On the other hand, Theorem 1 and the Córdoba–Fefferman inequality in [6] tell us that

$$\|T_{A_1,A_2}\|_{p,u} \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|f\|_{p,u};$$

and so

$$\|T^*_{A_1,A_2}f\|_{p,u} \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|f\|_{p,u}.$$

This finishes the proof of Theorem 1.

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Limit laws for products of free and independent random variables

by

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Abstract. We determine the distributional behavior of products of free (in the sense of Voiculescu) identically distributed random variables. Analogies and differences with the classical theory of independent random variables are then discussed.

1. Introduction. The concept of free independence introduced by D. Voiculescu has developed into a powerful noncommutative analogue of the classical notion of independence in probability theory. The book [8] provides an introduction to the area, showing in particular that some results about free random variables parallel in a rather striking fashion classical facts of probability theory. One instance of this parallelism occurs in our earlier work [2], where we studied the limiting behavior of sums of free, identically distributed infinitesimal random variables. More precisely, let $\{X_{ij} : i \ge 1,\ 1 \le j \le n_i\}$ be an array of classical independent random variables, and $\{Y_{ij} : i \ge 1,\ 1 \le j \le n_i\}$ an array of free random variables. Assume that $\lim_{i\to\infty} n_i = \infty$ and the variables $X_{i1},\ldots,X_{in_i},Y_{i1},\ldots,Y_{in_i}$ are identically distributed for every i. The main result of [2] states that the variables $\sum_{j=1}^{n_i} X_{ij}$ have a limit in distribution as $i\to\infty$ if and only if the variables $\sum_{j=1}^{n_i} Y_{ij}$ do. Moreover, the classical and free limits are related in a rather explicit manner.

Our purpose in this paper is to develop a similar result for products of positive random variables. Here the parallelism between freeness and independence is not as perfect. An instance of this phenomenon was already seen in [5], where it was shown that there exist two free multiplicative "Poisson" laws with no commutative analogues.

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The results in this paper will be written in terms of convolutions (as were those of [2]). Denote by $\mathcal{M}_+$ the Borel probability measures defined on $(0,\infty)$. The classical multiplicative convolution of two measures $\mu,\nu\in\mathcal{M}_+$ is denoted by $\mu\circledast\nu$. Thus, $\mu\circledast\nu$ is the probability distribution of XY, where X and Y are classical (commuting) independent random variables with distributions μ and ν, respectively. Free multiplicative convolution is another associative composition law on $\mathcal{M}_+$, denoted by $\boxtimes$. This was first defined by Voiculescu [7] (for compactly supported measures; see [5] for the general case). For $\mu,\nu\in\mathcal{M}_+$, $\mu\boxtimes\nu$ is the probability distribution of $X^{1/2}YX^{1/2}$, where X and Y are free random variables with distributions μ and ν, respectively.

The remainder of this paper is organized as follows. Section 2 contains notation and preliminaries concerning the calculation of multiplicative convolutions. Our main result about free multiplicative convolution is in Section 3, while Section 4 provides a comparison between the free limit theorem and its classical counterpart.

The authors would like to thank Rick Bradley for several useful discussions concerning the subject matter of this paper.

2. Preliminaries. We begin with the analytic method for the calculation of multiplicative free convolution discovered by Voiculescu [7] (cf. also [4], [5], and [8]). Denote by $\mathbb{C}$ the complex plane and set $\mathbb{C}^+=\{z\in\mathbb{C}:\Im z>0\}$, $\mathbb{C}^-=-\mathbb{C}^+$. For a measure $\nu\in\mathcal{M}_+$ one defines the analytic function ψ_ν by

$$\psi_\nu(z)=\int_{(0,\infty)}\frac{zt}{1-zt}\,d\nu(t)$$

for $z\in\mathbb{C}\setminus[0,\infty)$. The measure ν is completely determined by ψ_ν. The function ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$, and $\psi_\nu(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(-1,0)$; moreover, $\psi_\nu(i\mathbb{C}^+)\cap(-\infty,0]=(-1,0)$. If we set $\Omega_\nu=\psi_\nu(i\mathbb{C}^+)$, the function ψ_ν has an inverse

$$\chi_\nu:\Omega_\nu\to i\mathbb{C}^+.$$

Finally, define the *S-transform* of ν to be

$$S_\nu(z)=\frac{1+z}{z}\chi_\nu(z),\qquad z\in\Omega_\nu.$$

The following basic result is proved in [7] (see [5] for unbounded supports).

2.1. THEOREM. *Let $\mu,\nu\in\mathcal{M}_+$. Then*

$$S_{\mu\boxtimes\nu}(z)=S_\mu(z)S_\nu(z)$$

for every z in the connected component of the common domain of S_μ and S_ν containing the interval $(-1,0)$.

Recall that a measure $\nu\in\mathcal{M}_+$ is said to be $\boxtimes$-*infinitely divisible* if, for every natural number n, there exists a measure $\mu_n\in\mathcal{M}_+$ such that

$$\nu=\underbrace{\mu_n\boxtimes\ldots\boxtimes\mu_n}_{n\text{ times}}.$$

The notion of $\circledast$-*infinite divisibility* is defined analogously. $\boxtimes$-infinitely divisible measures were characterized in terms of their S-transforms [5]. Namely, a measure $\nu\in\mathcal{M}_+$ is $\boxtimes$-infinitely divisible if and only if S_ν can be written as $S_\nu(z)=\exp(v(z))$, where v is analytic in $\mathbb{C}\setminus[0,1]$, $v(\mathbb{C}^+)\subset\mathbb{C}^-\cup\mathbb{R}$, and $v(\overline{z})=\overline{v(z)}$. One can then use the Nevanlinna representation of functions with positive imaginary part in a half-plane to arrive at the following Lévy–Khinchin formula.

2.2. THEOREM. *A measure $\nu\in\mathcal{M}_+$ is $\boxtimes$-infinitely divisible if and only if there exist a finite positive Borel measure σ on the compact space $[0,\infty]$ and a real number γ such that $S_\nu(z)=\exp(v(z))$, where v is given by*

$$v\left(\frac{z}{1-z}\right)=\gamma-\beta z+\int_{[0,\infty)}\frac{1+tz}{z-t}\,d\sigma(t)$$

with $\beta=\sigma(\{\infty\})$.

We denote by $\nu_{\boxtimes}^{\gamma,\sigma}$ the $\boxtimes$-infinitely divisible measure determined by the above formula.

The study of $\circledast$-infinitely divisible probability measures reduces (by a change of variable) to the study of the usual infinitely divisible measures on $\mathbb{R}$. The Fourier transform needs to be replaced by the *Mellin–Fourier* transform of a measure $\nu\in\mathcal{M}_+$ defined by

$$\Phi_\nu(s)=\int_{(0,\infty)}t^{is}\,d\nu(t),\qquad s\in\mathbb{R}.$$

We have

$$\Phi_{\nu\circledast\mu}(s)=\Phi_\nu(s)\Phi_\mu(s),$$

and the classical Lévy–Khinchin formula is as follows (see [6] for the additive case) ([1]).

2.3. THEOREM. *A measure $\nu\in\mathcal{M}_+$ is $\circledast$-infinitely divisible if and only if there exist a finite positive Borel measure ϱ on the space $(0,\infty)$ and a real number δ such that*

$$\Phi_\nu(s)=\exp\left[i\delta s+\int_{(0,\infty)}\left(t^{-is}-1+\frac{is\log t}{\log^2t+1}\right)\frac{\log^2t+1}{\log^2t}\,d\varrho(t)\right].$$

([1]) We should remark that in [2] the classical Lévy–Khinchin formula for the additive case is written incorrectly. This error has no bearing on the validity of the results of that paper.

We denote by $\nu_{\circledast}^{\delta,\varrho}$ the $\circledast$-infinitely divisible measure determined by the above formula.

We conclude this section with a brief discussion of weak convergence on $(0,\infty)$. A sequence $\{\nu_n : n \geq 1\}$ in $\mathcal{M}_+$ converges weakly to $\nu \in \mathcal{M}_+$ if $\lim_{n\to\infty} \nu_n((a,b)) = \nu((a,b))$ whenever a and b are not atoms of ν. This condition implies the tightness of $\{\nu_n : n \geq 1\}$ on $(0,\infty)$, i.e., $\lim_{\varepsilon\to 0} \sup_{n\geq 1} \nu_n(\mathbb{R} \setminus (\varepsilon, 1/\varepsilon)) = 0$. If the measures μ_n are finite Borel measures (not necessarily of total mass one) then one moreover requires tightness of the sequence. This convergence can moreover be characterized by the requirement that

$$\lim_{n\to\infty} \int_{(0,\infty)} f(t)\, d\nu_n(t) = \int_{(0,\infty)} f(t)\, d\nu(t)$$

for every bounded continuous function f on $(0,\infty)$. We will also use weak convergence for measures on the compact space $[0,\infty]$. Thus, a sequence $\{\nu_n : n \geq 1\}$ of Borel measures on $[0,\infty]$ converges weakly to ν if

$$\lim_{n\to\infty} \nu_n([0,\infty]) = \nu([0,\infty]) \quad \text{and} \quad \lim_{n\to\infty} \nu_n((a,b)) = \nu((a,b))$$

whenever a and b are not atoms of ν. Equivalently,

$$\lim_{n\to\infty} \int_{[0,\infty]} f(t)\, d\nu_n(t) = \int_{[0,\infty]} f(t)\, d\nu(t)$$

for all continuous functions f on $[0,\infty]$. Here, of course,

$$\int_{[0,\infty]} f(t)\, d\nu(t) = \int_{[0,\infty)} f(t)\, d\nu(t) + L\nu(\{\infty\}),$$

where $L = f(\infty) = \lim_{t\to\infty} f(t)$.

Weak convergence of measures in $\mathcal{M}_+$ can be translated in terms of convergence of the corresponding S-transforms. The following proposition subsumes Propositions 6.4 and 6.5 in [5], and is more suitable for our purposes.

2.4. PROPOSITION. *Let $\{\nu_n : n \geq 1\}$ be a sequence in $\mathcal{M}_+$. The following assertions are equivalent*:

(i) *The sequence $\{\nu_n : n \geq 1\}$ converges weakly to a measure $\nu \in \mathcal{M}_+$.*

(ii) *There exist two positive numbers $0 < b < a < 1$ such that the disc D with diameter $[-a,-b]$ is contained in Ω_{ν_n} for all n, and the sequence S_{ν_n} converges uniformly on D to a function S.*

Moreover, if (i) *and* (ii) *are satisfied, we have $S = S_\nu$ in D.*

3. Limit laws for free multiplicative convolution. We are now ready for the main result of this paper.

3.1. THEOREM. *Let $\{\mu_n : n \geq 1\}$ be a sequence of measures in $\mathcal{M}_+$, and let $\{k_n : n \geq 1\}$ be natural numbers such that $\lim_{n\to\infty} k_n = \infty$. The following assertions are equivalent*:

(i) *The sequence $\mu_n \boxtimes \ldots \boxtimes \mu_n$ (k_n times) converges weakly to a measure $\nu \in \mathcal{M}_+$;*

(ii) *The measures*

$$d\sigma_n(t) = k_n \frac{(t-1)^2}{t^2+1} d\mu_n(1/t)$$

converge weakly in $[0,\infty]$ to a measure σ, and the limit

$$\gamma = \lim_{n\to\infty} k_n \int_{(0,\infty)} \frac{t^2-1}{t^2+1}\, d\mu_n(1/t)$$

exists.

If the equivalent conditions (i) *and* (ii) *are satisfied, we have $\nu = \nu_{\boxtimes}^{\gamma,\sigma}$.*

Proof. Assume first that (i) is satisfied. By virtue of Proposition 2.4 there is a closed disk $\overline{D}$ with diameter $[-a,-b] \subset (-1,0)$ such that

$$\lim_{n\to\infty} S_{\mu_n}(z)^{k_n} = S_\nu(z)$$

uniformly for $z \in \overline{D}$. The function $S_\nu(z)$ does not vanish in $\overline{D}$, and we conclude that there is a constant $c > 0$ such that

$$1/c \leq |S_{\mu_n}(z)^{k_n}| \leq c$$

for all n and all $z \in \overline{D}$. Therefore the sequence $|S_{\mu_n}(z)|$ converges uniformly to 1 for $z \in \overline{D}$. Since these functions are actually positive on $(-1,0)$, an application of the Vitali–Montel theorem shows that in fact $\lim_{n\to\infty} S_{\mu_n}(z) = 1$ uniformly in our disk.

A second application of Proposition 2.4 shows that the sequence μ_n converges weakly to the Dirac measure δ_1; indeed, $S_{\delta_1} = 1$. Now, since S_ν is not zero in $\overline{D}$, it can be written under the form $S_\nu(z) = \exp(v(z))$, where v is analytic in D and real-valued on $(-a,-b)$.

For sufficiently large n, the principal branch $\log S_{\mu_n}(z)$ is defined in D, and $k_n \log S_{\mu_n}(z)$ converges uniformly to $v(z)$. Since $\log(w)/(w-1) \to 1$ as $w \to 1$, we conclude that

$$\lim_{n\to\infty} k_n(S_{\mu_n}(z) - 1) = v(z)$$

uniformly for $z \in D$. We want to rewrite this relation in terms of the functions ψ_{μ_n}. In order to do this observe first that $\lim_{n\to\infty} \chi_{\mu_n}(z) = z/(1+z)$, $\lim_{n\to\infty} \chi'_{\mu_n}(z) = 1/(1+z)^2$ uniformly for $z \in D$, while $\lim_{n\to\infty} \psi_{\mu_n}(w) = w/(1-w)$ uniformly for w in a compact subset of $i\mathbb{C}^+$ (this follows easily from the weak convergence of μ_n to 1). The last relation implies that

$$\lim_{n\to\infty} \psi_{\mu_n}\left(\frac{z}{1+z}\right) = z$$

uniformly for $z \in D$.

We can now calculate another approximation of $v(z)$ by noting that

$$S_{\mu_n}(z) - 1 = \frac{1+z}{z}\chi_{\mu_n}(z) - 1 = \frac{1+z}{z}\left(\chi_{\mu_n}(z) - \chi_{\mu_n}\left(\psi_{\mu_n}\left(\frac{z}{1+z}\right)\right)\right).$$

Denote by Γ_n the line segment joining z and $\psi_{\mu_n}(z/(1+z))$. If the point z belongs to a smaller disk $D' \subset D$ centered on the real line, we must have $\Gamma_n \subset D$ for sufficiently large n. If we set

$$\chi'_{\mu_n}(z) = \frac{1}{(1+z)^2} + \varepsilon_n(z),$$

we have

$$\chi_{\mu_n}(z) - \chi_{\mu_n}\left(\psi_{\mu_n}\left(\frac{z}{1+z}\right)\right) = \int_{\Gamma_n}\left(\frac{1}{(1+\xi)^2} + \varepsilon_n(\xi)\right)d\xi$$
$$= \frac{z}{1+z} - \frac{\psi_{\mu_n}(z/(1+z))}{1+\psi_{\mu_n}(z/(1+z))} + \int_{\Gamma_n}\varepsilon_n(\xi)\,d\xi.$$

Now,

$$\frac{z}{1+z} - \frac{\psi_{\mu_n}(z/(1+z))}{1+\psi_{\mu_n}(z/(1+z))} = \frac{z-\psi_{\mu_n}(z/(1+z))}{(1+z)^2}(1+o(1))$$

uniformly in D', and

$$\left|\int_{\Gamma_n}\varepsilon_n(\xi)\,d\xi\right| \le \max_{\xi\in D}|\varepsilon_n(\xi)|\left|z - \psi_{\mu_n}\left(\frac{z}{1+z}\right)\right| = \left(z - \psi_{\mu_n}\left(\frac{z}{1+z}\right)\right)o(1)$$

uniformly in D' as well. Combining these relations yields

$$\chi_{\mu_n}(z) - \chi_{\mu_n}\left(\psi_{\mu_n}\left(\frac{z}{1+z}\right)\right) = \frac{z-\psi_{\mu_n}(z/(1+z))}{(1+z)^2}(1+o(1))$$

uniformly in D'. Thus, modulo a factor which is $1+o(1)$, $S_{\mu_n}(z)-1$ can be replaced by

$$\frac{1}{z(1+z)}\left(z - \psi_{\mu_n}\left(\frac{z}{1+z}\right)\right) = \int_{(0,\infty)}\frac{1-t}{1+z-zt}\,d\mu_n(t).$$

Introducing the variable $w = z/(1+z)$, changing the variable of integration, and exploiting the equality

$$\frac{(w-1)(t-1)}{w-t} = \frac{t^2-1}{t^2+1} + \frac{1+tw}{w-t}\cdot\frac{(t-1)^2}{t^2+1},$$

we conclude that

$$\lim_{n\to\infty}\left[k_n\int_{(0,\infty)}\frac{t^2-1}{t^2+1}\,d\mu_n(1/t) + \int_{(0,\infty)}\frac{1+tw}{w-t}\,d\sigma_n(t)\right] = v\left(\frac{w}{1-w}\right)$$

uniformly for $z \in D_0$, where $D_0 = \{z/(1+z) : z \in D'\}$ is another disk with real center.

We are now ready to prove that (ii) holds. Consider indeed the imaginary part

$$\Im v\left(\frac{w}{1-w}\right) = -\Im w\lim_{n\to\infty}\int_{(0,\infty)}\frac{t^2+1}{|w-t|^2}\,d\sigma_n(t).$$

The integrand stays bounded away from zero and infinity, and if w has nonzero imaginary part this shows that the measures σ_n are bounded, and have therefore a cluster point σ which is a measure on $[0,\infty]$. If we set $\beta = \sigma(\{\infty\})$, we then have

$$\Im v\left(\frac{w}{1-w}\right) = -\Im w\left[\beta + \int_{[0,\infty)}\frac{t^2+1}{|w-t|^2}\,d\sigma(t)\right].$$

This equality shows that the measure σ is uniquely determined, and hence the sequence σ_n converges weakly on $[0,\infty]$ to σ. Moreover, the difference

$$v\left(\frac{w}{1-w}\right) + \beta w - \int_{[0,\infty)}\frac{1+tw}{w-t}\,d\sigma(t)$$

must be real-valued, hence a constant γ whose value is

$$\lim_{n\to\infty} k_n\int_{(0,\infty)}\frac{t^2-1}{t^2+1}\,d\mu_n(1/t).$$

Thus (ii) holds, and the above considerations show that $\nu = \nu_{\boxtimes}^{\gamma,\sigma}$.

Conversely, assume now that (ii) is true. Then we have

$$\lim_{n\to\infty}\int_{(0,\infty)}\frac{t^2-1}{t^2+1}\,d\mu_n(1/t) = 0.$$

Since the integrand is bounded away from zero and infinity, except at $t=1$, we conclude immediately that the sequence μ_n converges weakly to δ_1. It follows that the estimates made in the first part of the proof still hold, and one can basically reverse the steps in order to arrive at (i). ∎

4. Comparison with the commutative case. The classical analogue of Theorem 3.1 is not usually stated because of its equivalence with additive results. The additive results from [6] have the following multiplicative form.

4.1. THEOREM. *Let $\{\mu_n : n \geq 1\}$ be a sequence of measures in $\mathcal{M}_+$, and let $\{k_n : n \geq 1\}$ be natural numbers such that $\lim_{n\to\infty} k_n = \infty$. The following assertions are equivalent*:

(i) *The sequence $\mu_n \circledast \ldots \circledast \mu_n$ (k_n times) converges weakly to a measure $\nu \in \mathcal{M}_+$;*

(ii) *The measures*

$$d\varrho_n(t) = k_n \frac{\log^2 t}{\log^2 t + 1}\, d\mu_n(1/t)$$

converge weakly in $(0,\infty)$ to a measure ϱ, and the limit

$$\delta = \lim_{n\to\infty} k_n \int_{(0,\infty)} \frac{-\log t}{\log^2 t + 1}\, d\mu_n(1/t)$$

exists.

If the equivalent conditions (i) *and* (ii) *are satisfied, we have $\nu = \nu_{\circledast}^{\delta,\varrho}$.*

Note that weak convergence of the measures ϱ_n requires the tightness of this sequence. Conditions (ii) of Theorems 3.1 and 4.1 are not equivalent. The following is however true.

4.2. THEOREM. *Let $\{\mu_n : n \geq 1\}$ be a sequence of measures in $\mathcal{M}_+$, and let $\{k_n : n \geq 1\}$ be natural numbers such that $\lim_{n\to\infty} k_n = \infty$. The following assertions are equivalent*:

(i) *The sequence $\mu_n \circledast \ldots \circledast \mu_n$ (k_n times) converges weakly to $\nu_{\circledast}^{\delta,\varrho}$;*

(ii) *The sequence $\mu_n \boxtimes \ldots \boxtimes \mu_n$ (k_n times) converges weakly to $\nu_{\boxtimes}^{\gamma,\sigma}$, and $\sigma(\{0\}) = \sigma(\{\infty\}) = 0$.*

If the equivalent conditions (i) *and* (ii) *are satisfied then the measures σ and ϱ are related by*

$$d\sigma(t) = \frac{\log^2 t + 1}{\log^2 t} \cdot \frac{(t-1)^2}{t^2+1}\, d\varrho(t),$$

while

$$\delta + \gamma = \int_{(0,\infty)} \left(\frac{t^2-1}{t^2+1} - \frac{\log^2 t}{\log^2 t + 1}\right) \frac{\log^2 t + 1}{\log^2 t}\, d\varrho(t).$$

Proof. Define measures σ_n and ϱ_n as in the statements of Theorems 3.1 and 4.1, respectively. Then we have

$$d\sigma_n(t) = \frac{\log^2 t + 1}{\log^2 t} \cdot \frac{(t-1)^2}{t^2+1}\, d\varrho_n(t),$$

and the function

$$\frac{\log^2 t + 1}{\log^2 t} \cdot \frac{(t-1)^2}{t^2+1}$$

is continuous and bounded away from zero on $(0,\infty)$. (The value of the function for $t = 1$ is $1/2$ and is defined to preserve continuity at that point.) It follows at once that the convergence on $(0,\infty)$ of σ_n to a measure σ is equivalent to convergence of ϱ_n to ϱ, and the two limit measures are related as in the statement of the theorem. Note in addition that the convergence on $(0,\infty)$ of σ_n is equivalent to the convergence of the sequence on the compact interval $[0,\infty]$ to a measure which assigns no mass to 0 and ∞. Assume now that the limits σ and ϱ exist on $(0,\infty)$, and set

$$\delta_n = k_n \int_{(0,\infty)} \frac{-\log t}{\log^2 t + 1}\, d\mu_n(1/t)$$

and

$$\gamma_n = k_n \int_{(0,\infty)} \frac{t^2-1}{t^2+1}\, d\mu_n(1/t),$$

so that $\gamma_n + \delta_n = \int_{(0,\infty)} f(t)\, d\varrho_n(t)$, where

$$f(t) = \left(\frac{t^2-1}{t^2+1} - \frac{\log t}{\log^2 t + 1}\right) \frac{\log^2 t + 1}{\log^2 t}.$$

Note again that (upon defining $f(1) = 0$) f is continuous and bounded on $(0,\infty)$. Therefore γ_n has a limit γ if and only if δ_n has a limit δ, and these limits are related as in the statement. The theorem now follows from the characterizations of weak convergence given in Theorems 3.1 and 4.1. ∎

It is now fairly easy to give examples of sequences $\mu_n \in \mathcal{M}_+$ such that $\mu_n \boxtimes \ldots \boxtimes \mu_n$ (n times) converges, but $\mu_n \circledast \ldots \circledast \mu_n$ (n times) does not. The reader will have no difficulty verifying this for the sequence

$$\mu_n = \left(1 - \frac{1}{n}\right)\delta_1 + \frac{1}{n}\delta_n.$$

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Non-regularity for Banach function algebras

by

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Abstract. Let A be a unital Banach function algebra with character space Φ_A. For $x \in \Phi_A$, let M_x and J_x be the ideals of functions vanishing at x and in a neighbourhood of x, respectively. It is shown that the hull of J_x is connected, and that if x does not belong to the Shilov boundary of A then the set $\{y \in \Phi_A : M_x \supseteq J_y\}$ has an infinite connected subset. Various related results are given.

1. Introduction. Let A be a Banach algebra and let $\mathrm{Prim}(A)$ be the set of primitive ideals of A. The *hull-kernel topology* on $\mathrm{Prim}(A)$ is defined by declaring the open sets to be those of the form $\{P \in \mathrm{Prim}(A) : P \not\supseteq I\}$ as I varies through the closed ideals of A. This topology is compact if A has an identity, but not usually Hausdorff, nor even T_1. Indeed it seems, in general, to have few useful properties, and it has not played a prominent part in the general theory of Banach algebras. An attempt to find a more useful topology has been made in [14].

The situation is different, however, for particular classes of Banach algebras, such as C*-algebras and certain L^1-group algebras. Here the hull-kernel topology does have good properties such as local compactness, the Baire property, and (for separable C*-algebras) second countability. These properties have been considerably exploited in C*-algebra theory and abstract harmonic analysis.

For commutative Banach algebras, the hull-kernel topology plays a secondary role. The primitive ideals of a (unital) commutative Banach algebra A are precisely the kernels of characters. Thus $\mathrm{Prim}(A)$ is in bijective correspondence with the character space Φ_A, which carries the compact, Hausdorff Gelfand topology. This is the topology usually employed in the study of commutative Banach algebras, but the hull-kernel topology (defined on Φ_A using the natural bijection) is also used from time to time. The hull-kernel topology is a T_1 topology in this case, and is weaker than the Gelfand topology. Thus the two topologies coincide if and only if the

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