[7] B. V. Gnedenko and A. N. Kolmogorov, *Multiplication of certain non-commuting random variables*, J. Operator Theory 18 (1987), 223–235.

[8] D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables*, CRM Monogr. Ser. 1, Amer. Math. Soc., Providence, RI, 1992.

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Non-regularity for Banach function algebras

by

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Abstract. Let A be a unital Banach function algebra with character space Φ_A. For $x \in \Phi_A$, let M_x and J_x be the ideals of functions vanishing at x and in a neighbourhood of x, respectively. It is shown that the hull of J_x is connected, and that if x does not belong to the Shilov boundary of A then the set $\{y \in \Phi_A : M_x \supseteq J_y\}$ has an infinite connected subset. Various related results are given.

1. Introduction. Let A be a Banach algebra and let $\mathrm{Prim}(A)$ be the set of primitive ideals of A. The *hull-kernel topology* on $\mathrm{Prim}(A)$ is defined by declaring the open sets to be those of the form $\{P \in \mathrm{Prim}(A) : P \not\supseteq I\}$ as I varies through the closed ideals of A. This topology is compact if A has an identity, but not usually Hausdorff, nor even T_1. Indeed it seems, in general, to have few useful properties, and it has not played a prominent part in the general theory of Banach algebras. An attempt to find a more useful topology has been made in [14].

The situation is different, however, for particular classes of Banach algebras, such as C*-algebras and certain L^1-group algebras. Here the hull-kernel topology does have good properties such as local compactness, the Baire property, and (for separable C*-algebras) second countability. These properties have been considerably exploited in C*-algebra theory and abstract harmonic analysis.

For commutative Banach algebras, the hull-kernel topology plays a secondary role. The primitive ideals of a (unital) commutative Banach algebra A are precisely the kernels of characters. Thus $\mathrm{Prim}(A)$ is in bijective correspondence with the character space Φ_A, which carries the compact, Hausdorff Gelfand topology. This is the topology usually employed in the study of commutative Banach algebras, but the hull-kernel topology (defined on Φ_A using the natural bijection) is also used from time to time. The hull-kernel topology is a T_1 topology in this case, and is weaker than the Gelfand topology. Thus the two topologies coincide if and only if the

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hull-kernel topology is Hausdorff, in which case the algebra is said to be *regular*. Even for non-regular algebras, however, it is known that every Gelfand clopen subset of the character space is hull-kernel clopen. This is the celebrated Shilov Idempotent Theorem (see [9; 3.5.13] for example), one of the deepest results in the whole theory. Another interesting result involving the hull-kernel topology is Neumann's characterization of the elements of a commutative Banach algebra which induce a decomposable multiplication operator—these are precisely the elements which are continuous with respect to the hull-kernel topology (see [8]).

The purpose of this paper is to analyse the failure of regularity in a non-regular algebra a little more closely. A first approach, as in C*-algebra theory, might be to regard non-regularity as the failure of the Hausdorff property for the hull-kernel topology. This would lead to the study of separated points (i.e. points which can be separated by disjoint hull-kernel open sets from any point not in their closure). For a separable C*-algebra, the local compactness and the second countability of the hull-kernel topology ensure the existence of a dense subset of separated points. For a separable commutative Banach algebra, however, there might not be any separated points, as the case of the disc algebra shows. Furthermore, this approach fails to take advantage of the Gelfand topology on the character space.

What we do, therefore, is to adopt the approach used in the study of spectral synthesis. Let A be a Banach function algebra on a compact Hausdorff space X, and for $x \in X$, let M_x and J_x be the ideals of functions vanishing at x and in a neighbourhood of x, respectively. The standard notion is that if J_x is dense in M_x then A is *strongly regular* at x. If A is not strongly regular at x, there is still the possibility that x is the only point in the hull of J_x, i.e. in the set $\{y \in X : M_y \supseteq J_x\}$. In this case we will say that x is an *R-point*. If x is not an R-point, so that the hull of J_x is non-trivial, then the properties of the hull of J_x become interesting. An investigation along these lines, for the non-regular algebra $H^\infty + C$, is conducted in [5]. One question that arose in that work, which we are able to answer, is whether the hull of J_x is necessarily connected.

As well as the hull of J_x, another set which it is natural to consider is the set which we call F_x, defined by $F_x = \{y \in x : M_x \supseteq J_y\}$. If F_x is a singleton, we say that x is a *point of continuity*. In Proposition 2.2 we show that a point x is a separated point in X if and only if it is both an R-point and a point of continuity. Thus our general approach, given a non-regular Banach function algebra, is to ask the following questions. Firstly, how badly does regularity fail—how many R-points and non-R-points are there, and how many points of continuity and discontinuity? Secondly, if x is a non-R-point, or a point of discontinuity, how large is the hull of J_x or the set F_x? Are they finite or infinite, countable or uncountable, connected or disconnected? As we shall see, the answers to these questions can vary depending on such things as whether the Banach function algebra is natural, or whether it is a uniform algebra.

The structure of the paper is as follows. In Section 2 we introduce the various definitions in more detail, and consider some conditions which ensure that there are an abundance of R-points, or points of continuity. In Section 3 we consider non-R-points. The main result is that if A is a natural Banach function algebra then the hull of J_x is connected for each point $x \in \Phi_A$. Thus the hull of J_x is either a singleton, in which case x is an R-point, or it is uncountable. In Section 4 we consider points of discontinuity. The main results are that if A is a natural Banach function algebra and x does not belong to the Shilov boundary of A then F_x has an infinite connected subset, while if A is a uniform algebra and x is a point of discontinuity then F_x has a non-empty perfect subset.

Let us conclude this introduction by mentioning an interesting consequence of our work. One striking difference between C*-algebras and commutative Banach algebras is that whereas there are simple examples of C*-algebras with hull-kernel topology which is non-Hausdorff, but very close to being Hausdorff, the standard examples of non-regular Banach function algebras all have highly non-Hausdorff hull-kernel topologies. For example, let A be the C*-algebra of all sequences of two-by-two complex matrices which converge to a diagonal matrix at infinity. Then $\mathrm{Prim}(A)$ is isomorphic to the set of natural numbers, with a double point at infinity, so there are only two non-separated points in $\mathrm{Prim}(A)$. The disc algebra, however, which is the most familiar non-regular Banach function algebra, has character space equal to the disc, and the hull-kernel topology on this space is only a little stronger than the cofinite topology (the weakest possible T_1 topology); see [6; p. 89] for a description. Part of the motivation for this paper was the search for a non-regular Banach function algebra with a hull-kernel topology which was close to being Hausdorff—perhaps with only a finite or countable number of non-separated points. Theorem 3.2 shows, however, that the set of points which cannot be separated from a given point is connected, and hence is either a singleton (the point itself) or uncountable. Thus if a Banach function algebra is not regular, its hull-kernel topology must be very far from Hausdorff.

2. Separated points, R-points, and points of continuity.

In this section we establish various basic results about separated points, R-points, and points of continuity, and consider some conditions which ensure an abundance of such points.

Let X be a topological space and let $x, y \in X$. Then $x \sim y$ if x and y cannot be separated by disjoint open sets in X. A point of X is a *separated point* if it can be separated from every point not in its closure.

Now suppose that A is a Banach function algebra on a compact, Hausdorff space X. In this setting, when discussing separated points, we will always work with the hull-kernel topology on X. Evidently, $x \sim y$ whenever either $M_x \supseteq J_y$ or $J_x \subseteq M_y$. Let us say that x is an *R-point* if for all $y \in X$ with $y \neq x$, $J_x \not\subseteq M_y$ (or in other words if, working on X, the hull of J_x is just $\{x\}$). We say that x is a *point of continuity* if for $y \neq x$, $J_y \not\subseteq M_x$ (or, in terms of the notation introduced earlier, if $F_x = \{x\}$). Recall that A is *regular on* X if every point of X is an R-point, or equivalently, if every point of X is a point of continuity. We say that the algebra A is *regular* if it is regular on Φ_A. The algebra A is *normal on* X if, for every pair of disjoint closed sets E, F contained in X, there is an $f \in A$ with $f(E) \subseteq \{0\}$ and $f(F) \subseteq \{1\}$; A is *normal* if it is normal on Φ_A. It is standard (see [16; 27.2] for example) that every regular Banach function algebra is normal.

With A and X as above, we denote the Shilov boundary of A by Γ_A. The algebra A can clearly be regarded as a Banach function algebra on Φ_A, or on Γ_A if we wish. However, we will also consider cases where X is neither equal to the character space nor the Shilov boundary. In the case where $X = \Phi_A$ we say that A is *natural on* X.

Unless otherwise specified, we shall only consider unital Banach function algebras.

LEMMA 2.1. *Let A be a Banach function algebra on a compact Hausdorff space X, and let $x \in X$. The following are equivalent*:

(i) *x is a point of continuity*,

(ii) *every Gelfand neighbourhood of x contains a hull-kernel neighbourhood of x*,

(iii) *every net in X which converges to x in the hull-kernel topology converges to x in the Gelfand topology.*

Proof. The equivalence of (ii) and (iii) is a simple matter of general topology. Suppose then that (i) holds. Let (x_α) be a net in X converging to x in the hull-kernel topology. Suppose for a contradiction that (x_α) does not converge to x in the Gelfand topology. Then by passing to a subnet if necessary, we may suppose that (x_α) converges to y in the Gelfand topology, for some $y \in X$ with $y \neq x$. Since x is a point of continuity, there exists $f \in A$ such that $f \in J_y$ and $f(x) \neq 0$. Hence eventually $f(x_\alpha) = 0$, since $f \in J_y$ and (x_α) converges to y in the Gelfand topology. On the other hand, eventually $f(x_\alpha) \neq 0$, since (x_α) converges to x in the hull-kernel topology. This contradiction shows that (x_α) must converge to x in the Gelfand topology after all. Hence (iii) holds.

Finally, suppose that (ii) holds. Let $y \in X$ with $y \neq x$. Let N be a Gelfand neighbourhood of x such that y is not in the Gelfand closure of N.

By assumption there exists $f \in A$ with f vanishing outside N and such that $f(x) \neq 0$. Hence $f \in J_y$, so $M_x \not\supseteq J_y$. Thus (i) holds. ∎

It follows from Lemma 2.1(ii) that if $\Gamma_A \subseteq X$ then every point of continuity is contained in the Shilov boundary of A. In particular, this is the case if A is natural on X, or if A is a uniform algebra on X.

Recall that for a Banach function algebra on a compact, Hausdorff space X, a point $x \in X$ is an *independent point* if for every ε with $0 < \varepsilon < 1$ and every neighbourhood N of x, there exists $f \in A$ with $f(x) = 1$ and $|f|_{X\setminus N} < \varepsilon$ (where $|f|_{X\setminus N}$ is defined to be $\sup\{|f(x)| : x \in X\setminus N\}$). Lemma 2.1(ii) shows that every point of continuity is an independent point.

PROPOSITION 2.2. *Let A be a Banach function algebra on a compact Hausdorff space X. Then a point $x \in X$ is a separated point if and only if it is both an R-point and a point of continuity.*

Proof. Suppose first that x is a separated point. Let $y \in X$ with $y \neq x$. Then there exist $f, g \in A$ with $fg = 0$ and $f(x) \neq 0$, $g(y) \neq 0$. Thus $f \in J_y$ but $f \notin M_x$, while $g \in J_x$ but $g \notin M_y$. Hence $M_x \not\supseteq J_y$ and $M_y \not\supseteq J_x$. Since this is true for all $y \neq x$, x is an R-point and a point of continuity.

Now suppose that x is both an R-point and a point of continuity. Let $y \in X$ with $y \neq x$. Since x is an R-point, $M_y \not\supseteq J_x$, so there exists $g \in J_x$ such that $g(y) \neq 0$. But x is a point of continuity, so by Lemma 2.1(ii) the Gelfand neighbourhood of x on which g vanishes must contain a hull-kernel neighbourhood of x. Thus there exists $f \in A$ such that $fg = 0$, and $f(x) \neq 0$. Hence $x \not\sim y$. Since this is true for all $y \neq x$, x is a separated point. ∎

Recall that a Banach function algebra A on a compact Hausdorff space X is *weakly regular on* X if every non-empty Gelfand open subset of X contains a non-empty hull-kernel open set. If A is weakly regular and uniform on X, then X is necessarily the Shilov boundary of A. The standard example of a weakly regular algebra which is not regular is the "tomato-can algebra", which is the uniform algebra of continuous functions on a solid cylinder which are analytic on the base of the cylinder. This is weakly regular on its character space, which is the solid cylinder.

For a subset U of X, let $\overline{U}^{h\text{-}k}$ and $\overline{U}^{G}$ denote the closures of U in the hull-kernel and Gelfand topologies, respectively.

THEOREM 2.3. *Let A be a Banach function algebra on a compact metrizable space X. Then A is weakly regular on X if and only if the set of points of continuity is Gelfand dense in X. In this case the set of points of continuity contains a dense G_δ of X in the Gelfand topology.*

Proof. Suppose first that the set of points of continuity is dense in X in the Gelfand topology. Then Lemma 2.1(ii) shows that A is weakly regular on X.

Conversely, suppose that A is weakly regular on X. Let $(U_i)_{i=1}^{\infty}$ be a base for the Gelfand topology on X. For each i, the set $\overline{U}_i^{h\text{-}k}\setminus U_i$ is a hull-kernel closed set with no hull-kernel interior, and hence no Gelfand interior. Thus $Y := \bigcup_{i=1}^{\infty} \overline{U}_i^{h\text{-}k}\setminus U_i$ is a Gelfand meagre subset of X. Let $y \in X\setminus Y$. Then for any $x \in X$ with $x \neq y$, there exists a U_i containing x but not y. Since $y \notin \overline{U}_i^{h\text{-}k}$, there exists a function $f \in A$ which is non-zero at y but vanishes on the Gelfand neighbourhood U_i of x. Hence $M_y \not\supseteq J_x$. This shows that y is a point of continuity. ∎

It follows from Theorem 2.3 that if A is weakly regular on a metrizable space X then the hull of J_x has empty interior for each $x \in X$. It would be interesting to know whether Theorem 2.3 can be improved to show that X has to have a dense G_δ of separated points.

THEOREM 2.4. *Let A be a Banach function algebra on a compact Hausdorff space X, and let $y \in X$. If the hull-kernel topology is first countable at y then F_y has no Gelfand interior. If the hull-kernel topology is second countable on X then the set of R-points contains a dense G_δ of X in the Gelfand topology.*

Proof. Let V be a Gelfand open subset of Φ_A not containing y. Let $(U_i)_{i=1}^{\infty}$ be a base for the hull-kernel topology at y. For each i, set $V_i = V \cap U_i$. Then for each $x \in V$ there is an i such that $x \notin V_i$. Hence $\bigcap_{i=1}^{\infty} V_i$ is empty. Since V is a Baire space, it follows that at least one V_i is not dense in V in the Gelfand topology. Thus there exists $f \in A$ such that $f(y) \neq 0$ but f vanishes on a Gelfand open subset, W say, of V. Thus for $x \in W$, $x \notin F_y$. It follows that F_y has empty Gelfand interior.

Now let $(U_i)_{i=1}^{\infty}$ be a base for the hull-kernel topology on X. Then for each i, $\overline{U}_i^{G}\setminus U_i$ is a Gelfand closed set with no Gelfand interior. Hence $Y := \bigcup_{i=1}^{\infty} \overline{U}_i^{G}\setminus U_i$ is a meagre subset of X. Let $y \in X\setminus Y$. Then since $(U_i)_{i=1}^{\infty}$ is a base for the hull-kernel topology on X, for any $x \in X$ there exists a U_i containing x but not y. Since $y \notin \overline{U}_i^{G}$, there exists a function $f \in A$ which is non-zero at x but vanishes in a Gelfand neighbourhood of y. Hence $M_x \not\supseteq J_y$. This shows that y is an R-point. ∎

Lemma 2.1(ii) shows that the hull-kernel topology is first countable at every point of continuity, provided that the space X is first countable in the Gelfand topology.

EXAMPLE. Let $X = \{0, 1, 1/2, 1/3, \ldots\}$, and let A be the restriction to X of the disc algebra. By the identity principle, the restriction map is an isomorphism, and so A is a Banach function algebra on X (where the norm is the uniform norm of the functions on the closed unit disc). The hull-kernel topology on X is simply the cofinite topology, which is second countable because X is countable. By Theorem 2.4, therefore, the set of R-points is dense in X. In fact, every point except 0 is an R-point, while 0 is the only point of continuity.

We do not have an example of a natural, non-regular Banach function algebra for which the hull-kernel topology is second countable, or even first countable. It could well be that such things do not exist. A partial result in this direction is given in Section 4.

3. Non-R-points. In this section we consider what happens when there is a non-R-point x. We show that the hull of J_x is connected, provided one is working on the character space. Thus there must be uncountably many points of discontinuity, and also uncountably many points which cannot be separated from x in the hull-kernel topology by disjoint open sets. We also show that, although there might not be a net in X converging to each point of the hull of J_x in the hull-kernel topology, if A is a separable Banach function algebra then the set of points x which do have this property contains a dense G_δ subset of X.

PROPOSITION 3.1. *Let A be a natural, unital Banach function algebra. Suppose that (x_α) is a net in Φ_A such that every hull-kernel cluster point of (x_α) is a hull-kernel limit point, and such that that there is an $x \in \Phi_A$ to which (x_α) converges in the Gelfand topology. Then the set of hull-kernel limit points of (x_α) is Gelfand connected.*

Proof. Let L be the set of hull-kernel limit points of (x_α). Note that L is hull-kernel closed, and hence also Gelfand closed. Suppose, for a contradiction, that L is a disjoint union of two non-empty Gelfand closed sets, M and N say. Let S and T be disjoint Gelfand open subsets of Φ_A containing M and N, respectively. For each α, let K_α be the hull-kernel closure of the set $\{x_\beta : \beta \geq \alpha\}$. Then $L = \bigcap_\alpha K_\alpha$ [7; Theorem 2.7], and each K_α is Gelfand closed, so a simple topological argument shows that there is an α_0 such that for all $\alpha \geq \alpha_0$, $K_\alpha \subseteq S \cup T$. If we suppose that $x \in M \subseteq S$ then there is a $\gamma \geq \alpha_0$ such that for all $\alpha \geq \gamma$, $x_\alpha \in S$. The quotient Banach algebra $A/I(K_\gamma)$ has the disconnected maximal ideal space K_γ. By the Shilov Idempotent Theorem there exists $f \in A$ such that f is zero on $K_\gamma \cap S$, but f equals one on $K_\gamma \cap T$. But then $f(x_\alpha) = 0$ for all $\alpha \geq \gamma$. Since the zero set of f is hull-kernel closed, this contradicts the hull-kernel convergence of (x_α) to points in N. Hence L is connected. ∎

THEOREM 3.2. *Let A be a natural, unital Banach function algebra. Let $x \in \Phi_A$. Then*

(i) *the hull of J_x is Gelfand connected,*
(ii) *the set $\{y \in \Phi_A : x \sim y\}$ is Gelfand connected.*

Proof. (i) Let E be the hull of J_x. Suppose that $y \in E$ with $y \neq x$. Then every hull-kernel neighbourhood of y has non-empty intersection with every Gelfand neighbourhood of x. Thus there is a net (x_α) in Φ_A converging to x in the Gelfand topology, and to y in the hull-kernel topology. By passing to a universal subnet we may assume that every hull-kernel cluster point of (x_α) is a limit point. Thus the set L of hull-kernel limit points of (x_α) is Gelfand connected, by Proposition 3.1. But if $z \in L$ then z cannot have a hull-kernel neighbourhood disjoint from a Gelfand neighbourhood of x, so we must have $M_z \supseteq J_x$. Thus L is a Gelfand connected subset of E, and $y \in L$. Hence every point of E is in the same Gelfand connected component of E as x, so E is Gelfand connected.

(ii) Let H be the set $\{z \in \Phi_A : x \sim z\}$. Let $y \in \Phi_A \backslash \{x\}$ with $y \sim x$. Let (x_α) be a net in Φ_A converging to both y and x in the hull-kernel topology. By passing to a universal subnet we may suppose that (x_α) is Gelfand convergent, and that every hull-kernel cluster point of (x_α) is a limit point. Thus the set L of hull-kernel limit points of (x_α) is connected, by Proposition 3.1. But for each $z \in L$, $z \sim x$. Thus L is a Gelfand connected subset of H, and $y \in L$. As above it follows that H is Gelfand connected. ∎

Theorem 3.2(i) answers one of the questions posed by Gorkin and Mortini [5].

One immediate consequence of Theorem 3.2 is the following.

COROLLARY 3.3. *Let A be a unital Banach function algebra. Then Φ_A has no isolated points of discontinuity.*

We now give examples to show that for a non-natural Banach function algebra the hull of J_x does not have to be connected, and there may be isolated points of discontinuity.

EXAMPLES. Let A be the disc algebra on the countable space X described in the Example after Theorem 2.4. Then the hull of J_0 is neither connected nor uncountable. The only point of continuity is 0, so every other point of X is an isolated point of discontinuity.

It is possible for a uniform algebra to have a solitary point of discontinuity. For example, let A be the uniform algebra obtained by restricting H^∞ (the algebra of bounded functions on the disc which are analytic on the open disc) to the fibre of its maximal ideal space associated with a point on the unit circle (see [6; p. 187 ff.]). Then A is regular on its Shilov boundary Γ_A, but A is not normal. Now consider A as a uniform algebra on $X = \Gamma_A \cup \{y\}$, where y is any point of $\Phi_A \backslash \Gamma_A$. Then y is the solitary point of discontinuity for A on X (although there must be many non-R-points—see the results in Section 4).

This example also shows that it is possible for the hull of J_x to have exactly two points, because if x is any element of $F_y \backslash \{y\}$ then the hull of J_x is precisely the set $\{x, y\}$. An easy modification produces an example of a uniform algebra on its Shilov boundary with a J_x having a two-point hull. Simply form a new compact space Y by gluing an interval to X with endpoints at x and y above, and take the uniform algebra of all continuous functions on Y whose restriction to X is in A.

Here is another consequence of Theorem 3.2.

COROLLARY 3.4. *Let A be a Banach function algebra with $\Phi_A = [0,1]$. If A is weakly regular then A is normal.*

Proof. By Theorem 2.3, the set of points of continuity contains a dense G_δ of $[0,1]$ in the Gelfand topology. Suppose that $x \in [0,1]$ with x not an R-point. Then the hull of J_x is a connected subset of $[0,1]$ by Theorem 3.2(i), and hence is an interval. Thus it contains points of continuity other than x itself, contradicting the definition of a point of continuity. Thus every point of $[0,1]$ is an R-point, so A is regular, and hence normal. ∎

If the condition on the character space is dropped, there are non-regular, weakly regular uniform algebras on $[0,1]$. For example, let A be the non-trivial uniform algebra on the Cantor set described in [17; 9.3]. The character space of A is the whole of the Riemann sphere (see [17; 9.2, 9.3]), so [16; 27.3] shows that A is not normal on the Cantor set. Let B be the algebra of continuous functions on $[0,1]$ whose restrictions to the Cantor set lie in A. Then B is a non-normal uniform algebra on $[0,1]$. Each point of $[0,1]$ not in the Cantor set is a separated point, so B is weakly regular on $[0,1]$. The character space of B is not equal to $[0,1]$ since it contains a copy of the Riemann sphere.

Now let A be a Banach function algebra on a compact Hausdorff space X, and let $x \in X$. Then for y in the hull of J_x there is a net converging to both x and y in the hull-kernel topology. It is not necessarily the case, however, that a net can be found converging to an arbitrary pair of points in the hull of J_x. Consider the following example.

EXAMPLE. Let A be the disc algebra on the disc X. Let x be a fixed point in X, and let $B = \{(f,g) \in A \oplus A : f(x) = g(x)\}$. Then the character space of B consists of two copies of the disc glued at the point x. The ideal J_x is the zero ideal, so the hull of J_x is the whole of Φ_B. But if y and z belong to different copies of the disc then there is no net in Φ_B converging simultaneously to both y and z in the hull-kernel topology.

To show that there are x's for which the hull of J_x is contained in a hull-kernel limit set, we require the notion of a primal ideal. Recall that an ideal I in a commutative ring R is *primal* if whenever $a_1, \ldots, a_n \in R$

with $a_1 \dots a_n = 0$, then $a_i \in I$ for at least one $i \in \{1, \dots, n\}$ (see [13] for example). It is a straightforward piece of general topology to show that if A is a Banach function algebra and I is a closed primal ideal of A, then there is a net in Φ_A converging to every point in the hull of I in the hull-kernel topology (see [1; 3.2]).

We shall show that if A is a separable Banach function algebra then, in the sense of Baire category, most of the ideals $\overline{J}_x$ are primal. First we observe that every closed primal ideal contains a J_x.

LEMMA 3.5. *Let A be a Banach function algebra on a compact Hausdorff space X. Let P be a closed primal ideal of A. Then there exists $x \in X$ such that $P \supseteq J_x$.*

Proof. Suppose for a contradiction that $P \not\supseteq J_x$ for each $x \in X$. Thus for each $x \in X$ there exists $f \in A$ such that f vanishes in a neighbourhood of x, but $f \notin P$. By a compactness argument, there exist a finite number of functions $f_1, \dots, f_n$ such that $f_1 \dots f_n = 0$, with $f_i \notin P$ $(1 \le i \le n)$. This contradicts the assumption that P is primal. Hence P contains J_x for some $x \in X$. ∎

LEMMA 3.6. *Let A be a Banach function algebra on a compact Hausdorff space X. For $f \in A$ the function $x \mapsto \|f + \overline{J}_x\|$ is upper semicontinuous on X.*

Proof. Let $f \in A$, $x \in X$, and let $\varepsilon > 0$ be given. Then there exists $g \in J_x$ such that $\|f - g\| < \|f + \overline{J}_x\| + \varepsilon$. But $g \in J_y$ for all y in a neighbourhood N of x, so $\|f + \overline{J}_y\| \le \|f - g\| < \|f + \overline{J}_x\| + \varepsilon$ for all $y \in N$. Thus the norm function $x \mapsto \|f + \overline{J}_x\|$ is upper semicontinuous on X. ∎

For an upper semicontinuous function f on a Baire space X, the set of points of continuity of f contains a dense G_δ of X (see [3; B18] for example). Suppose now that A is a separable Banach function algebra on a compact Hausdorff space X, and let $\{f_i\}_{i=1}^{\infty}$ be a countable dense subset of A. Then X is a Baire space, so there is a dense G_δ Y of X consisting of points at which all the norm functions $x \mapsto \|f_i + \overline{J}_x\|$ are continuous. But is is straightforward to check that the continuity of these norm functions, at a particular point, for a dense subset of A forces the continuity of all the norm functions at that point. Thus if A is separable there is a dense G_δ of X consisting of points at which every norm function $x \mapsto \|f + \overline{J}_x\|$ is continuous.

PROPOSITION 3.7. *Let A be a separable Banach function algebra on a compact Hausdorff space X. Then the set of $x \in X$ for which $\overline{J}_x$ is primal contains a dense G_δ in X.*

Proof. By the remarks above, there is a dense G_δ $Y \subseteq X$ such that for each $y \in Y$ every norm function is continuous at y. Let $y \in Y$. We show that $\overline{J}_y$ is primal. Suppose that $f_1, \dots, f_n$ are a finite number of elements of A which are not in $\overline{J}_y$. By continuity of the norm functions at y, there is a neighbourhood N of y in X such that $f_i \notin \overline{J}_x$ for $x \in N$. For each $i \in \{1, \dots, n\}$, let $N_i = \{x \in N : f_i(x) \neq 0\}$. Then N_i is a dense open subset of N, so $\bigcap_{i=1}^{n} N_i$ is non-empty. Let z belong to this intersection. Then $f_i(z) \neq 0$ for $1 \le i \le n$, so $f_1 \dots f_n \neq 0$. This shows that $\overline{J}_y$ is primal. ∎

4. Points of discontinuity.

In this section we consider what happens when there is a point y of discontinuity. The general aim is to show that F_y has to be big, and hence that there have to be many non-R-points. The main results are that if A is a natural unital Banach function algebra and $y \notin \Gamma_A$ then F_y has an infinite, connected subset, while if A is any uniform algebra then F_y has a perfect subset. Thus in both cases there are uncountably many non-R-points. A simple example was given after Corollary 3.3 of a non-natural Banach function algebra with a solitary non-R-point, but we are unable to say whether this phenomenon can occur for a natural Banach function algebra, or for a Banach function algebra on its Shilov boundary.

DEFINITION. Let A be a Banach function algebra on a compact, Hausdorff space X. Then A is *local on* X if the following condition holds: a function on X belongs to A if it agrees in a neighbourhood of each point of X with an element of A. The algebra A is 2-*local on* X if the following condition holds: a function on X belongs to A if there are elements g_1 and g_2 in A so that every point of X has neighbourhood on which f agrees with either g_1 or g_2. If X is the character space of A above, then A is said to be *local* or 2-*local*, respectively.

Every normal Banach function algebra is local, and every local Banach function algebra is, of course, 2-local. Most commonly met uniform algebras are local. For instance, if X is a compact subset of the plane then the uniform algebras $A(X)$ and $R(X)$ are local. All approximately normal uniform algebras (hence all uniform algebras with character space equal to $[0,1]$) are 2-local [18; 2.4, 3.1]. It follows from [4; Theorem 11] that the algebra B after Corollary 3.4 is not 2-local on $[0,1]$. For any compact plane set X, the Banach function algebras on X introduced by Dales and Davie in [2] are local on X.

In the first result of this section we do not require a norm on our algebra. In fact, the result is valid for multiplicative sub-semigroups of $C(X)$ (where $C(X)$ denotes the algebra of continuous, complex-valued functions on a compact Hausdorff space X).

PROPOSITION 4.1. *Let X be a compact Hausdorff space and let A be a subalgebra of $C(X)$. Suppose that A is 2-local on X. Let $y \in X$, and recall that $F_y = \{x \in X : J_x \subseteq M_y\}$. Then F_y is connected.*

Proof. Suppose, for a contradiction, that F_y is not connected. Since F_y is closed, we may write $F_y = E_1 \cup E_2$ where E_1 and E_2 are non-empty, disjoint, closed subsets of F_y. Suppose that $y \in E_2$. Choose a closed neighbourhood N of E_1 such that $N \cap F_y = E_1$, and let K be the boundary of N. Then K is a compact subset of $X \backslash F_y$. Thus, for each $t \in K$ we can find an $f \in J_t$ with $f(y) = 1$. By compactness, multiplying finitely many of these functions together, we can find a function $g \in A$ which vanishes on a neighbourhood of K and such that $g(y) = 1$ (if K is empty, simply take g to be zero on N, and 1 elsewhere). Define $h(t)$ to be 0 for $t \in N$, and set $h(t) = g(t)$ for other values of t. Then $h \in A$ since A is 2-local. But then $h \in J_x$ and $h(y) = 1$, contradicting the choice of x. The result follows. ■

COROLLARY 4.2. *Let A be a 2-local Banach function algebra on $[0,1]$. If the hull-kernel topology on $[0,1]$ is first countable then A is regular on $[0,1]$. If $\Phi_A = [0,1]$ and the set of points at which the hull-kernel topology is first countable is dense in $[0,1]$, then A is normal.*

Proof. Let y be a point of first countability for the hull-kernel topology. Then F_y is a connected set with no interior, by Proposition 4.1 and Theorem 2.4, and hence is a singleton. Thus $F_y = \{y\}$, so y is a point of continuity. Thus if the hull-kernel topology is first countable, every point is a point of continuity, so A is regular on $[0,1]$. The proof of the second statement is as in Corollary 3.4. ■

Corollary 4.2 shows that non-regular Dales–Davie algebras on $[0,1]$ must have hull-kernel topology which is not first countable.

We now use Proposition 4.1 to show that if A is a natural unital Banach function algebra and $y \notin \Gamma_A$ then F_y has an infinite, connected subset.

Let B be a uniform algebra on its character space Φ_B. Then there is a smallest local uniform algebra C on Φ_B containing B, which is obtained from B as follows. Set $L^0(B) = B$, and for each ordinal σ define $L^{\sigma+1}(B)$ inductively to be the uniform closure of the functions belonging locally to $L^\sigma(B)$. If σ is a limit ordinal, let L^σ be the uniform closure of $\bigcup\{L^{\sigma'} : 0 \le \sigma' < \sigma\}$. The process must terminate (with $C(\Phi_B)$, if not before), so eventually $L^\sigma = L^{\sigma+1}$. For this ordinal σ, set $C = L^\sigma$. Then evidently C is the smallest local uniform algebra on Φ_B containing B. Examples are given in [12] showing that, with Φ_B metrizable, the process can terminate at any given stage before the first uncountable ordinal.

LEMMA 4.3. *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an increasing family of uniform algebras, all with the same character space X. Let B be the uniform closure of their union in $C(X)$. Suppose that $\Gamma_{A_\lambda} = Y$ for all λ, where Y is a Gelfand closed subset of X. Then $\Phi_B = X$ and $\Gamma_B = Y$.*

Proof. It is easy to see that $\Phi_B = X$, and it is also clear that $\Gamma_B \supseteq Y$, since Γ_B is a boundary for each A_λ. Let $x \in X$ be a strong boundary point for B. Let N be any Gelfand neighbourhood of x. Then there exists $f \in B$ such that $|f(x)| > 3/4$ while $|f|_{X\backslash N} < 1/4$ (cf. [16; 7.18]). Hence for some λ there exists $g \in A_\lambda$ such that $|g(x)| > 1/2$ while $|g|_{X\backslash N} < 1/2$. This shows that $Y = \Gamma_{A_\lambda}$ meets N, and hence that $x \in Y$ since N was an arbitrary neighbourhood of x. Thus $\Gamma_B \subseteq Y$, since the set of strong boundary points is dense in Γ_B [16; 7.24]. Hence $\Gamma_B = Y$. ■

LEMMA 4.4. *Let A be a unital Banach function algebra and let C be the smallest local uniform subalgebra of $C(\Phi_A)$ containing A. Then $\Phi_A = \Phi_C$ and $\Gamma_A = \Gamma_C$.*

Proof. First, let B be the uniform closure of A in $C(\Phi_A)$. It is straighforward that $\Phi_A = \Phi_B$. We now argue as in Lemma 4.3. Since Γ_B is a boundary for A, we have $\Gamma_B \supseteq \Gamma_A$. On the other hand, suppose that $x \in \Phi_A$ is a strong boundary point for B. Let N be any Gelfand neighbourhood of x. Then there exists $f \in B$ such that $|f(x)| > 3/4$ while $|f|_{\Phi_A\backslash N} < 1/4$ (cf. [16; 7.18]). Hence there exists $g \in A$ such that $|g(x)| > 1/2$ while $|g|_{\Phi_A\backslash N} < 1/2$. This shows that Γ_A meets N, and hence that $x \in \Gamma_A$ since N was an arbitrary neighbourhood of x. Thus $\Gamma_A = \Gamma_B$.

Now let C be the smallest local uniform subalgebra of $C(\Phi_A)$ containing B. Then Stolzenberg showed that $\Phi_{L^\sigma(B)} = \Phi_{L^{\sigma+1}(B)}$ for each ordinal σ (cf. [15]), and he also mentioned in a remark that $\Gamma_{L^\sigma(B)} = \Gamma_{L^{\sigma+1}(B)}$ for each ordinal σ (this follows, as with the maximal ideal space result, by considering the intermediate uniform algebras generated by $L^\sigma(B)$ together with a single function f which is locally in $L^\sigma(B)$). Thus it follows from Lemma 4.3 that $\Phi_C = \Phi_B$ and $\Gamma_C = \Gamma_B$, and hence that $\Phi_C = \Phi_A$ and $\Gamma_C = \Gamma_A$. ■

We are now ready for the main result of this section.

THEOREM 4.5. *Let A be a unital Banach function algebra and let $y \in \Phi_A \backslash \Gamma_A$. Then F_y has an infinite connected subset.*

Proof. Let C be the smallest local uniform subalgebra of $C(\Phi_A)$ containing B. Then we saw in Lemma 4.4 that $\Phi_C = \Phi_A$ and that $\Gamma_C = \Gamma_A$. Hence $y \notin \Gamma_C$, so Lemma 2.1(ii) shows that y is not a point of continuity for C. Thus the set $F_{y,C} = \{x \in \Phi_C : J_x \subseteq M_y\}$ (where J_x and M_y are here defined relative to the algebra C) has more than one point, and is connected by Proposition 4.1. Since $F_y \supseteq F_{y,C}$, the result follows. ■

For uniform algebras a weaker but more general statement is true.

THEOREM 4.6. *Let A be a uniform algebra on a compact Hausdorff space X and let $y \in X$. Then either y is a point of continuity or F_y has a non-empty perfect subset.*

Proof. Suppose, for a contradiction, that F_y has no non-empty perfect subsets, and that there exists $x \in F_y \backslash \{y\}$. Choose $f \in A$ with $f(x) = 0$ and $f(y) = 1$. By [10], $f(F_y)$ is a countable, compact set. Thus it is easy to choose two disjoint closed rectangles R_1 and R_2 in the complex plane such that $0 \in R_1$, $1 \in R_2$, and such that $f(F_y)$ is contained in the interior, U, of $R_1 \cup R_2$. By Runge's theorem [11; 13.9] we can choose a sequence (p_n) of polynomials such that $p_n \to 0$ uniformly on R_1, but $p_n \to 1$ uniformly on R_2, as $n \to \infty$. Also, $X \backslash f^{-1}(U)$ is a compact subset of $X \backslash F_y$, so by compactness (as in Proposition 4.1) we can find a function g in A which vanishes on a neighbourhood of $X \backslash f^{-1}(U)$, but such that $g(y) = 1$. Set $h_n = g p_n(f)$. Then $h_n \in A$, and the sequence (h_n) converges uniformly on X to a function which vanishes on a neighbourhood of x but which is 1 at y. This contradicts the choice of x. Thus $F_y \backslash \{y\}$ must be empty, as claimed. ∎

The non-trivial uniform algebra A on the Cantor set, mentioned after Corollary 3.4, is not regular on the Cantor set, because it is an integral domain. Thus for a point y of discontinuity, the set F_y is not connected. This example is, however, 2-local on its character space (which is in fact equal to the Riemann sphere) so the set F_y would be connected if we were working on the character space.

The next corollary is used in [4].

COROLLARY 4.7. *Let A be a uniform algebra on a compact Hausdorff space X. Suppose that A is not regular on X. Then the set of non-R-points has a non-empty perfect subset.*

Proof. Since A is not regular on X, there exists $y \in X$ such that the set $\{x \in X \backslash \{y\} : J_x \subseteq M_y\}$ is non-empty. But all points of this set are non-R-points, and, by Theorem 4.6, this set contains a non-empty perfect set. The result follows. ∎

We saw in Section 3 that a Banach function algebra can have a solitary non-R-point. We do not know, however, whether this can happen for a natural Banach function algebra, nor for a Banach function algebra on its Shilov boundary. Proposition 4.1 shows that such an example would have to be non-2-local. The next theorem shows that the difficult case is when every point is an independent point.

THEOREM 4.8. *Let A be a Banach function algebra on a compact Hausdorff space X and let y be a non-independent point of X. Then F_y has a non-empty perfect subset.*

Proof. Let B be the uniform closure of A in $C(X)$. Then it is easy to see that y is not a point of continuity for B on X, so the set $F_{y,B} = \{x \in X : J_x \subseteq M_y\}$ (where J_x and M_y are here defined relative to the algebra B) has an infinite perfect subset by Theorem 4.6. Since $F_y \supseteq F_{y,B}$, the result follows. ∎

Recall that a Banach function algebra A on a compact Hausdorff space X is *approximately regular on* X if whenever Y is a closed subset of X, and $x \in X \backslash Y$, then for any $\varepsilon > 0$ there exists $f \in A$ such that $f(x) = 1$ and $|f|_Y < \varepsilon$. This is clearly equivalent to every point of X being an independent point. It is also clear that if A is a uniform algebra and A is approximately regular on X then X must be the Shilov boundary of A.

The disc algebra is approximately regular on the circle, but the tomato-can algebra is not approximately regular on its character space.

LEMMA 4.9. *Let A be a uniform algebra on a compact Hausdorff space X. Let $x \in X$ be an independent point, and let $y \in X$. Then x cannot be an isolated point in F_y, unless $x = y$ and y is a point of continuity.*

Proof. Suppose, for a contradiction, that x is an isolated point of F_y and that $F_y \backslash \{x\}$ is non-empty. Since x is an independent point there is a function $f' \in A$ taking the value 1 at x and nearly zero on $F_y \backslash \{x\}$. Hence $f = 1 - f'$ is zero at x and nearly 1 on $F_y \backslash \{x\}$. We now argue as in Theorem 4.6.

It is easy to choose two disjoint, closed rectangles R_1 and R_2 in the complex plane such that $0 \in R_1$, $1 \in R_2$, and such that $f(F_y)$ is contained in the interior, U, of $R_1 \cup R_2$. By Runge's theorem [11; 13.9] we can choose a sequence (p_n) of polynomials such that $p_n \to 0$ uniformly on R_1, but $p_n \to 1$ uniformly on R_2. Also, $X \backslash f^{-1}(U)$ is a compact subset of $X \backslash F_y$, so by compactness (as in Proposition 4.1) we can find a function g in A which is 0 on a neighbourhood of $X \backslash f^{-1}(U)$, but such that $g(y) = 1$. Set $h_n = g p_n(f)$. Then $h_n \in A$, and the sequence (h_n) converges uniformly on X to a function h which is 0 on a neighbourhood of x but which is 1 on a neighbourhood of $F_y \backslash \{x\}$.

The function h shows that $x = y$, for otherwise we have a function vanishing in a neighbourhood of x but non-zero at y, which contradicts the fact that $x \in F_y$. ∎

As an immediate consequence of Lemma 4.9, we have the following.

THEOREM 4.10. *Let A be a uniform algebra. If A is approximately regular on $X = \Gamma_A$, and $y \in \Gamma_A$, then $F_y \backslash \{y\}$ has no isolated points. Hence A has no isolated non-R-points.*

We conclude with an example showing that a uniform algebra on its Shilov boundary can have an isolated non-R-point. We do not know if this is possible for a uniform algebra on its character space.

EXAMPLE. Let X be the closed unit disc and $Y = T \cup \{0, 1/2, 1/3, \ldots\}$. Let A be the uniform algebra of all functions in $C(X)$ whose restriction to Y is in the restriction to Y of the disc algebra. It is easy to see that X is the Shilov boundary of A, and that the only non-R-points for A are the points of T and the point 0. Thus 0 is an isolated non-R-point for A. In fact, for $y \in Y$, $F_y = \{0, y\} \cup T$. All other points of X are points of continuity for A.

References

[1] R. J. Archbold and C. J. K. Batty, *On factorial states of operator algebras, III*, J. Operator Theory 15 (1986), 53–81.

[2] H. G. Dales and A. M. Davie, *Quasianalytic Banach function algebras*, J. Funct. Anal. 13 (1973), 28–50.

[3] J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1982.

[4] J. F. Feinstein and D. W. B. Somerset, *Strong regularity for uniform algebras*, in: Proc. 3rd Function Spaces Conference (Edwardsville, IL, 1998), Contemp. Math. 232, Amer. Math. Soc., 1999, 139–149.

[5] P. Gorkin and R. Mortini, *Synthesis sets in $H^\infty + C$*, Indiana Univ. Math. J., to appear.

[6] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[7] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, NJ, 1955.

[8] M. M. Neumann, *Commutative Banach algebras and decomposable operators*, Monatsh. Math. 113 (1992), 227–243.

[9] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras*, Vol. 1, Cambridge Univ. Press, New York, 1994.

[10] W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. 8 (1957), 39–42.

[11] —, *Real and Complex Analysis*, Tata McGraw-Hill, New Delhi, 1974.

[12] S. Sidney, *More on high-order non-local uniform algebras*, Illinois J. Math. 18 (1974), 177–192.

[13] D. W. B. Somerset, *Minimal primal ideals in Banach algebras*, Math. Proc. Cambridge Philos. Soc. 115 (1994), 39–52.

[14] —, *Ideal spaces of Banach algebras*, Proc. London Math. Soc. (3) 78 (1999), 369–400.

[15] G. Stolzenberg, *The maximal ideal space of functions locally in an algebra*, Proc. Amer. Math. Soc. 14 (1963), 342–345.

[16] E. L. Stout, *The Theory of Uniform Algebras*, Bogden and Quigley, New York, 1971.

[17] J. Wermer, *Banach algebras and analytic functions*, Adv. Math. 1 (1961), 51–102.

[18] D. R. Wilken, *Approximate normality and function algebras on the interval and the circle*, in: Function Algebras (New Orleans, 1965), Scott-Foresman, Chicago, IL, 1966, 98–111.

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Dirichlet series and uniform ergodic theorems for linear operators in Banach spaces

by

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Abstract. We study the convergence properties of Dirichlet series for a bounded linear operator T in a Banach space X. For an increasing sequence $\mu = \{\mu_n\}$ of positive numbers and a sequence $f = \{f_n\}$ of functions analytic in neighborhoods of the spectrum $\sigma(T)$, the Dirichlet series for $\{f_n(T)\}$ is defined by

$$D[f, \mu; z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T), \qquad z \in \mathbb{C}.$$

Moreover, we introduce a family of summation methods called Dirichlet methods and study the ergodic properties of Dirichlet averages for T in the uniform operator topology.

1. Introduction. In this paper we attempt to study the Dirichlet series in the ergodic theory setting for a bounded linear operator T in a Banach space X with a view to making up for a gap in the structural properties of the resolvent $R(\lambda; T)$ of T. In particular, the abscissa of uniform convergence of such Dirichlet series is investigated in an operator-theoretical sense. Moreover, we introduce a new summation method of what is called Dirichlet's type generalizing the Abel method and show that when $\|T^n\|/n \to 0$, the uniform $(C, 1)$ ergodicity of T is equivalent to the uniform ergodicity of Dirichlet's type.

Let X be a complex Banach space and let $B[X]$ denote the Banach algebra of bounded linear operators from X to itself. For a given $T \in B[X]$, the *resolvent set* of T, denoted by $\varrho(T)$, is the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ exists as an operator in $B[X]$ with domain X. The *spectrum* of T is the complement of $\varrho(T)$ and is denoted by $\sigma(T)$. $\varrho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a nonempty bounded closed subset of $\mathbb{C}$. So the spectral radius $\gamma(T)$ of T is well defined: in fact $\gamma(T) = \sup|\sigma(T)| = \lim_{n\to\infty} \|T^n\|^{1/n}$. The function $R(\lambda; T)$ defined by $R(\lambda; T) = (\lambda I - T)^{-1}$ for $\lambda \in \varrho(T)$ is called the *resolvent* of T. It is well known ([3], [10]) that $R(\lambda; T)$ is analytic in $\varrho(T)$

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