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Weighted weak type (1,1) estimates for oscillatory singular integrals

by

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Abstract. We consider the A_1-weights and prove the weighted weak type $(1,1)$ inequalities for certain oscillatory singular integrals.

1. Introduction. Let $K \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfy

$$|K(x)| \le c|x|^{-n}, \quad |\nabla K(x)| \le c|x|^{-n-1}; \tag{1.1}$$

$$\int\limits_{a<|x|<b} K(x)\,dx = 0 \quad \text{for all } a, b \ (0 < a < b). \tag{1.2}$$

The smallest constant for which (1.1) holds will be denoted by $C(K)$.

We consider an oscillatory singular integral operator:

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int\limits_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\,dy \\ &= \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} e^{iP(x,y)} K(x-y) f(y)\,dy, \end{aligned}$$

initially defined for $f \in \mathfrak{S}(\mathbb{R}^n)$ (the Schwartz space), where P is a real-valued polynomial:

$$P(x,y) = \sum_{|\alpha|\le M,\ |\beta|\le N} a_{\alpha\beta} x^\alpha y^\beta. \tag{1.3}$$

The following results are known.

Theorem A (Ricci–Stein [9]). *Let $1 < p < \infty$. Then T is bounded on $L^p(\mathbb{R}^n)$ with the operator norm bounded by a constant depending only on the total degree of P, $C(K)$, p and the dimension n.*

Theorem B (Chanillo–Christ [2]). *The operator T is bounded from $L^1(\mathbb{R}^n)$ to the weak $L^1(\mathbb{R}^n)$ space:*

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$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}| \le c\|f\|_{L^1},$$

with a constant c depending only on the total degree of P, C(K) and the dimension n.

See also [1] and [3] for the weighted weak type (1, 1) estimates for convolution operators with oscillating kernels.

Let w be a locally integrable positive function on $\mathbb{R}^n$. We say that $w \in A_1$ if there is a constant c such that

$$M(w)(x) \le cw(x) \quad \text{a.e.} \tag{1.4}$$

where M denotes the Hardy–Littlewood maximal operator. The smallest constant for which (1.4) holds will be denoted by $C_1(w)$.

In this note we shall prove that T is bounded from L^1_w to $L^{1,\infty}_w$ (the weak L^1_w space) for $w \in A_1$:

THEOREM. *There exists a constant c depending only on the total degree of P, C(K), $C_1(w)$ and the dimension n such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}) \le c\|f\|_{L^1_w},$$

where $w(E) = \int_E w(x)\,dx$ *and* $\|f\|_{L^1_w} = \int |f(x)|w(x)\,dx$.

The theorem will be proved by a double induction, as in [9] and [2]. Let P be a polynomial as in (1.3). We assume that there exists a multi-index α such that $|\alpha| = M$ and $a_{\alpha\beta} \neq 0$ for some β. We write

$$P(x, y) = \sum_{|\alpha|\le M} x^\alpha Q_\alpha(y) \tag{1.5}$$

and define

$$L = \max\{\deg(Q_\alpha) : Q_\alpha \neq 0,\ |\alpha| = M\}.$$

Then $0 \le L \le N$. We assume that $L \ge 1$ and

$$\max_{\substack{|\alpha|=M\\ |\beta|=L}} |a_{\alpha\beta}| = 1.$$

Under this assumption on the polynomial P, we define

$$T_\infty(f)(x) = \int_{|x-y|>1} e^{iP(x,y)} K(x-y) f(y)\,dy.$$

To prove the Theorem, we shall use the following result in the induction.

PROPOSITION. *Let $\eta > 0$. There exists a constant c depending only on the total degree of P, η and the dimension n such that if $C(K), C_1(w) \le \eta$, then*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |T_\infty(f)(x)| > \lambda\}) \le c\|f\|_{L^1_w}.$$

REMARK 1. By the Theorem and the extrapolation theorem of Rubio de Francia, we get the L^p_w-boundedness of T for all $p \in (1, \infty)$ and all $w \in A_p$, where L^p_w is the space of all those measurable functions f which satisfy $\|f\|_{L^p_w} = (\int |f(x)|^p w(x)\,dx)^{1/p} < \infty$ and A_p denotes the weight class of Muckenhoupt.

We shall give the outlines of the proofs of the Theorem and the Proposition in Sections 2 and 4, respectively. Our proof of the Proposition is based on the techniques used in Christ [5] to prove the weak type (1, 1) estimates for rough operators (see also Christ [6], Christ–Rubio de Francia [7] and Sato [10]). We also use the geometrical argument of Chanillo–Christ [2]. We have to prove a key estimate (Lemma 7 in §5) in the unweighted case in order to apply the method of Vargas [11] involving an interpolation with change of measure. To prove Lemma 7, we need a geometrical result for polynomials (Lemma 5 in §5). We shall prove Lemma 5 in §7 by using the results and arguments appearing in the proof of Chanillo–Christ [2, Lemma 4.1].

Finally, we note that in this paper, the constants with the same notation are not necessarily the same at each occurrence.

2. Outline of proof of the Theorem. To apply the induction argument of [9] we need some preparation. We may assume that $M \ge 1$ and $N \ge 1$; otherwise the Theorem reduces to a well-known fact that the operator

$$A(f)(x) = \text{p.v.}\int K(x-y)f(y)\,dy$$

is bounded from L^1_w to $L^{1,\infty}_w$ (see, for example, [8]).

We write a polynomial in (1.3) as follows:

$$P(x, y) = \sum_{j=0}^{M} \sum_{|\alpha|=j} x^\alpha Q_\alpha(y) = \sum_{j=0}^{M} P_j(x, y),$$

say. We further decompose P_j as follows:

$$P_j(x, y) = \sum_{t=0}^{N} \sum_{\substack{|\alpha|=j\\ |\beta|=t}} a_{\alpha\beta} x^\alpha y^\beta = \sum_{t=0}^{N} P_{jt}(x, y),$$

say. For $j = 1, \ldots, M$ and $k = 0, 1, \ldots, N$, define

$$R_{jk}(x, y) = \sum_{s=0}^{j-1} P_s(x, y) + \sum_{t=0}^{k} P_{jt}(x, y). \tag{2.1}$$

Note that $R_{jN} = \sum_{s=0}^{j} P_s$ $(j = 1, \ldots, M)$.

For $j = 1, \ldots, M$ and $k = 0, 1, \ldots, N$, we consider the following propositions.

PROPOSITION $A(j,k)$. *Let $\eta>0$. There exists a constant c depending only on η, j, N and the dimension n such that if $C(K), C_1(w)\le\eta$ and if R_{jk} is a polynomial as in* (2.1), *then*

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : |T_{jk}(f)(x)|>\lambda\})\le c\|f\|_{L^1_w},$$

where

$$T_{jk}(f)(x)=\text{p.v.}\int_{\mathbb{R}^n} e^{iR_{jk}(x,y)}K(x-y)f(y)\,dy.$$

Then the Theorem follows from Proposition $A(M,N)$. We shall prove it by double induction. We first note that $A(1,0)$ follows from the L^1_w-$L^{1,\infty}_w$ boundedness of the operator A.

Next, we observe that if $M\ge2$ and if $A(j,N)$ $(1\le j\le M-1)$ is true, so is $A(j+1,0)$ since

$$R_{j+1,0}(x,y)=R_{jN}(x,y)+\sum_{|\alpha|=j+1}a_{\alpha0}x^\alpha$$

and hence $|T_{j+1,0}(f)(x)|=|T_{jN}(f)(x)|$. Thus, to complete the induction starting from $A(1,0)$ and arriving at $A(M,N)$, it suffices to prove $A(j,k+1)$ by assuming $A(j,k)$ $(0\le k<N,\ 1\le j\le M)$. To achieve this, put $R=R_{j,k+1}$, $R_0=R_{jk}$, $T_{j,k+1}=S$. We note that

$$R(x,y)=R_0(x,y)+\sum_{\substack{|\alpha|=j\\|\beta|=k+1}}a_{\alpha\beta}x^\alpha y^\beta.$$

We may assume $C_{jk}=\max_{|\alpha|=j,\,|\beta|=k+1}|a_{\alpha\beta}|\ne0$. Then by a suitable dilation we may assume $C_{jk}=1$. This can be seen as follows. We first note that, for $a>0$,

$$S(f)(ax)=\text{p.v.}\int e^{iR(ax,ay)}K_a(x-y)f(ay)\,dy,$$

where $K_a(x)=a^nK(ax)$. Assume the boundedness of S for the case $C_{jk}=1$. Then, choosing a to satisfy $a^{j+k+1}C_{jk}=1$, and using the dilation invariance of both the class A_1 and the class of kernels satisfying (1.1) and (1.2), we get

$$\begin{aligned}w(\{x\in\mathbb{R}^n : |S(f)(x)|>\lambda\})&=w_a(\{x\in\mathbb{R}^n : |S(f)(ax)|>\lambda\})\\&\le c\lambda^{-1}\int|f(ax)|a^nw(ax)\,dx\\&=c\lambda^{-1}\|f\|_{L^1_w}.\end{aligned}$$

We split the kernel K as $K=K_0+K_\infty$, where $K_0(x)=K(x)$ if $|x|\le1$ and $K_\infty(x)=K(x)$ if $|x|>1$. Assuming $C_{jk}=1$, we consider the corresponding splitting $S=S_0+S_\infty$:

$$S_0(f)(x)=\text{p.v.}\int e^{iR(x,y)}K_0(x-y)f(y)\,dy,$$

$$S_\infty(f)(x)=\int e^{iR(x,y)}K_\infty(x-y)f(y)\,dy.$$

In the next section, we shall prove

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : |S_0(f)(x)|>\lambda\})\le c\|f\|_{L^1_w},\tag{2.2}$$

while by the Proposition we have

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : |S_\infty(f)(x)|>\lambda\})\le c\|f\|_{L^1_w}.\tag{2.3}$$

Combining (2.2) and (2.3), we shall complete the proof of $A(j,k+1)$, which will finish the proof of the Theorem.

3. Estimate for S_0. In this section, we shall prove, under the assumption made in §2, that if $C(K), C_1(w)\le\eta$ $(\eta>0)$, then S_0 satisfies (2.2) with a constant c depending only on j, N, η and n.

First, we shall prove

$$w(\{x\in B(0,1) : |S_0(f)(x)|>\lambda\})\le c\lambda^{-1}\int_{|y|<2}|f(y)|w(y)\,dy,\tag{3.1}$$

where $B(x,r)$ denotes the closed ball with center x and radius $r>0$.

LEMMA 1. *Let $w\in A_1$. Let T be an operator of the form*

$$T(f)(x)=\text{p.v.}\int_{\mathbb{R}^n}K(x,y)f(y)\,dy=\lim_{\varepsilon\to0}\int_{|x-y|>\varepsilon}K(x,y)f(y)\,dy\quad a.e.$$

for $f\in L^1_w$, where the kernel K satisfies $|K(x,y)|\le c_0|x-y|^{-n}$. For $\varepsilon>0$, put

$$T_\varepsilon(f)(x)=\text{p.v.}\int_{|x-y|<\varepsilon}K(x,y)f(y)\,dy.$$

Suppose

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : |T(f)(x)|>\lambda\})\le c_w\|f\|_{L^1_w}.$$

Then there exists a constant c depending only on the dimension n such that

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : |T_\varepsilon(f)(x)|>\lambda\})\le c(c_w+c_0C_1(w))\|f\|_{L^1_w}.$$

Proof. The proof is similar to that of the Lemma in [9, p. 187]. We shall prove

$$\begin{aligned}w(\{x\in B(h,\varepsilon/4) : |T_\varepsilon(f)(x)|>\lambda\})\\ \le(2c_w+cc_0C_1(w))\lambda^{-1}\int_{|y-h|<5\varepsilon/4}|f(y)|w(y)\,dy\end{aligned}\tag{3.2}$$

uniformly in $h\in\mathbb{R}^n$. Integrating both sides of the inequality in (3.2) with respect to h, we get the conclusion of Lemma 1.

Split f into 3 pieces: $f = f_1+f_2+f_3$, where $f_1(y) = f(y)$ if $|y-h| < \varepsilon/2$, $f_1(y) = 0$ otherwise; $f_2(y) = f(y)$ if $\varepsilon/2 \le |y-h| < 5\varepsilon/4$, $f_2(y) = 0$ otherwise; $f_3(y) = f(y)$ if $|y-h| \ge 5\varepsilon/4$, $f_3(y) = 0$ otherwise. Note that if $|x-h| \le \varepsilon/4$, then $T_\varepsilon(f_1)(x) = T(f_1)(x)$ since $|y-h| < \varepsilon/2$ and $|x-h| \le \varepsilon/4$ imply $|x-y| < \varepsilon$. So by the assumption on T, we have

$$\begin{aligned} w(\{x \in B(h,\varepsilon/4) : |T_\varepsilon(f_1)(x)| > \lambda\}) &= w(\{x \in B(h,\varepsilon/4) : |T(f_1)(x)| > \lambda\}) \\ &\le w(\{x : |T(f_1)(x)| > \lambda\}) \\ &\le c_w\lambda^{-1}\|f_1\|_{L^1_w} \\ &\le c_w\lambda^{-1} \int\limits_{|y-h|<5\varepsilon/4} |f(y)|w(y)\,dy. \end{aligned}$$

Next, if $|x-h| \le \varepsilon/4$ and $\varepsilon/2 \le |y-h| < 5\varepsilon/4$, then $\varepsilon/4 \le |x-y| < 3\varepsilon/2$, and so

$$|T_\varepsilon(f_2)(x)| \le cc_0\varepsilon^{-n} \int\limits_{|y-h|<5\varepsilon/4} |f_2(y)|\,dy.$$

Hence, by Chebyshev's inequality,

$$\begin{aligned} w(\{x \in B(h,\varepsilon/4) : |T_\varepsilon(f_2)(x)| > \lambda\}) &\le cc_0\lambda^{-1}w(B(h,\varepsilon/4))\varepsilon^{-n} \int\limits_{|y-h|<5\varepsilon/4} |f_2(y)|\,dy \\ &\le cc_0C_1(w)\lambda^{-1} \int\limits_{|y-h|<5\varepsilon/4} |f(y)|w(y)\,dy. \end{aligned}$$

Finally, if $|x-h| \le \varepsilon/4$ and $|y-h| \ge 5\varepsilon/4$, then $|x-y| \ge \varepsilon$, and so $T_\varepsilon(f_3)(x) = 0$. Combining these, we get (3.2). This completes the proof of Lemma 1.

Now we turn to the proof of (3.1). If $|x| \le 1$ and $|y| \le 2$, then

$$\Big|\exp(iR(x,y)) - \exp\Big(i\Big(R_0(x,y) + \sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}y^{\alpha+\beta}\Big)\Big)\Big| \le c|x-y|,$$

where c depends only on k, j and n.

Hence, if $|x| \le 1$, then

$$|S_0(f)(x)| \le \Big|U\Big(\exp\Big(i \sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}y^{\alpha+\beta}\Big)f(y)\Big)(x)\Big| + cI(f)(x),$$

where

$$U(f)(x) = \text{p.v.}\int e^{iR_0(x,y)}K_0(x-y)f(y)\,dy,$$

$$I(f)(x) = \int\limits_{|x-y|<1} |x-y|^{-n+1}|f(y)|\,dy.$$

Note that $U(f)(x) = U(f\chi_{B(0,2)})(x)$, $I(f)(x) = I(f\chi_{B(0,2)})(x)$ if $|x| < 1$. By the induction hypothesis $A(j,k)$ and Lemma 1, we see that U is bounded from L^1_w to $L^{1,\infty}_w$. On the other hand, since

$$\begin{aligned} \int\limits_{|x-y|<1} |x-y|^{-n+1}w(x)\,dx &= \sum_{j\le 0} \int\limits_{2^{j-1}\le|x-y|\le 2^j} |x-y|^{-n+1}w(x)\,dx \\ &\le c\sum_{j\le 0} 2^j2^{-jn} \int\limits_{|x-y|\le 2^j} w(x)\,dx \le cM(w)(y), \end{aligned}$$

by Chebyshev's inequality we have

$$\begin{aligned} w(\{x \in B(0,1) : I(f)(x) > \lambda\}) &\le \lambda^{-1} \int\limits_{|y|<2} \Big(\int\limits_{|x-y|<1} |x-y|^{-n+1}w(x)\,dx\Big)|f(y)|\,dy \\ &\le cC_1(w)\lambda^{-1} \int\limits_{|y|<2} |f(y)|w(y)\,dy. \end{aligned}$$

Combining these results, we get (3.1).

Similarly we can prove

$$w(\{x \in B(h,1) : |S_0(f)(x)| > \lambda\}) \le c\lambda^{-1} \int\limits_{|y-h|<2} |f(y)|w(y)\,dy, \tag{3.3}$$

where c is independent of $h \in \mathbb{R}^n$. To see this, we first note that

$$S_0(f)(x+h) = \text{p.v.}\int e^{iR(x+h,y+h)}K_0(x-y)f(y+h)\,dy$$

and

$$R(x+h,y+h) = R_1(x,y,h) + \sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}x^\alpha y^\beta.$$

We can apply the induction hypothesis $A(j,k)$ to the operator

$$\text{p.v.}\int e^{iR_1(x,y,h)}K(x-y)f(y)\,dy$$

to get its boundedness from L^1_w to $L^{1,\infty}_w$. Thus, by the same argument that leads to (3.1) we get

$$
\begin{aligned}
w(\{x \in B(h,1) : |S_0(f)(x)| > \lambda\}) \\
&= \tau_h w(\{x \in B(0,1) : |S_0(f)(x+h)| > \lambda\}) \\
&\le c\lambda^{-1} \int_{|y|<2} |f(y+h)| w(y+h)\, dy \\
&\le c\lambda^{-1} \int_{|y-h|<2} |f(y)| w(y)\, dy,
\end{aligned}
$$

where $\tau_h w(x) = w(x+h)$, and we have used the translation invariance of the class A_1. Integrating both sides of the inequality (3.3) with respect to h, we get (2.2).

4. Outline of proof of the Proposition. We may assume $f \in \mathfrak{S}(\mathbb{R}^n)$. By Calderón–Zygmund decomposition at height $\lambda > 0$ we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g, b such that

$$f = g + b; \tag{4.1}$$

$$\lambda \le |Q|^{-1} \int_Q |f| \le c\lambda; \tag{4.2}$$

$$v\Big(\bigcup Q\Big) \le c_v \|f\|_{L^1_v}/\lambda \quad \text{for all } v \in A_1; \tag{4.3}$$

$$\|g\|_\infty \le c\lambda; \tag{4.4}$$

$$\|g\|_{L^1_v} \le c_v \|f\|_{L^1_v} \quad \text{for all } v \in A_1; \tag{4.5}$$

$$b = \sum_Q b_Q; \tag{4.6}$$

$$\operatorname{supp}(b_Q) \subset Q; \tag{4.7}$$

$$\int b_Q = 0; \tag{4.8}$$

$$\|b_Q\|_{L^1} \le c\lambda |Q|. \tag{4.9}$$

Remark 2. In this note we do not use (4.8).

Let a polynomial P be as in the Proposition. We assume, as we may, that $M \ge 1$ as in the outline of the proof of the Theorem in §2. We write P as in (1.5). Then let $q(y) = \sum_{|\beta| \le L} c_\beta y^\beta$ be the coefficient of x_1^M. By a rotation of coordinates and a normalization, and by discarding a negligible difference, we see that to prove the Proposition we may study T_∞ assuming $\max_{|\beta|=L} |c_\beta| = 1$; in this case the condition $\max_{|\alpha|=M,\, |\beta|=L} |a_{\alpha\beta}| = 1$ may not hold (see [2, p. 151] and Sublemma 2 in §7).

We pick a non-negative $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\varphi) \subset \{1/2 \le |x| \le 2\}, \qquad \sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \quad \text{if } |x| \ge 1.$$

Put $K_j(x,y) = \varphi(2^{-j}(x-y)) K_\infty(x,y)$, where $K_\infty(x,y) = e^{iP(x,y)} K_\infty(x-y)$ ($K_\infty(x)$ is as in §2) and decompose $K_\infty(x,y)$ as $K_\infty(x,y) = \sum_{j=0}^\infty K_j(x,y)$.

Define

$$V_j(f)(x) = \int K_j(x,y) f(y)\, dy \quad \text{for } j \ge 0$$

and put

$$V(f)(x) = \sum_{j=1}^{\infty} V_j(f)(x).$$

Then $T_\infty = V_0 + V$. In the following, we study V only, since we easily see that V_0 is bounded on L^1_w ($w \in A_1$).

We set (see [5–7])

$$B_i = \sum_{|Q| = 2^{in}} b_Q \quad (i \ge 1), \qquad B_0 = \sum_{|Q| \le 1} b_Q.$$

Put $\mathcal{U} = \bigcup \widetilde{Q}$, where $\widetilde{Q}$ denotes the cube with the same center as Q and with sidelength 100 times that of Q. Here and below all cubes we consider have sides parallel to the coordinate axes.

When $x \in \mathbb{R}^n \setminus \mathcal{U}$, we observe that

$$
\begin{aligned}
V(b)(x) = V\Big(\sum_{i\ge 0} B_i\Big)(x) &= \sum_{i \ge 0} \sum_{j \ge 1} \int K_j(x,y) B_i(y)\, dy \\
&= \sum_{i\ge 0} \sum_{j \ge i+1} \int K_j(x,y) B_i(y)\, dy \\
&= \sum_{s \ge 1} \sum_{j \ge s} \int K_j(x,y) B_{j-s}(y)\, dy = \sum_{s\ge 1} \sum_{j \ge s} V_j(B_{j-s})(x).
\end{aligned} \tag{4.10}
$$

In §5 we shall prove the following.

Lemma 2. *Suppose $w \in A_1$. There exists an $\varepsilon > 0$ such that, for any positive integer s,*

$$\Big\| \sum_{j \ge s} V_j(B_{j-s}) \Big\|^2_{L^2_w} \le c 2^{-\varepsilon s} \lambda \|f\|_{L^1_w}.$$

In §6 we shall prove the following.

Lemma 3. *Suppose $w \in A_1$. Let $\|\cdot\|_{2,w}$ denote the operator norm on L^2_w. Then there exist constants $c, \delta > 0$ such that*

$$\|V_j\|_{2,w} \le c 2^{-\delta j} \quad \textit{for all } j \ge 1.$$

Assuming Lemmas 2 and 3, we now prove the Proposition. From Lemma 3 we easily see that V is bounded on L^2_w. By this boundedness, (4.1), (4.4), (4.5), (4.10) and Lemma 2 we have

$$(4.11)\quad w(\{x \in \mathbb{R}^n \setminus \mathcal{U} : |V(f)(x)| > \lambda\}) \le w(\{x \in \mathbb{R}^n \setminus \mathcal{U} : |V(g)(x)| > \lambda/2\}) + w(\{x \in \mathbb{R}^n \setminus \mathcal{U} : |V(b)(x)| > \lambda/2\})$$
$$\le c\lambda^{-2}\|g\|_{L^2_w}^2 + c\lambda^{-2}\Big\|\sum_{s\ge1}\sum_{j\ge s} V_j(B_{j-s})\Big\|_{L^2_w}^2$$
$$\le c\lambda^{-1}\|f\|_{L^1_w} + c\lambda^{-2}\Big(\sum_{s\ge1}\lambda^{1/2}2^{-\varepsilon s/2}\|f\|_{L^1_w}^{1/2}\Big)^2$$
$$\le c\lambda^{-1}\|f\|_{L^1_w}.$$

On the other hand, by (4.3) we see that

$$(4.12)\quad w(\mathcal{U}) \le c_w\lambda^{-1}\|f\|_{L^1_w}.$$

Combining (4.11) and (4.12), we get the boundedness of V from L^1_w to $L^{1,\infty}_w$. This completes the proof of the Proposition.

5. Proof of Lemma 2.

For $k, m \ge 1$, put

$$(5.1)\quad H_{km}(x,y) = \int \overline{K}_k(z,x)K_m(z,y)\,dz = \int e^{-iP(z,x)+iP(z,y)}\overline{K}(z-x)K(z-y)\varphi_k(z-x)\varphi_m(z-y)\,dz.$$

Then $V_k^*V_m(f)(x) = \int H_{km}(x,y)f(y)\,dy$, where V_k^* denotes the adjoint of V_k.

LEMMA 4. *Let $k \ge m \ge 1$. Then $H_{km}(x,y) = 0$ unless $|x-y| \le 42^k$; and*

(1) $|H_{km}(x,y)| \le c2^{-kn}$,

(2) $|H_{km}(x,y)| \le c2^{-kn}2^{-m}|q(x)-q(y)|^{-1/M}$.

Proof. We prove the estimate (2) only since the other assertion immediately follows from (5.1). We first note that

$$(\partial/\partial z_1)^M(P(z,x) - P(z,y)) = M!(q(x) - q(y)).$$

Hence, from van der Corput's lemma it follows that

$$\Big|\int_a^b e^{i(P(z,x)-P(z,y))}\,dz_1\Big| \le c|q(x)-q(y)|^{-1/M},$$

for any a and b (see [2, p. 152]).

Therefore by integration by parts in variable z_1 in (5.1), and by using the estimates in (1.1), we easily get the conclusion.

For the rest of this note $P(x)$ will denote a real-valued polynomial on $\mathbb{R}^n$.

DEFINITION 1. For a polynomial $P(x) = \sum_{|\alpha|\le N} a_\alpha x^\alpha$ of degree N, define

$$\|P\| = \max_{|\alpha|=N}|a_\alpha|.$$

DEFINITION 2. For a polynomial P and $\beta > 0$, let

$$\mathcal{R}(P,\beta) = \{x \in \mathbb{R}^n : |P(x)| \le \beta\}.$$

Let $d(E,F)$ denote the distance between sets E and F. We now state a geometrical lemma for polynomials which will be proved in §7.

LEMMA 5. *Let k, m be integers such that $k \ge m$. Suppose $N \ge 1$. Then, for any polynomial P of degree N satisfying $\|P\| = 1$ and any $\gamma > 0$, there exists a positive constant $C_{n,N,\gamma}$ depending only on n, N and γ such that*

$$|\{x \in B(a,2^k) : d(x,\mathcal{R}(P,2^{Nm})) \le \gamma 2^m\}| \le C_{n,N,\gamma}2^{(n-1)k}2^m$$

uniformly in $a \in \mathbb{R}^n$.

Let $\lambda > 0$ and let $\{\mathcal{B}_j\}_{j\ge0}$ be a family of measurable functions such that

$$(5.2)\quad \int_Q |\mathcal{B}_j| \le \lambda|Q|$$

for all cubes Q in $\mathbb{R}^n$ with sidelength $\ell(Q) = 2^j$.

Then we have the following.

LEMMA 6. *Let the kernels H_{ji} be as in Lemma 4. Then we can find a constant c such that*

$$\sum_{i=s}^{j}\sup_{x\in\mathbb{R}^n}\Big|\int \mathcal{B}_{i-s}(y)H_{ji}(x,y)\,dy\Big| \le c\lambda 2^{-s}$$

for all integers j and s such that $0 < s \le j$.

DEFINITION 3. For $m \in \mathbb{Z}$ (the set of all integers), let $\mathcal{D}_m$ be the family of all closed dyadic cubes Q with sidelength $\ell(Q) = 2^m$.

Proof of Lemma 6. Fix $x \in \mathbb{R}^n$. Let

$$\mathcal{F} = \{Q \in \mathcal{D}_{i-s} : Q \cap B(x,2^{j+2}) \ne \emptyset\} \quad (0 < s \le i \le j).$$

Then clearly $\sum_{Q\in\mathcal{F}}|Q| \le c2^{jn}$.

Decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where

$$\mathcal{F}_0 = \{Q \in \mathcal{F} : Q \cap \mathcal{R}(q(\cdot) - q(x), 2^{L(i-s)}) \ne \emptyset\}$$

and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. Then by Lemma 5 we have

$$(5.3)\quad \sum_{Q\in\mathcal{F}_0}|Q| \le c2^{(n-1)j}2^{i-s}.$$

By Lemma 4(1), (5.2) and (5.3), we see that

$$(5.4)\quad \sum_{Q\in\mathcal{F}_0}\int_Q |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy \le c2^{-jn}\sum_{Q\in\mathcal{F}_0}\int_Q |\mathcal{B}_{i-s}(y)|\,dy$$
$$\le c2^{-jn}\lambda\sum_{Q\in\mathcal{F}_0}|Q|$$
$$\le c2^{-jn}\lambda 2^{(n-1)j}2^{i-s}$$
$$= c\lambda 2^{i-j-s}.$$

Next, by Lemma 4(2), (5.2) and the estimate $\sum_{Q\in\mathcal{F}_1}|Q|\le c2^{jn}$, we have

$$(5.5)\quad \sum_{Q\in\mathcal{F}_1}\int_Q |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy$$
$$\le c2^{-jn}2^{-i}2^{-L(i-s)/M}\sum_{Q\in\mathcal{F}_1}\int_Q |\mathcal{B}_{i-s}(y)|\,dy$$
$$\le c2^{-jn}2^{-i}2^{-L(i-s)/M}\lambda\sum_{Q\in\mathcal{F}_1}|Q| \le c\lambda 2^{-i}2^{-L(i-s)/M}.$$

From (5.4) and (5.5) it follows that

$$\int |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy = \sum_{Q\in\mathcal{F}}\int_Q |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy$$
$$= \sum_{\nu=0}^{1}\sum_{Q\in\mathcal{F}_\nu}\int_Q |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy$$
$$\le c\lambda(2^{i-j-s}+2^{-i}2^{-L(i-s)/M}).$$

Thus we see that

$$\sum_{i=s}^{j}\sup_{x\in\mathbb{R}^n}\int |\mathcal{B}_{i-s}(y)H_{ji}(x,y)|\,dy \le c\lambda\sum_{i=s}^{j}(2^{i-j-s}+2^{-i}2^{-L(i-s)/M})$$
$$\le c\lambda 2^{-s}.$$

This completes the proof of Lemma 6.

By Lemma 6 we readily get the following.

LEMMA 7. *Let $\{\mathcal{B}_j\}_{j\ge 0}$ be as in Lemma 6. Suppose $\sum_{j\ge 0}\|\mathcal{B}_j\|_{L^1}<\infty$. Then, for any positive integer s, we have*

$$\Big\|\sum_{j\ge s}V_j(\mathcal{B}_{j-s})\Big\|_{L^2}^2 \le c\lambda 2^{-s}\sum_{j\ge 0}\|\mathcal{B}_j\|_{L^1}.$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the inner product in L^2. Using Lemma 6, we see that

$$\Big\|\sum_{j\ge s}V_j(\mathcal{B}_{j-s})\Big\|_{L^2}^2 \le 2\sum_{j\ge s}\sum_{i=s}^{j}|\langle V_j(\mathcal{B}_{j-s}),V_i(\mathcal{B}_{i-s})\rangle|$$
$$\le 2\sum_{j\ge s}\sum_{i=s}^{j}|\langle \mathcal{B}_{j-s},V_j^*V_i(\mathcal{B}_{i-s})\rangle|$$
$$\le 2\sum_{j\ge s}\sum_{i=s}^{j}\|\mathcal{B}_{j-s}\|_{L^1}\|V_j^*V_i(\mathcal{B}_{i-s})\|_{L^\infty}$$
$$\le c\lambda 2^{-s}\sum_{j\ge s}\|\mathcal{B}_{j-s}\|_{L^1}.$$

This completes the proof.

DEFINITION 4. For each $j\ge 0$, let $\mathcal{G}_j$ be a family of non-overlapping closed dyadic cubes Q such that $\ell(Q)\le 2^j$. We suppose that if $Q\in\mathcal{G}_j$, $R\in\mathcal{G}_k$ and $j\ne k$, then Q and R are non-overlapping and that $\sum_{j\ge 0}\sum_{Q\in\mathcal{G}_j}|Q|<\infty$. Put $\mathcal{G}=\bigcup_{j\ge 0}\mathcal{G}_j$.

Let $\lambda>0$. With each $Q\in\mathcal{G}$ we associate $f_Q\in L^1$ such that

$$\int |f_Q|\le\lambda|Q|,\qquad \operatorname{supp}(f_Q)\subset Q.$$

We define

$$\mathcal{A}_i=\sum_{Q\in\mathcal{G}_i}f_Q.$$

LEMMA 8. *Let v be a locally integrable positive function and let s be a positive integer. Then*

$$\Big\|\sum_{j\ge s}V_j(\mathcal{A}_{j-s})\Big\|_{L_v^2}^2 \le c\lambda^2\sum_{Q\in\mathcal{G}}|Q|\inf_Q MM(v),$$

where $\inf_Q f=\inf_{x\in Q}f(x)$.

Proof. The proof we give here is essentially the same as that in [11]. We include it for completeness. We may assume $\sum_{Q\in\mathcal{G}}|Q|\inf_Q MM(v)<\infty$. Let $\langle\cdot,\cdot\rangle_v$ denote the inner product in L_v^2. Then, if $s\le i\le j$, we see that

$$\langle V_j(\mathcal{A}_{j-s}),V_i(\mathcal{A}_{i-s})\rangle_v$$
$$=\int\Big(\int K_j(x,y)\mathcal{A}_{j-s}(y)\,dy\int\overline{K}_i(x,z)\overline{\mathcal{A}}_{i-s}(z)\,dz\Big)v(x)\,dx$$
$$=\int\mathcal{A}_{j-s}(y)\int\overline{\mathcal{A}}_{i-s}(z)\Big(\int K_j(x,y)\overline{K}_i(x,z)v(x)\,dx\Big)\,dz\,dy.$$

Put

$$v(y,z;i,j)=2^{-in}2^{-jn}\int\limits_{B(y,2^{j+2})\cap B(z,2^{i+2})} v(x)\,dx.$$

Let c_Q denote the center of a cube Q. If $Q\in\mathcal{G}_{j-s}$, $R\in\mathcal{G}_{i-s}$ and if $B(y,2^{j+2})$ intersects $B(z,2^{i+2})$ for some $y\in Q$ and some $z\in R$, then $R\subset B(c_Q,n^{1/2}2^{j+10})$. Thus we have

$$\begin{aligned}\sum_{i=s}^{j}|\langle V_j(\mathcal{A}_{j-s}),V_i(\mathcal{A}_{i-s})\rangle_v| &\le c\sum_{i=s}^{j}\int|\mathcal{A}_{j-s}(y)|\int|\mathcal{A}_{i-s}(z)|v(y,z;i,j)\,dz\,dy\\ &\le c\sum_{Q\in\mathcal{G}_{j-s}}\int|f_Q(y)|\sum_{i=s}^{j}\sum_{R\in\mathcal{G}_{i-s}}\int|f_R(z)|v(y,z;i,j)\,dz\,dy\\ &\le c\sum_{Q\in\mathcal{G}_{j-s}}2^{-jn}\int|f_Q(y)|\,dy\sum_{i=s}^{j}\sum_{\substack{R\in\mathcal{G}_{i-s}\\ R\subset B(c_Q,n^{1/2}2^{j+10})}}\inf_R M(v)\int|f_R(z)|\,dz\\ &=I,\end{aligned}$$

say. Since

$$\inf_R M(v)\int|f_R(z)|\,dz\le\lambda|R|\inf_R M(v)\le\lambda\int\limits_R M(v)(z)\,dz$$

and cubes in $\mathcal{G}$ are non-overlapping,

$$\begin{aligned}I&\le c\lambda\sum_{Q\in\mathcal{G}_{j-s}}2^{-jn}\int\limits_{B(c_Q,n^{1/2}2^{j+10})}M(v)(z)\,dz\int|f_Q(y)|\,dy\\ &\le c\lambda\sum_{Q\in\mathcal{G}_{j-s}}\inf_Q MM(v)\int|f_Q(y)|\,dy\le c\lambda^2\sum_{Q\in\mathcal{G}_{j-s}}|Q|\inf_Q MM(v).\end{aligned}$$

Therefore, we get the conclusion by summing over $j\ge s$, since

$$\Big\|\sum_{j\ge s}V_j(\mathcal{A}_{j-s})\Big\|_{L^2_v}^2\le 2\sum_{j\ge s}\sum_{i=s}^{j}|\langle V_j(\mathcal{A}_{j-s}),V_i(\mathcal{A}_{i-s})\rangle_v|.$$

We prove Lemma 2 by the interpolation argument of [11] between the estimates of Lemmas 7 and 8.

LEMMA 9. *Let $\mathcal{F}$ denote the family of dyadic cubes arising from the Calderón–Zygmund decomposition in §4. Then, for all $t>0$, we have*

$$\int\Big|\sum_{j\ge s}V_j(B_{j-s})(x)\Big|^2\min(v(x),t)\,dx\le c\lambda^2\sum_{Q\in\mathcal{F}}|Q|\min(t2^{-s},\inf_Q MM(v)),$$

where s is a positive integer and v is a locally integrable positive function.

Proof. We define

$$\mathcal{F}_t=\{Q\in\mathcal{F}:\inf_Q MM(v)<t2^{-s}\}$$

and $\mathcal{F}_t^*=\mathcal{F}\setminus\mathcal{F}_t$. For $j\ge1$, put

$$B_j'=\sum_{\substack{|Q|=2^{jn}\\ Q\in\mathcal{F}_t}}b_Q,\quad B_j''=\sum_{\substack{|Q|=2^{jn}\\ Q\in\mathcal{F}_t^*}}b_Q$$

and

$$B_0'=\sum_{\substack{|Q|\le1\\ Q\in\mathcal{F}_t}}b_Q,\quad B_0''=\sum_{\substack{|Q|\le1\\ Q\in\mathcal{F}_t^*}}b_Q.$$

Then $B_j=B_j'+B_j''$ for $j\ge0$. Hence

$$\begin{aligned}\int\Big|\sum_{j\ge s}V_j(B_{j-s})(x)\Big|^2\min(v(x),t)\,dx &\le 2\int\Big|\sum_{j\ge s}V_j(B_{j-s}')(x)\Big|^2\min(v(x),t)\,dx\\ &\quad+2\int\Big|\sum_{j\ge s}V_j(B_{j-s}'')(x)\Big|^2\min(v(x),t)\,dx\\ &\le 2\int\Big|\sum_{j\ge s}V_j(B_{j-s}')(x)\Big|^2v(x)\,dx+2\int\Big|\sum_{j\ge s}V_j(B_{j-s}'')(x)\Big|^2t\,dx\\ &=I+II,\end{aligned}$$

say.

Applying Lemma 8 with $\mathcal{A}_j=c_1B_j'$ (see (4.7) and (4.9)), we get

$$I\le c\lambda^2\sum_{Q\in\mathcal{F}_t}|Q|\inf_Q MM(v)=c\lambda^2\sum_{Q\in\mathcal{F}_t}|Q|\min(t2^{-s},\inf_Q MM(v)).$$

By Lemma 7 with $\mathcal{B}_j=c_2B_j''$ (see (4.7) and (4.9)), we have

$$\begin{aligned}II&\le c\lambda t2^{-s}\sum_{j\ge0}\|B_j''\|_{L^1}\le c\lambda^2t2^{-s}\sum_{Q\in\mathcal{F}_t^*}|Q|\\ &=c\lambda^2\sum_{Q\in\mathcal{F}_t^*}|Q|\min(t2^{-s},\inf_Q MM(v)).\end{aligned}$$

(Here c_1 and c_2 are normalizing constants.) Combining the estimates for I and II, we get the conclusion.

Now we finish the proof of Lemma 2. Multiplying both sides of the inequality in Lemma 9 by $t^{-\theta}$ ($\theta \in (0,1)$), then integrating them on $(0,\infty)$ with respect to the measure dt/t and using

$$\int_0^\infty \min(A,t)t^{-\theta}\,\frac{dt}{t} = c_\theta A^{1-\theta} \quad (A>0) \quad \text{for some } c_\theta > 0,$$

we get

$$(5.6)\qquad \int \Big|\sum_{j\ge s} V_j(B_{j-s})(x)\Big|^2 v(x)^{1-\theta}\,dx$$
$$\le c\lambda^2 \sum_{Q\in\mathcal{F}} |Q|2^{-\theta s}\inf_Q MM(v)^{1-\theta}$$
$$\le c\lambda 2^{-\theta s}\sum_{Q\in\mathcal{F}}\int_Q |f(x)|\,dx \inf_Q MM(v)^{1-\theta}$$
$$\le c\lambda 2^{-\theta s}\int |f(x)|MM(v)(x)^{1-\theta}\,dx,$$

where the second inequality follows from (4.2).

If $w\in A_1$, then $w^{1+\delta}\in A_1$ for some $\delta>0$; so substituting $w^{1+\delta}$ for v and putting $\theta=\delta/(1+\delta)$ in (5.6), we get Lemma 2.

6. Proof of Lemma 3

LEMMA 10. *Let $\|\cdot\|_2$ denote the operator norm on L^2. Then, for $j\ge 1$,*

$$\|V_j\|_2 \le \begin{cases} C_{M,L}2^{-j/2-\min(L/M,M/L)j/2} & (M\ne L),\\ C_M j^{1/2}2^{-j} & (M=L).\end{cases}$$

Estimates of this kind have been obtained in Ricci–Stein [9]. Here we give an alternative proof.

Proof of Lemma 10. Fix x. Let

$$E=\mathcal{R}(q(\cdot)-q(x),2^{-jM})\cap B(x,2^{j+2}) \quad\text{and}\quad F=B(x,2^{j+2})\setminus E.$$

Define

$$\mathcal{E}=\{Q\in\mathcal{D}_{-[jM/L]} : Q\cap E\ne\emptyset\},$$

where $[a]$ denotes the greatest integer not exceeding a. Then by Lemmas 4(1) and 5 we have

$$(6.1)\qquad \int_E |H_{jj}(x,y)|\,dy \le \sum_{Q\in\mathcal{E}}\int_Q |H_{jj}(x,y)|\,dy \le c2^{-jn}\sum_{Q\in\mathcal{E}}|Q|$$
$$\le c2^{-j}2^{-jM/L}.$$

For $\nu=0,1,\ldots,$ let

$$F_\nu = B(x,2^{j+2})\cap(\mathcal{R}(q(\cdot)-q(x),2^{-jM+\nu+1})\setminus\mathcal{R}(q(\cdot)-q(x),2^{-jM+\nu})).$$

Then $F=\bigcup_{\nu=0}^\infty F_\nu$. For $0\le\nu\le j(M+L)-1$, let

$$\mathcal{F}_\nu=\{Q\in\mathcal{D}_{-[(jM-\nu-1)/L]} : Q\cap F_\nu\ne\emptyset\}.$$

Then by Lemma 5 we have

$$|F_\nu|\le\sum_{Q\in\mathcal{F}_\nu}|Q|\le c2^{j(n-1)}2^{-[(jM-\nu-1)/L]}.$$

So by Lemma 4(2) we see that

$$\int_{F_\nu}|H_{jj}(x,y)|\,dy\le c2^{-jn}2^{-j}2^{j-\nu/M}|F_\nu|\le c2^{-j}2^{-\nu/M}2^{-[(jM-\nu-1)/L]}$$
$$\le c2^{-j}2^{-jM/L}2^{-\nu(1/M-1/L)}.$$

Thus, if $M\ne L$, then

$$(6.2)\qquad \sum_{\nu=0}^{j(M+L)-1}\int_{F_\nu}|H_{jj}(x,y)|\,dy$$
$$\le c2^{-j}2^{-jM/L}(2^{-(1/M-1/L)}-1)^{-1}(2^{-j(M+L)(1/M-1/L)}-1)$$
$$= c2^{-j}(2^{-(1/M-1/L)}-1)^{-1}(2^{-jL/M}-2^{-jM/L});$$

and if $M=L$, then

$$(6.3)\qquad \sum_{\nu=0}^{j(M+L)-1}\int_{F_\nu}|H_{jj}(x,y)|\,dy\le c2^{-2j}j(M+L).$$

Finally, by Lemma 4(2) we have

$$(6.4)\qquad \int_{\bigcup_{\nu\ge j(M+L)}F_\nu}|H_{jj}(x,y)|\,dy\le c2^{-jn}2^{-j}2^{-jL/M}\Big|\bigcup_{\nu\ge j(M+L)}F_\nu\Big|$$
$$\le c2^{-j}2^{-jL/M}.$$

By (6.1)–(6.4) we see that

$$\sup_{x\in\mathbb{R}^n}\int|H_{jj}(x,y)|\,dy\le\begin{cases}C_{M,L}2^{-j-\min(L/M,M/L)j} & (M\ne L),\\ C_M j2^{-2j} & (M=L).\end{cases}$$

We have the same estimate for $\sup_y\int|H_{jj}(x,y)|\,dx$. From these results we get the conclusion since

$$\|V_j^*V_j(f)\|_{L^2}\le\Big(\sup_x\int|H_{jj}(x,y)|\,dy\Big)^{1/2}\Big(\sup_y\int|H_{jj}(x,y)|\,dx\Big)^{1/2}\|f\|_{L^2}$$

and $\|V_j^*V_j\|_2=\|V_j\|_2^2$.

Now we prove Lemma 3. It is easy to see that $\|V_j\|_{2,w} \le c_w$. By interpolation with change of measure between this estimate and that of Lemma 10, we get

$$\|V_j\|_{2,w^\theta} \le c_{w,\theta} 2^{-(1-\theta)j/2} \tag{6.5}$$

for all $\theta \in (0,1)$. We have $w^{1+\varepsilon} \in A_1$ for some $\varepsilon > 0$; so substituting $w^{1+\varepsilon}$ for w and putting $\theta = 1/(1+\varepsilon)$ in (6.5), we get the desired estimate.

7. Proof of Lemma 5. Our proof is an application of the methods appearing in the proof of [2, Lemma 4.1]. We use some tools and results given in [2].

DEFINITION 5. Suppose $n \ge 2$. Let

$$S_m = \{Q_m + (0,\ldots,0,j) : j \in \mathbb{Z}\},$$

where $m = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $Q_m = [0,1]^n + (m_1, \ldots, m_{n-1}, 0)$. We call S_m a *strip*.

DEFINITION 6. Suppose $n \ge 2$. For $m \in \mathbb{Z}^{n-1}$, we define

$$I_m = \{Q_m + (0,\ldots,0,j) : j_1 < j < j_2\},$$

where $j_1, j_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$ and Q_m is as in Definition 5. We call I_m an *interval*.

DEFINITION 7. For a set $E \subset \mathbb{R}^n$, we put

$$\mathcal{N}(E) = \{x \in \mathbb{R}^n : d(x,E) \le 1\}.$$

Let P be a polynomial of degree N as in Lemma 5. We consider $\mathcal{R}(P,\beta)$ for $\beta > 0$ (see Definition 2).

LEMMA 11. *Suppose that $n \ge 2$ and $N \ge 1$. There exists a positive integer $C_{n,N}$ depending only on n and N such that for $i = 1, \ldots, C_{n,N}$ we can find $U_i \in O(n)$ (the orthogonal group) and families of cubes $J_{m,i} \subset S_m$ ($m \in \mathbb{Z}^{n-1}$) so that*

(1) $\mathcal{N}(\mathcal{R}(P,\beta)) \subset \bigcup_{i=1}^{C_{n,N}} U_i(\mathcal{L}_i)$, *where*

$$\mathcal{L}_i = \bigcup \Big\{Q : Q \in \bigcup_{m \in \mathbb{Z}^{n-1}} J_{m,i}\Big\};$$

(2) $\mathrm{card}(J_{m,i}) \le c$ *for some constant c depending only on n, N and β.*

REMARK 3. If Lemma 11 holds, then we have, for any $\gamma > 0$,

$$\{x : d(x, \mathcal{R}(P,\beta)) \le \gamma\} \subset \bigcup_{i=1}^{C_{n,N,\gamma}} U_i(\mathcal{L}_i)$$

for some positive integer $C_{n,N,\gamma}$ depending only on n, N and γ, where U_i and $\mathcal{L}_i$ are as in Lemma 11. This can be proved by considering a finite number of polynomials which are defined by translating P and by applying Lemma 11 to each of them. (See [2, p. 149].)

To prove Lemma 11, we need the following results given in [2].

SUBLEMMA 1. *Suppose $n \ge 2$. For any positive integer N, there exists a positive integer $C_{n,N}$ depending only on n and N such that for any strip S, any polynomial P of degree N and any $\gamma > 0$,*

$$\{Q \in S : Q \cap \mathcal{R}(P,\gamma) \ne \emptyset\}$$

is a union of at most $C_{n,N}$ intervals. (See Lemma 4.2 of [2].)

SUBLEMMA 2. *Suppose $n \ge 2$. For any positive integer N, there exist positive constants $A_{n,N}$ and $B_{n,N}$ depending only on n and N such that*

$$A_{n,N}\|P\| \le \|P \circ \Xi\| \le B_{n,N}\|P\|$$

for every polynomial P of degree N and every $\Xi \in O(n)$, where $P \circ \Xi(x) = P(\Xi x)$.

SUBLEMMA 3. *Suppose $n \ge 2$. For any positive integer N, there exists a positive constant $C_{n,N}$ depending only on n and N such that for any polynomial P of degree N we can find $\Theta \in O(n)$ so that*

$$\min_{1 \le j \le n} \|D_j(P \circ \Theta)\| \ge C_{n,N}\|P \circ \Theta\|,$$

where $D_j = \partial/\partial x_j$.

Now we prove Lemma 11. We use induction on the polynomial degree N. Let $A(N)$ be the assertion of Lemma 11 for polynomials of degree N.

Proof of $A(1)$. Let $P(x) = \sum_{i=1}^n a_i x_i + b$. First, we consider the case $|a_n| = 1$. Now we show that if I is an interval such that each cube of I intersects $\mathcal{R}(P,\beta)$, then $\mathrm{card}(I) \le c$ for some c depending only on n and β. Let $y \in Q \in I$ satisfy $|P(y)| \le \beta$. We note that

$$P(y + de_n) - P(y) = da_n \quad \text{for } d \in \mathbb{R},$$

where e_j is the element of $\mathbb{R}^n$ whose jth coordinate is 1 and whose other coordinates are all 0. Therefore, if $y + de_n \in Q' \in I$, we see that

$$\inf_{z \in Q'} |P(z)| \ge |P(y + de_n)| - \sum_{i=1}^n |a_i| \ge |da_n| - \beta - \sum_{i=1}^n |a_i| \ge |d| - \beta - n.$$

This easily implies that $\mathrm{card}(I) \le c$.

By this and Sublemma 1, there exists a constant c depending only on n and β such that

$$\mathrm{card}(\{Q \in S : Q \cap \mathcal{R}(P,\beta) \ne \emptyset\}) \le c$$

for all strips S. Therefore, if we put

$$J_m = \{Q \in S_m : d(Q, \mathcal{R}(P,\beta)) \le 1\},$$

then $\mathrm{card}(J_m) \le c$ for some c depending only on n and β; and $\mathcal{N}(\mathcal{R}(P,\beta)) \subset \mathcal{L}$, where

$$\mathcal{L} = \bigcup \Big\{ Q : Q \in \bigcup_{m\in\mathbb{Z}^{n-1}} J_m \Big\}.$$

Next, we consider any polynomial P of degree 1 such that $\|P\| = 1$. Then if $P_1(x) = P(Ux)$ for a suitable $U \in O(n)$, we have $D_n P_1 = 1$. Hence, by what we have already proved we get $\mathcal{N}(\mathcal{R}(P_1,\beta)) \subset \mathcal{L}$. It follows that $\mathcal{N}(\mathcal{R}(P,\beta)) \subset U(\mathcal{L})$ since $\mathcal{N}(\mathcal{R}(P\circ U,\beta)) = U^{-1}\mathcal{N}(\mathcal{R}(P,\beta))$. This completes the proof of $A(1)$.

Now we assume $A(N-1)$ $(N \ge 2)$ and prove $A(N)$. For a polynomial P of degree N such that $\|P\| = 1$, we choose $\Theta \in O(n)$ as in Sublemma 3. Put

$$E_0 = \mathcal{R}(P\circ\Theta,\beta) \cap \Big(\bigcup_{j=1}^{n} \mathcal{R}(D_j(P\circ\Theta),\beta) \Big);$$

and for $\kappa = (\kappa_1,\ldots,\kappa_n) \in \{-1,1\}^n$ put

$$E_\kappa = \{x \in \mathcal{R}(P\circ\Theta,\beta) : \kappa_j D_j(P\circ\Theta)(x) > \beta \text{ for } j = 1,\ldots,n\}.$$

Then

$$\mathcal{R}(P\circ\Theta,\beta) = E_0 \cup \bigcup_{\kappa\in\{-1,1\}^n} E_\kappa$$

and so

$$\mathcal{N}(\mathcal{R}(P\circ\Theta,\beta)) = \mathcal{N}(E_0) \cup \bigcup_{\kappa\in\{-1,1\}^n} \mathcal{N}(E_\kappa). \tag{7.1}$$

We separately treat the $2^n + 1$ sets on the right hand side.

First, clearly

$$\mathcal{N}(E_0) \subset \bigcup_{j=1}^{n} \mathcal{N}(\mathcal{R}(D_j(P\circ\Theta),\beta)). \tag{7.2}$$

Since $C_j = \|D_j(P\circ\Theta)\| \sim 1$ (this means that $c^{-1} \le \|D_j(P\circ\Theta)\| \le c$ for some $c > 1$ depending only on n and N) and $\mathcal{R}(D_j(P\circ\Theta),\beta) = \mathcal{R}(C_j^{-1}D_j(P\circ\Theta), C_j^{-1}\beta)$, we can apply the induction hypothesis $A(N-1)$ to the right hand side of (7.2).

Next, we fix κ and consider $\mathcal{N}(E_\kappa)$. Pick $O_\kappa \in O(n)$ such that $O_\kappa(e_n) = n^{-1/2}\kappa$. Define

$$\mathcal{D}_0^* = \mathcal{D}_0 \setminus \Big\{ Q \in \mathcal{D}_0 : \Big(\bigcup_{j=1}^{n} \mathcal{R}((D_j(P\circ\Theta))\circ O_\kappa,\beta) \Big) \cap Q \ne \emptyset \Big\}.$$

Since $\|(D_j(P\circ\Theta))\circ O_\kappa\| \sim 1$ by Sublemmas 2 and 3, we can apply the hypothesis $A(N-1)$ along with Remark 3 to

$$G = \bigcup \Big\{ Q \in \mathcal{D}_0 : \Big(\bigcup_{j=1}^{n} \mathcal{R}((D_j(P\circ\Theta))\circ O_\kappa,\beta) \Big) \cap Q \ne \emptyset \Big\}$$

to get

$$\mathcal{N}(G) \subset \bigcup_i U_i'(\mathcal{L}_i') \tag{7.3}$$

for some $U_i' \in O(n)$ and some $\mathcal{L}_i'$ such that

$$\mathcal{L}_i' = \bigcup \Big\{ Q : Q \in \bigcup_{m\in\mathbb{Z}^{n-1}} J_{m,i}' \Big\}$$

for some $J_{m,i}'$ $(\subset S_m)$ satisfying $\mathrm{card}(J_{m,i}') \le c$.

We have to study $O_\kappa^{-1}(E_\kappa) \cap \bigcup \mathcal{D}_0^*$. First, we note that if $O_\kappa^{-1}(E_\kappa)$ intersects Q, $Q \in \mathcal{D}_0^*$, then

$$\min_{1\le j\le n} \kappa_j D_j(P\circ\Theta)(O_\kappa y) > \beta \quad \text{for all } y \in Q. \tag{7.4}$$

This can be seen as follows. Suppose that there are j_0 and $y_0 \in Q$ such that $\kappa_{j_0} D_{j_0}(P\circ\Theta)(O_\kappa y_0) \le \beta$. Then, since we have $\kappa_{j_0} D_{j_0}(P\circ\Theta)(O_\kappa x) > \beta$ for some $x \in Q$, by the intermediate value theorem we can find $z \in Q$ such that $|D_{j_0}(P\circ\Theta)(O_\kappa z)| \le \beta$. This contradicts the fact that $Q \in \mathcal{D}_0^*$.

By (7.4) we have

$$\begin{aligned} O_\kappa^{-1}(E_\kappa) \cap \bigcup \mathcal{D}_0^* &\subset \bigcup \{Q \in \mathcal{D}_0 : \min_{1\le j\le n} \kappa_j D_j(P\circ\Theta)(O_\kappa y) > \beta \text{ for all } y \in Q \\ &\qquad \text{and } \mathcal{R}(P\circ\Theta\circ O_\kappa,\beta) \cap Q \ne \emptyset\}. \end{aligned} \tag{7.5}$$

For a strip S, put

$$\begin{aligned} \mathcal{E} = \{Q \in S : \min_{1\le j\le n} \kappa_j D_j(P\circ\Theta)(O_\kappa y) > \beta \text{ for all } y \in Q \\ \text{and } \mathcal{R}(P\circ\Theta\circ O_\kappa,\beta) \cap Q \ne \emptyset\}. \end{aligned}$$

We shall show $\mathrm{card}(\mathcal{E}) \le C_{n,N}$.

We first see that $\mathcal{E}$ is a union of at most $C_{n,N}$ intervals. Put

$$\begin{aligned} \mathcal{E}' = \{Q \in S : \min_{1\le j\le n} |D_j(P\circ\Theta)(O_\kappa y)| > \beta \text{ for all } y \in Q \\ \text{and } \mathcal{R}(P\circ\Theta\circ O_\kappa,\beta) \cap Q \ne \emptyset\}. \end{aligned}$$

Then

$$\mathcal{E}' = \Big(\bigcap_{j=1}^{n}(S\setminus\{Q\in S:\mathcal{R}((D_j(P\circ\Theta))\circ O_\kappa,\beta)\cap Q\neq\emptyset\})\Big)$$
$$\cap\{Q\in S:\mathcal{R}(P\circ\Theta\circ O_\kappa,\beta)\cap Q\neq\emptyset\}.$$

We observe that the complement of a finite union of intervals in a strip S is also a finite union of intervals, and the intersection of finite unions of intervals is also a finite union of intervals. Hence, by Sublemma 1 we see that $\mathcal{E}'$ is a union of at most $C_{n,N}$ intervals: $\mathcal{E}'=\bigcup_i J_i$.

Consider any J_i. Then by the intermediate value theorem we have either

$$\min_{1\le j\le n}\kappa_j D_j(P\circ\Theta)(O_\kappa y)>\beta \quad \text{for all } y\in\bigcup\{Q:Q\in J_i\}$$

or

$$\min_{1\le j\le n}\kappa_j D_j(P\circ\Theta)(O_\kappa y)<-\beta \quad \text{for all } y\in\bigcup\{Q:Q\in J_i\}.$$

Thus $\mathcal{E}$ is a union of a subfamily $\{I_i\}$ of $\{J_i\}:\mathcal{E}=\bigcup_i I_i$.

Let I be any interval in $\{I_i\}$. We need the following (see [2, p. 151]).

SUBLEMMA 4. *There exists a constant c_n depending only on n such that if $x,y\in I$ and $y_n-x_n\ge c_n$, then*

$$y-x=\sum_{i=1}^{n}\lambda_i O_\kappa^{-1}e_i$$

for some $\lambda_i\in\mathbb{R}$ such that $\kappa_i\lambda_i\ge 3$.

Proof. We see that

$$O_\kappa(y-x)=\sum_{i=1}^{n}(y_i-x_i)O_\kappa e_i=\sum_{i=1}^{n-1}(y_i-x_i)O_\kappa e_i+(y_n-x_n)n^{-1/2}\kappa$$
$$=\sum_{i=1}^{n}(n^{-1/2}(y_n-x_n)\kappa_i+b_i)e_i$$

for some $b_i\in\mathbb{R}$ such that $|b_i|\le c$, which is feasible since $|y_i-x_i|\le 1$ for $i=1,\ldots,n-1$. This readily implies the conclusion.

Put $Y=P\circ\Theta\circ O_\kappa$. Then $\nabla Y(x)=O_\kappa^{-1}(\nabla(P\circ\Theta)(O_\kappa x))$; so, if $x,y\in I$ and $y_n-x_n\ge c_n$, by Sublemma 4 we have

$$Y(y)-Y(x)=\int_0^1\langle y-x,(\nabla Y)(x+t(y-x))\rangle\,dt$$
$$=\int_0^1\sum_{i=1}^{n}\lambda_i\langle O_\kappa^{-1}e_i,O_\kappa^{-1}(\nabla(P\circ\Theta)(O_\kappa(x+t(y-x))))\rangle\,dt$$

$$=\int_0^1\sum_{i=1}^{n}\lambda_i D_i(P\circ\Theta)(O_\kappa(x+t(y-x)))\,dt$$
$$\ge\sum_{i=1}^{n}\lambda_i\kappa_i\beta\ge 3n\beta>3\beta,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Since $\mathcal{R}(Y,\beta)\cap Q\neq\emptyset$ for all $Q\in I$, we can conclude that $\mathrm{card}(I)\le c_n+3$.

Combining the above results, we have $\mathrm{card}(\mathcal{E})\le C_{n,N}$ as claimed. From this and (7.5) we easily see that

$$\mathcal{N}\Big(O_\kappa^{-1}(E_\kappa)\cap\bigcup\mathcal{D}_0^*\Big)\subset\mathcal{L}, \tag{7.6}$$

where $\mathcal{L}=\bigcup\{Q:Q\in\bigcup_{m\in\mathbb{Z}^{n-1}}J_m\}$ for some $J_m\subset S_m$ with $\mathrm{card}(J_m)\le C_{n,N}$.

By (7.3) and (7.6) we have

$$\mathcal{N}(O_\kappa^{-1}(E_\kappa))\subset\mathcal{N}(G)\cup\mathcal{N}\Big(O_\kappa^{-1}(E_\kappa)\cap\bigcup\mathcal{D}_0^*\Big)$$
$$\subset\Big(\bigcup_i U_i'(\mathcal{L}_i')\Big)\cup\mathcal{L};$$

and so, observing $\mathcal{N}(O_\kappa^{-1}(E_\kappa))=O_\kappa^{-1}\mathcal{N}(E_\kappa)$,

$$\mathcal{N}(E_\kappa)\subset\Big(\bigcup_i O_\kappa U_i'(\mathcal{L}_i')\Big)\cup O_\kappa(\mathcal{L}). \tag{7.7}$$

Since $\mathcal{N}(\mathcal{R}(P\circ\Theta,\beta))=\Theta^{-1}\mathcal{N}(\mathcal{R}(P,\beta))$, by (7.1), (7.2) with $A(N-1)$ and (7.7) we get $A(N)$. This completes the proof of Lemma 11.

Proof of Lemma 5. We see that $\mathcal{R}(P,2^{Nm})=2^m\mathcal{R}(\widetilde{P},1)$, where

$$\widetilde{P}(x)=2^{-Nm}P(2^m x).$$

Note that $\|\widetilde{P}\|=1$. (See [2, p. 151].) This observation enables us to assume $m=0$ to prove Lemma 5. Clearly, we may also assume $\gamma=1$.

Thus it suffices to show, for $k\ge 0$,

$$|\{x\in B(a,2^k):d(x,\mathcal{R}(P,1))\le 1\}|\le C_{n,N}2^{(n-1)k} \tag{7.8}$$

uniformly in $a\in\mathbb{R}^n$.

If $n=1$, (7.8) easily follows from Chanillo–Christ [2, Lemma 3.2] (see also [4]). Suppose $n\ge 2$. Then (7.8) follows from Lemma 11 with $\beta=1$ and the obvious estimate

$$|B(a,2^k)\cap U_i(\mathcal{L}_i)|\le c2^{(n-1)k},$$

where $U_i(\mathcal{L}_i)$ is as in Lemma 11. This completes the proof of Lemma 5.

References

[1] S. Chanillo, *Weighted norm inequalities for strongly singular convolution operators*, Trans. Amer. Math. Soc. 281 (1984), 77–107.

[2] S. Chanillo and M. Christ, *Weak (1, 1) bounds for oscillatory singular integrals*, Duke Math. J. 55 (1987), 141–155.

[3] S. Chanillo, D. S. Kurtz and G. Sampson, *Weighted weak (1, 1) and weighted L^p estimates for oscillating kernels*, Trans. Amer. Math. Soc. 295 (1986), 127–145.

[4] M. Christ, *Hilbert transforms along curves, I: Nilpotent groups*, Ann. of Math. 122 (1985), 575–596.

[5] —, *Weak type (1, 1) bounds for rough operators*, ibid. 128 (1988), 19–42.

[6] —, *Weak type endpoint bounds for Bochner–Riesz multipliers*, Rev. Mat. Iberoamericana 3 (1987), 25–31.

[7] M. Christ and J. L. Rubio de Francia, *Weak type (1, 1) bounds for rough operators, II*, Invent. Math. 93 (1988), 225–237.

[8] J.-L. Journé, *Calderón–Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Math. 994, Springer, 1983.

[9] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals, I*, J. Funct. Anal. 73 (1987), 179–194.

[10] S. Sato, *Some weighted weak type estimates for rough operators*, Math. Nachr. 187 (1997), 211–240.

[11] A. Vargas, *Weighted weak type (1, 1) bounds for rough operators*, J. London Math. Soc. (2) 54 (1996), 297–310.

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A variant sharp estimate for multilinear singular integral operators

by

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Abstract. We establish a variant sharp estimate for multilinear singular integral operators. As applications, we obtain the weighted norm inequalities on general weights and certain $L\log^+ L$ type estimates for these multilinear operators.

1. Introduction. We will work on $\mathbb{R}^n$, $n\ge 1$. Let m_1, m_2 be two positive integers and $m = m_1 + m_2$. Suppose that $K\in C^1(\mathbb{R}^n\setminus\{0\})$ is homogeneous of degree $-n$ and satisfies

$$|K(x)|\le C|x|^{-n} \quad\text{and}\quad |\nabla K(x)|\le C|x|^{-n-1} \quad \text{for } |x|\ne 0,$$
$$\int_{|x|=1} K(x)x^\gamma\,dx = 0 \quad \text{for any } |\gamma|\le m.$$

Let A_j be a function on $\mathbb{R}^n$ whose derivatives of order m_j belong to the space $\mathrm{BMO}(\mathbb{R}^n)$ for $j=1,2$. Define the multilinear singular integral operator T_{A_1,A_2} by

$$T_{A_1,A_2}f(x) = \text{p.v.}\int_{\mathbb{R}^n} K(x-y)\frac{\prod_{j=1}^2 P_{m_j+1}(A_j;x,y)}{|x-y|^m} f(y)\,dy, \tag{1}$$

where $P_{m_j+1}(A_j;x,y)$ denotes the (m_j+1)th order Taylor series remainder of A_j at x about y, precisely,

$$P_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j}\frac{1}{\alpha!}D^\alpha A_j(y)(x-y)^\alpha. \tag{2}$$

It is well known that the operators of this type have been studied by many authors (see [2], [4], [5] and [9]). We point out that the first result in this direction was established by Coifman, Rochberg and Weiss in [5]. The

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