_{m+1} \text{Re}(z-x_0)} \right\}
\]

which approaches zero as \(n, m \to \infty\). Consequently, \(D[f; \mu; z](T)\) converges in the uniform operator topology. The proof of Theorem 1 is complete.

When \(0 < a_\mu(f; T) < \infty\), we say that the number \(a_\mu(f; T)\) is the \textit{abscissa of uniform convergence} of the Dirichlet series \(D[f; \mu; z](T)\).

**Example 2.** If \(T \in B[X]\) satisfies \(\sup_{n \geq 1} \| T^n \|/\| n^{\omega} = C < \infty\) for some real \(\omega > 0\), then \(\gamma(T) \leq 1\), and this yields the uniform convergence of the series for the resolvent \(R(\lambda; T)\) for \(|\lambda| > 1\). This fact can also be restated in terms of \(a_\mu(f; T)\). Indeed, if \(f = \{f_n\}, f_n(T) = T^n\), and \(\mu = \{\mu_n\}, \mu_n = n + 1\), then

\[
a_\mu(f; T) \leq \lim_{n \to \infty} \sum_{k=0}^{n} T^k \leq \lim_{n \to \infty} \sum_{k=0}^{n} \left| T^k \right| + Cn^{\omega} = 0,
\]

where \(N\) is a positive integer sufficiently large such that \(|T^n| \leq Cn^{\omega}\) for all \(n > N\). Hence Theorem 1 is applicable to yield the uniform convergence of \(\sum_{n=0}^{\infty} e^{-(n+1)^2} T^n\).

**Example 3.** If \(T \in B[X]\) satisfies \(\sup_{n \geq 1} \| T^n \|/(n + 1)^\omega = D < \infty\) for some real \(\omega > 0\), then for \(z = s + it\) with \(s > 1 + \omega\) we have, with
\[ \varepsilon = s - (1 + \omega) > 0, \]
\[ \frac{\|T^n\|}{(n+1)^{s}} = \frac{\|T^n\|}{(n+1)^{\omega}} \cdot \frac{1}{(n+1)^{\varepsilon}} \leq \frac{D}{(n+1)^{1+\varepsilon}}, \]

which yields the uniform convergence of \( \sum_{n=0}^{\infty} T^n/(n+1)^{2} \). This fact can also be restated in terms of \( a_{\mu}(f; T) \). Indeed, if \( f = \{f_n\} \), \( f_n(T) = T^n \), and \( \mu = \{\mu_n\} \), then
\[ a_{\mu}(f; T) \leq \limsup_{n \to \infty} \frac{\log \| \sum_{k=0}^{n} T^k \|}{\log(n+1)} \]
\[ \leq \limsup_{n \to \infty} \frac{\log(n+1) + \log(\sup_{0 \leq k \leq N} \| T^k \| + D(n+1)^{\omega})}{\log(n+1)} \]
\[ = 1 + \omega, \]

where \( N \) is a positive integer so large that \( \|T^n\| \leq D(n+1)^{\omega} \) for all \( n > N \). Hence Theorem 1 is applicable to yield the uniform convergence of \( \sum_{n=0}^{\infty} e^{-\log(n+1)} T^n \).

As mentioned in the introduction, we now introduce a summation method of Dirichlet's type with a view to relating the properties of \( D[f; \mu; x](T) \) for an operator \( T \in B[X] \) and the uniform ergodic theorem for \( T \).

Let \( \mu = \{\mu_n\} \) \( n \geq 0 \) be a sequence of real numbers satisfying the following conditions:

(i) \( \mu_0 \geq 0 \) and \( \inf_{n \geq 0} \{\mu_{n+1} - \mu_n\} = \delta \) for some \( \delta > 0 \),

(ii) \( \sup_{s > 0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n \{e^{-\mu_{n+s}} - e^{-\mu_{n+s+\delta}}\} < \infty, \)

where \( g(s) = \sum_{n=0}^{\infty} e^{-\mu_{n+s}} \) converges for \( s > 0 \). The basic assumption is (i) and it also implies the strict monotonicity of \( \{\mu_n\} \) and \( \mu_n \geq n\delta + \mu_0 \).

Moreover, it follows that \( \lim_{s \to +0} g(s) = \infty \), because
\[ \lim_{s \to +0} g(s) \geq \lim_{s \to +0} \sum_{n=0}^{N-1} e^{-\mu_{n+s}} = N \]

for every integer \( N > 0 \). Condition (ii) is needed whenever we deal with operators which satisfy \( \|T^n\|/n^\omega \to 0 \) for some \( 0 < \omega \leq 1 \). Such a sequence \( \mu = \{\mu_n\} \) determines a strongly regular method of summability (Dirichlet summability) which will be called a \( (D, \mu) \)-method in what follows. Then we can define the so-called \textit{Dirichlet averages} \( D^{(\mu)}(T) \) for \( T \) by the formula
\[ D^{(\mu)}(T) = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_{n+s}} T^n, \quad s > 0, \]

where \( a_{\mu}(f; T) \leq 0 \) with \( f = \{f_n\} \), \( f_n(T) = T^n \), \( n \geq 0 \).

For example, let \( 1 \leq \alpha < \infty \) and define \( \mu^{(\alpha)}_n = (a + b)^\alpha \) for some \( a > 0 \) and \( b \geq 0 \). Clearly
\[ \mu^{(\alpha)}_n = b^\alpha \geq 0, \]
\[ \inf_{n \geq 0} \{\mu^{(\alpha)}_{n+1} - \mu^{(\alpha)}_n\} = (a + b)^\alpha - b^\alpha = \delta > 0 \]

and
\[ \sup_{s > 0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n \{e^{-(a+b)^\alpha + s} - e^{-(a(n+1)+b)^\alpha + s}\} < \infty. \]

In particular, when \( \mu_n = n+1 \), we get the Abel averages \( (1-r) \sum_{n=0}^{\infty} r^n T^n \), \( 0 < r < 1 \) (or, equivalently, \( (\lambda - 1)R(\lambda; T) \), \( \lambda > 1 \)).

The study of Dirichlet methods is particularly natural, appropriate and interesting because they contain the Abel method as a special case. We are in particular interested in the connection between uniform convergence of Dirichlet averages and Cesáro averages of order \( \alpha \).

**Theorem 4.** Let \( T \in B[X] \) satisfy \( \| T^n \|/n \to 0 \) as \( n \to \infty \). Then the following are equivalent:

(i) \( n^{-1} \sum_{k=0}^{n-1} r^{k} \) converges, as \( n \to \infty \), in the uniform operator topology.

(ii) \( (1-r) \sum_{n=0}^{\infty} r^n T^n \) converges, as \( r \to 1^- \), in the uniform operator topology.

(iii) For every \( (D, \mu) \)-method, \( D^{(\mu)}(T) \) converges, as \( s \to 0^+ \), in the uniform operator topology.

(iv) For some \( (D, \mu) \)-method, \( D^{(\mu)}(T) \) converges, as \( s \to 0^+ \), in the uniform operator topology.

**Proof.** The proof starts with (iv). Assume that for some \( (D, \mu) \)-method \( \mu = \{\mu_n\} \), \( D^{(\mu)}(T) \) converges, as \( s \to 0^+ \), to some \( E \in B[X] \) in the uniform operator topology. Given small \( \varepsilon > 0 \), choose a number \( N = N(\varepsilon) \geq 1 \) such that \( \|T^n\| < \varepsilon n! \) for all \( n > N \). Then we have
\[ \frac{1}{g(s)} \left| (I - T) \sum_{n=0}^{\infty} e^{-\mu_{n+s}} T^n \right| \leq \frac{1}{g(s)} \left[ e^{-\mu_{0+s}} + \sum_{n=1}^{\infty} \{e^{-\mu_{n-1+s}} - e^{-\mu_{n+s}}\} \|T^n\| \right] \]
\[ \leq \frac{1}{g(s)} \left[ e^{-\mu_{0+s}} + \sum_{n=1}^{\infty} \{e^{-\mu_{n-1+s}} - e^{-\mu_{n+s}}\} \|T^n\| \right] \]
\[ + \varepsilon \left( \sum_{n=N+1}^{\infty} \{e^{-\mu_{n-1+s}} - e^{-\mu_{n+s}}\} \right), \]

(\textsuperscript{1}) The statement of Theorem 4 is due to the referee's suggestion. This theorem remains valid even if \( X \) is a real Banach space.
which tends to zero by first letting $s \to 0+$ and then $\varepsilon \to 0$. (We use the fact that $\sup_{s > 0} \|(1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s T^n}\| < \infty$.) That implies that $E = T E = E T$ and

$$E = \lim_{s \to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s T^n} E = E^2.$$  

Hence $E$ is a projection operator and $EX = N(I - T)$. Now it follows that

$$\frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} (I - T^n) = \frac{1}{g(s)} (I - T) \sum_{k=0}^{\infty} \left( \sum_{n=k+1}^{\infty} e^{-\mu_n s} \right) T^k.$$  

Let $x \in X$ and $\overline{x} = x - E x$. Clearly $E x$ is an element of $N(I - T)$. On the other hand,

$$\overline{x} = (s) \lim_{s \to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} (I - T^n) x = (s) \lim_{s \to 0+} \frac{1}{g(s)} (I - T) \sum_{k=0}^{\infty} \left( \sum_{n=k+1}^{\infty} e^{-\mu_n s} \right) T^k x \in R(I - T).$$  

We claim that $N(I - T) \cap R(I - T) = \{0\}$. Let $e > 0$ be given as before. If $x$ is of the form $x = (I - T) y + y_0$, $y, y_0 \in X$, $\|y_0\| < e$, then

$$\left\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s T^n} x \right\| = \left\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s T^n} (I - T) y + \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s T^n} y_0 \right\| \\
\leq \frac{1}{g(s)} \left\{ e^{-\mu_0 s} \|y\| + \sum_{n=1}^{\infty} (e^{-\mu_{n-1}s} - e^{-\mu_n s}) \|T^n y\| + e \sum_{n=0}^{\infty} e^{-\mu_n s} \|T^n\| \right\} \\
\leq \frac{\|y\|}{g(s)} \left\{ e^{-\mu_0 s} + \sum_{n=1}^{N} (e^{-\mu_{n-1}s} - e^{-\mu_n s}) \|T^n\| + e \sum_{n=N+1}^{\infty} (e^{-\mu_{n-1}s} - e^{-\mu_n s}) n \right\} \\
+ e \left\{ \sum_{n=0}^{N} e^{-\mu_n s} \|T^n\| + \sum_{n=N+1}^{\infty} e^{-\mu_n s} n \right\},$$

so that $\|(1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s T^n} x\| \to 0$ as $s \to 0+$. This means that

$$\left( \lim_{s \to 0+} \left( \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s T^n} \right) \right) R(I - T) = \theta,$$

where $\theta$ denotes the zero operator in $B[X]$. Consequently, if $x \in N(I - T) \cap R(I - T)$, we get $x = E x = 0$ as asserted. Evidently $R(I - T)$ is invariant under $T$ and we let $S = T R(I - T)$. Then on $R(I - T)$,

$$\left( \lim_{s \to 0+} \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n \right) \theta = \theta.$$  

Thus for a fixed $s$ sufficiently small, $I - (1/g(s)) \sum_{n=0}^{\infty} e^{-\mu_n s} S^n$ is invertible on $R(I - T)$. Hence, so is the operator $I - S$ and $R(I - T)$ must be closed because we have

$$I - \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n = \frac{1}{g(s)} (I - S) \sum_{k=0}^{\infty} \left( \sum_{n=k+1}^{\infty} e^{-\mu_n s} \right) S^k.$$  

We have thus proved that

$$X = N(I - T) \oplus R(I - T), \quad R(I - T) \text{ is closed}.$$  

Hence we may apply Dunford’s uniform ergodic theorem ([2], Theorem 3.16) to conclude that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k = E,$$

and (i) follows.

(i) implies (ii) by Hille’s theorem ([4], Theorem 6).

(ii) implies the ergodic decomposition (special case of (iv)). One gets


It turns out that $I - T$ is a bijection of $R(I - T)$ onto itself and $R(I - T)$ is invariant under $T$. Let $S = T R(I - T)$. Then $I - S$ is invertible on $R(I - T)$.

Since by assumption, $\mu_n \geq \mu_0$ and $\|T^n\|/n \to 0$ as $n \to \infty$, it follows that $a_n(f; T) \leq 0$, where $f = \{f_n\}$, $f_n(T) = T^n$. Taking into account that $\sum_{n=0}^{\infty} e^{-\mu_n s T^n}$ converges in $B[X]$ for $s > 0 \geq a_n(f; T)$ in virtue of Theorem 1, all that is to show is

$$\lim_{s \to 0+} \left\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n \right\| = 0.$$  

Now for sufficiently small $e > 0$, there exists by assumption a positive integer $N = N(e)$ such that $\|S^n\|/n < e$ for all $n > N$. Then

$$\left\| \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} S^n \right\| \leq \frac{1}{g(s)} \sup_{n=0}^{\infty} \left\| (I - S)^{-1} \sum_{n=0}^{\infty} e^{-\mu_n s} (I - S) S^n \right\| \\
\leq \frac{1}{g(s)} \sup_{n=0}^{\infty} \left\| (I - S)^{-1} \left( e^{-\mu_0 s} + \sum_{n=1}^{\infty} e^{-\mu_n s - e^{-\mu_n s}} \right) \right\|.$$
\[
\leq \frac{1}{g(s)} \| (I - S)^{-1} \left( e^{-\mu a} + \sum_{n=1}^{N} (e^{-\mu_{n+1} a} - e^{-\mu_{n} a}) S^n \right) + \varepsilon \sum_{n=N+1}^{\infty} (e^{-\mu_{n+1} a} - e^{-\mu_{n} a}) n, \]
\]
whence the required convergence to 0 by first letting \( s \to 0^+ \) and then \( s \to 0 \)
since \( \lim_{s \to 0^+} g(s) = \infty \) and \( \sum_{n=1}^{\infty} (e^{-\mu_{n+1} a} - e^{-\mu_{n} a}) n \) converges uniformly for \( s > 0 \). We have thus proved that (ii) implies (iii). This completes the proof of the theorem.

Next let \( 0 < \alpha < \infty \) and let \( A_{n}^{(a)} \), \( n \geq 0 \), denote the \((C,\alpha)\) coefficients of order \( \alpha \), which means that
\[
(1 - r)^{-(\alpha + 1)} = \sum_{n=0}^{\infty} A_{n}^{(a)} r^n, \quad 0 < r < 1.
\]
Then the \( (C,\alpha) \) averages \( C_{n}^{(a)}[T], n \geq 0 \), of the sequence of powers \( T^n \) are defined by
\[
C_{n}^{(a)}[T] = \frac{1}{A_{n}^{(a)}} \sum_{k=0}^{n} A_{n-k}^{(a - 1)} T^k, \quad n \geq 0.
\]

As early as 1945 E. Hille obtained, as applications of Abelian and Tauberian theorems to ergodic theorems, the uniform (strong) ergodic theorems for \( T \in B[X] \) with a view to relating the uniform (strong) \((C,\alpha)\) ergodic theorems and the properties of \( R(\lambda; T) \) (see [4], Theorems 6 and 7). In particular, the fact that the uniform (strong) convergence of \( \{ T^n \} \) as \( \lambda \to 1^+ \) implies the uniform (strong) \((C,\alpha)\) convergence of \( \{ T^n \} \) has been established on supposing the power-boundedness of \( T \). Hille’s uniform ergodic theorem has recently been improved by the author [11], where the power-boundedness of \( T \) is replaced by the condition \( \lim_{n \to \infty} \| T^n \|/n^{\omega} = 0 \) with \( \omega = \min(1, \alpha) \). Using this fact, we have the following theorem which is a further extension of Theorem 4.

**Theorem 5.** Let \( 0 < \alpha < \infty \) and let \( T \in B[X] \) satisfy \( \| T^n \|/n^{\omega} \to 0 \) as \( n \to \infty \), where \( \omega = \min(1, \alpha) \). Then the following are equivalent:

(i) \( C_{n}^{(a)}[T] \) converges, as \( n \to \infty \), in the uniform operator topology.
(ii) \( (1 - r) \sum_{n=0}^{\infty} r^n T^n \) converges, as \( r \to 1^- \), in the uniform operator topology.
(iii) For every \((D,\mu)\)-method, \( D_{\mu}^{(a)}[T] \) converges, as \( s \to 0^+ \), in the uniform operator topology.
(iv) For some \((D,\mu)\)-method, \( D_{\mu}^{(a)}[T] \) converges, as \( s \to 0^+ \), in the uniform operator topology.

**Proof.** The equivalence of (i) and (ii) follows from the author’s extension of Hille’s uniform ergodic theorem ([4], Theorem 1). On the other hand, since \( \| T^n \|/n \to 0 \) as \( n \to \infty \), it follows from Theorem 4 that (ii)–(iv) are equivalent. Hence the theorem follows.

**Theorem 6.** Let \( 0 < \alpha < \infty \) and let \( T \in B[X] \) satisfy \( \| T^n \|/n^{\omega} \to 0 \) (as \( n \to \infty \)) for all \( x \in X \), where \( \omega = \min(1, \alpha) \). Suppose \( \sup_{n \geq 0} \| C_{n}^{(a)}[T]x \| < \infty \) for all \( x \in (I - T)X \). Then the following are equivalent:

(i) For all \( x \in X \), \( C_{n}^{(a)}[T]x \) converges strongly as \( n \to \infty \).
(ii) For all \( x \in X \), \( (1 - r) \sum_{n=0}^{\infty} r^n T^n x \) converges strongly as \( r \to 1^- \).
(iii) For every \((D,\mu)\)-method and all \( x \in X \), \( D_{\mu}^{(a)}[T]x \) converges strongly as \( s \to 0^+ \).
(iv) For some \((D,\mu)\)-method and all \( x \in X \), \( D_{\mu}^{(a)}[T]x \) converges strongly as \( s \to 0^+ \).

**Proof.** The equivalence of (i) and (ii) follows from the author’s extension of Hille’s strong ergodic theorem ([4], Theorem 2). Next, instead of \( \{ T^n \} \), we consider the sequence \( \{ T^n x \} \) for every \( x \in X \). Then the proof of the equivalence of (ii)–(iv) follows exactly the same lines as the proof of Theorem 4. The theorem follows.

Following Laursen and Mbekhta [5], we say that \( T \in B[X] \) is a quasi-Fredholm operator if there exist two closed \( T \)-invariant subspaces \( M \) and \( N \) of \( X \) such that

(i) \( X = N \oplus M \);
(ii) \( T[N] \) is nilpotent;
(iii) \( (T[M])^\ast(M) \) is closed;
(iv) \( (T[M])^\ast(M) \) contains all subspaces \( N((T[M])^\ast) \), \( n \geq 1 \).

Using the uniform ergodic theorems in Dunford [2], Lin [6], Mbekhta and Zemánek [8] and Laursen and Mbekhta [5] together with our Theorem 4 we have the following theorem which shows that the uniform ergodic theorem of Dirichlet’s type has a close connection with the usual uniform ergodic theorems and the spectral theory of bounded linear operators on \( X \).
form $z = (I - D_1^{(\mu)}[T])u$, $u \in X$,

$$D_1^{(\mu)}[T]z = D_1^{(\mu)}[T]u - D_1^{(\mu)}[T]D_1^{(\mu)}[T]u$$

$$= \left\{ 1 - \frac{1}{g(t)} \frac{e^{-at}}{e^{at} - e^{at}} \right\} D_1^{(\mu)}[T]u - \frac{1}{g(t)} \frac{e^{ab} - e^{at}}{e^{at} - e^{at}} D_1^{(\mu)}[T]u,$$

which approaches zero in norm as $t \to 0+$ since $\sup_{t < t_1} \|D_1^{(\mu)}[T]\| \leq M$.

Moreover, the same result is obviously true for all $z \in R(I - D_1^{(\mu)}[T])$. Hence $D_1^{(\mu)}[T](x - y) \to 0$ in norm whenever $x - y \in R(I - D_1^{(\mu)}[T])$. Now suppose on the contrary that $x - y$ does not belong to $R(I - D_1^{(\mu)}[T])$. Then there exists an $x_0^* \in X^*$ such that $x_0^*(x - y) = 1$ and $x_0^*(z) = 0$ for all $z \in R(I - D_1^{(\mu)}[T])$. Since $u - D_1^{(\mu)}[T]u \in R(I - D_1^{(\mu)}[T])$ for any $u \in X$, we have $x_0^*(u - D_1^{(\mu)}[T]u) = 0$, i.e., $x_0^*(D_1^{(\mu)}[T]u) = x_0^*(u)$. It follows that

$$x_0^*(D_1^{(\mu)}[T]x) = x_0^*(D_1^{(\mu)}[T]x)$$

$$= \frac{e^{ab} - e^{at}}{e^{at} - e^{at}} \frac{1}{g(t)} x_0^*(D_1^{(\mu)}[T]x) - \frac{e^{ab} - e^{at}}{e^{at} - e^{at}} \frac{1}{g(t)} x_0^*(x),$$

so that since $g(t) = e^{(a+b)t}/(e^{at} - 1)$, we have $x_0^*(D_1^{(\mu)}[T]x) = x_0^*(x)$ for all $k$. In the limit as $k \to \infty$ we obtain $x_0^*(y) = x_0^*(x)$ and this contradicts the assumption that $x_0^*(x - y) = 1$. Hence $x - y \in R(I - D_1^{(\mu)}[T])$ and (s) $\lim_{t \to +} D_1^{(\mu)}[T](x - y) = 0$. This finishes the proof of the theorem.

**Theorem 9.** Let $T \in B[X]$ satisfy $\|T\|/n^\omega \to 0$ (as $n \to \infty$) for some $0 < \omega \leq 1$ and let $\mu = \{\mu_n\}$ be a $(\mu, \nu)$-method. Suppose that for each $x \in X$, $\{D_1^{(\mu)}[T]x : s > 0\}$ is weakly relatively compact. Then for each $x \in X$, $D_1^{(\mu)}[T]x$ converges strongly to $Ex$, where $E$ is the projection of $X$ onto the null space $N(I - T)$ of $I - T$.

**Proof.** Let $x \in X$ be arbitrarily fixed. There exists a sequence $\{s_k\}$ with $s_k > 0$, $s_k \to k \to \infty$, and an element $y \in X$ such that $\lim_{k \to +} D_1^{(\mu)}[T]y = y$. Using the argument applied in the proof of Theorem 4, we see that $y \in N(I - T)$, $x - y \in (I - T)x$, and

$$x = N(I - T) \oplus (I - T)x.$$

All that remains is to show that (s) $\lim_{\epsilon \to 0} D_1^{(\mu)}[T] = \theta$ on $(I - T)x$. Assume $z \in (I - T)x$, then for given $\epsilon > 0$ we can find $u \in X$ such that
\[ \|z - (I - T)u\| < \varepsilon. \] Thus, writing \( w = z - (I - T)u \), we get
\[
D_s^{(u)}[T]z = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n (u - Tu + w) \\
= \frac{1}{g(s)} \left\{ e^{-\mu_n s} u + \sum_{n=0}^{\infty} (e^{-\mu_{n+1} s} - e^{-\mu_n s}) T^{n+1} u \right\} + D_s^{(u)}[T]w.
\]
Since \( \{D_s^{(u)}[T]x\} \) is assumed to be weakly relatively compact for each \( x \in X \), it turns out that the operators \( \{D_s^{(u)}[T]\} \) are uniformly bounded. Take an integer \( N \) so large that \( \|T^n\| < \varepsilon n^N \) for all \( n > N \). The uniform boundedness of \( \{D_s^{(u)}[T]\} \) and the strong regularity of the \((D, \mu)\)-method give
\[
\|D_s^{(u)}[T]z\| \leq \frac{1}{g(s)} \left\{ e^{-\mu_n s} \|u\| + \sum_{n=0}^{N} (e^{-\mu_{n+1} s} - e^{-\mu_n s}) \|T^{n+1} u\| \right\} + \varepsilon M,
\]
where \( M \) is a positive constant independent of \( s \) and \( \varepsilon \). Hence we have \( (s) \lim_{s \to 0^+} D_s^{(u)}[T]z = 0 \) by first letting \( s \to 0^+ \) and then \( \varepsilon \to 0 \). The proof is complete.

**Example 10.** Let \( C_0[0,1] \) be the space of functions \( f = f(t) \) continuous for \( 0 \leq t \leq 1 \) which vanish at 0, with \( \|f\| = \max \|f(t)\| \). Let \( \beta > 0 \) be any real number. Following Hille [4], we define \( Q_{\beta}f = (I - J_\beta)f \) for \( f \in C_0[0,1] \), where
\[
(J_\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta-1} f(u) \, du, \quad 0 \leq t \leq 1.
\]
Then for each \( n \geq 1 \), the iterate \( Q_{\beta}^n f \) has the form
\[
(Q_{\beta}^n f)(t) = f(t) - \int_0^t P_n(t-u, \beta) f(u) \, du,
\]
where
\[
P_n(w, \beta) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{w^{k-1} \beta}{\Gamma(k \beta)}.
\]
If \( 0 < \beta \leq 1 \), then by Hille’s theorems ([4], Theorems 7 and 11), \( Q_{\beta} \) is strongly (but not uniformly) \((C, \alpha)\)-ergodic for \( \alpha > 1/2 \). Therefore the operators \( \{C_0(\alpha)\} \) are uniformly bounded and \( \|Q_{\beta}^n\|/n^\alpha \to 0 \) as \( n \to \infty \).

From this and Theorem 6 we see that for \( 0 < \beta \leq 1 \), \( Q_{\beta} \) is strongly (but not uniformly) \((D, \mu)\)-ergodic.

Next we define \( T_\beta = \Gamma(\beta + 1)Q_1J_\beta \) for \( \beta \geq 3/2 \). Then \( \|T_\beta^n\| = O(n^{\beta/4}) \) and \( T_\beta \) is uniformly \((C, \alpha)\)-ergodic for \( \alpha > 1/4 \) (see [11]). From this and Theorem 5 it now follows that \( T_\beta \) is uniformly \((D, \mu)\)-ergodic for \( \beta \geq 3/2 \).

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