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A class of Fourier multipliers on $H^1(\mathbb{R}^2)$

by

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Abstract. An integral criterion for being an $H^1(\mathbb{R}^2)$ Fourier multiplier is proved. It is applied in particular to suitable regular functions which depend on the product of variables.

We give integral criteria for a function of two real variables to be a Fourier multiplier on the space $H^1(\mathbb{R}^2)$. The main result deals with functions whose support is contained in the set $\{\xi : |\xi_1\xi_2| \le c\}$, and all the other results are its direct consequences.

THEOREM 1. *Suppose that* $m \in \mathcal{S}'(\mathbb{R}^2)$ *is such that* $\operatorname{supp}\widehat{m} \subset \{\xi : |\xi_1\xi_2| < C\}$ *and it has locally integrable derivatives* $\frac{\partial^{\alpha+\beta}}{\partial\xi_1^\alpha\partial\xi_2^\beta}\widehat{m}(\xi)$ *for* $\alpha, \beta \in \{0,1,2\}$ *which satisfy for every* $r > 1$,

$$I_{\alpha,\beta}(r) = r^{\alpha-\beta} \int\limits_{\substack{r<\xi_1<2r\\ \xi_1>\xi_2}} \left| \frac{\partial^{\alpha+\beta}}{\partial\xi_1^\alpha\partial\xi_2^\beta}\widehat{m}(\xi)\right| d\xi < C, \tag{1}$$

and a symmetrically modified inequality for $\xi_2 > \xi_1$. *Then* $\widehat{m}$ *is a Fourier multiplier on* $H^1(\mathbb{R}^2)$.

The condition (1) is much weaker than Hörmander's integral condition, which in general is not satisfied by the multipliers considered in this paper. In general only the multidimensional Marcinkiewicz multiplier theorem can be applied to multipliers of this type, but it gives no information about their L^1-norm behavior. Moreover, as shown in [W2], operators given by multipliers described in Theorem 1 which do not tend to 0 at infinity may not be of weak type $(1,1)$. On the other hand, at least for sufficiently regular multipliers which depend on the product of variables, one can check using the method developed in [W1] that they are bounded on the multiparameter Hardy space $H^1(\mathbb{R}\times\mathbb{R})$. This in turn implies that the norm of such a

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multiplier operator on $L^p(\mathbb{R}^2)$ is majorized by $(p-1)^{-2}$ as $p \to 1$. Clearly Theorem 1 allows us to improve this bound to $(p-1)^{-1}$.

The proof of Theorem 1 is based on some refinement of the Stein theorem on Fourier multipliers on $H^1(\mathbb{R})$ given in Lemma 1, the Littlewood–Paley theory which is used in Lemmas 2 and 3 to construct, using Lemma 1, some specific auxiliary Fourier multiplier on $H^1(\mathbb{R}^2)$ and finally on the theorem of Marcinkiewicz type for multipliers on H^1 spaces on product domains (cf. [W1]). Notice that we use the theory of multiparameter Hardy spaces to obtain a result concerning the classical Hardy space. Roughly speaking, using the multiplier constructed in Lemma 3 and the Littlewood–Paley theory, we are able factorize the multiplier from Theorem 1 through some other multiplier m' acting on $H^1(\mathbb{R}\times\mathbb{R})$. Condition (1) yields that m' satisfies an integral condition of Marcinkiewicz type, precisely that which is the assumption of the result of [W1].

Theorem 1 is applied first of all to multipliers which depend on the product of variables.

COROLLARY 1. *Let $f:\mathbb{R}\to\mathbb{R}$ have a bounded support and $f^{(k)}\in L^1(\mathbb{R})$ for $k=0,1,2,3,4$. Let $m\in\mathcal{S}'(\mathbb{R}^2)$ be such that $\widehat{m}(\xi)=f(\xi_1\xi_2)$. Then $\widehat{m}$ is a Fourier multiplier on $H^1(\mathbb{R}^2)$.*

Notice that the norm of the multiplier from Corollary 1 does not depend only on the Sobolev norms of derivatives, but also on the size of the support of f. Nevertheless one can remove the bounded support assumption at the price of strengthening the integral conditions on f.

COROLLARY 2. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ satisfies*

$$\int_{\mathbb{R}}(1+|t|)^{k-1}|f^{(k)}(t)|\,dt<\infty \quad \text{for } k=0,1,2,3,4.$$

Let $\widehat{m}(\xi)=f(\xi_1\xi_2)$ for some $m\in\mathcal{S}'(\mathbb{R}^2)$. Then $\widehat{m}$ is a Fourier multiplier on $H^1(\mathbb{R}^2)$.

As a consequence we get immediately an answer to the question stated in [BBPW].

COROLLARY 3. *The function $m(\xi)=\frac{\xi_1\xi_2}{1+(\xi_1\xi_2)^2}$ is a Fourier multiplier on $H^1(\mathbb{R}^2)$.*

Elements of $\mathbb{R}^2$ are denoted by $x=(x_1,x_2)$, $y=(y_1,y_2)$, and Greek characters $\xi=(\xi_1,\xi_2)$, $\eta=(\eta_1,\eta_2)$ stand for elements of the dual group. The symbol $\mathbb{R}_+$ stands for positive real numbers. A function $\psi\in L^\infty(\mathbb{R}^2)$ is called a *Fourier multiplier* on a translation invariant function space $X(\mathbb{R}^2)$ if the formula $T_\psi f=(\widehat{f}\psi)^\vee$ defines a bounded operator on $X(\mathbb{R}^2)$. Here the symbols "$\wedge$" and "$\vee$" denote the Fourier transform and its inverse. $H^1(\mathbb{R}^2)$ stands for the space of functions f such that $f, R_1f, R_2f\in L^1(\mathbb{R}^2)$, where

R_i is the Riesz transform with symbol $\xi_i/|\xi|$. By $H^1(\mathbb{R}\times\mathbb{R})$ we mean in this paper the subspace of $L^1(\mathbb{R}^2)$ consisting of functions with Fourier transforms supported by the first quadrant.

The proof of Theorem 1 is based on a series of lemmas. Lemmas 1 and 2 deal with Fourier multipliers on $H^1(\mathbb{R})$.

Suppose that $\psi:\mathbb{R}\to\mathbb{R}$ is such that $\operatorname{supp}\widehat{\psi}\subset(3/4,3/2)$ and

$$(2)\qquad \begin{aligned}&\int_{s>t}|\psi(s)|\,ds<C_1t^{-a} \quad \text{for some } a>0 \text{ and every } t>0,\\ &\int_{\mathbb{R}}|\psi'(s)|\,ds=C_2<\infty.\end{aligned}$$

Notice that the estimate in (2) holds if, for example, $\widehat{\psi}$ has square integrable derivative. Let $\psi_\lambda,\psi_{\lambda,n}\in\mathcal{S}'(\mathbb{R})$ satisfy $\widehat{\psi_\lambda}(t)=\widehat{\psi}(1+(t-1)\lambda)$ for $\lambda\ge 1$ and

$$(3)\qquad \widehat{\psi}_{\lambda,n}(t)=\widehat{\psi}_\lambda(t/2^n),$$

and let $h_\lambda\in\mathcal{S}'(\mathbb{R})$ be such that

$$(4)\qquad \widehat{h}_\lambda=\sum_{n=1}^{\infty}\widehat{\psi}_{\lambda,n}.$$

LEMMA 1. *The norm of $\widehat{h}_\lambda$ as a Fourier multiplier on $H^1(\mathbb{R})$ is uniformly bounded for $\lambda\ge1$.*

REMARK. The lemma is a refinement of Stein's theorem on H^1 multipliers which gives the boundedness of these multipliers. The refinement is the uniform boundedness in λ.

Proof (based on the proof of Hörmander's theorem, cf. [Hö], Th. 7.9.5). We have $|\psi_\lambda(s)|=\lambda^{-1}|\psi(\lambda^{-1}s)|$ and

$$(5)\qquad \int_{s>t}|\psi_\lambda(s)|\,ds=\int_{v>\lambda^{-1}t}|\psi(v)|\,dv<C_1\lambda^at^{-a}.$$

Similarly

$$\int_{\mathbb{R}}|\psi_\lambda'(s)|\,ds=\lambda^{-1}\int_{\mathbb{R}}|\psi'(v)|\,dv=C_2\lambda^{-1},$$

which implies

$$(6)\qquad \int_{\mathbb{R}}|\psi_\lambda(s+y)-\psi_\lambda(s)|\,ds<C_2\lambda^{-1}|y|.$$

Let $I(a,t)=\{x:|x-a|\le t\}$. We will show that

$$(7)\qquad \int_{\mathbb{R}\setminus I^*}|h_\lambda*w|\,dx\le C\int|w|\,dx \quad \text{for } w\in C_0^\infty(I) \text{ with } \int w\,dx=0$$

where $I^* = I(a, 2t)$. It is enough to prove (7) for $a = 0$. Since the Fourier transform of $\psi_\lambda(Rx)R$ is $\widehat{\psi}_\lambda(\xi/R)$, (3) and (4) yield

$$h_\lambda(x) = \sum_{j=1}^{\infty} \psi_\lambda(2^j x)2^j$$

with $\mathcal{S}'$ convergence. Since $\operatorname{supp} w \subset I$, (5) and (6) give

$$\begin{aligned}\int_{x\notin I^*} R|\psi_\lambda(\cdot R) * w|\,dx &\le \int_{|x|>t} |\psi_\lambda(Rx)|\,d(Rx)\int |w|\,dx \\ &\le C_1\lambda^a (tR)^{-a}\int |w|\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{x\notin I^*} R|\psi_\lambda(\cdot R) * w|\,dx &\le \iint |w(y)(\psi_\lambda((x-y)R) - \psi_\lambda(xR))|R\,dx\,dy \\ &\le C_2\lambda^{-1} tR\int |w|\,dx.\end{aligned}$$

Hence the triangle inequality gives

$$\begin{aligned}\int_{x\notin I^*} |h_\lambda * w|\,dx &\le C\Big(\lambda^a \sum_{2^j t\ge\lambda} (2^j t)^{-a} + \lambda^{-1}\sum_{2^j t<\lambda} 2^j t\Big)\int |w|\,dx \\ &\le C'\int |w|\,dx,\end{aligned}$$

which proves (7).

Let now $u \in H^1(\mathbb{R})$. By [T, §XIV, Th. 1.10], u admits an atomic decomposition

$$u = \sum_{j=1}^{\infty} \lambda_j u_j$$

where $\sum |\lambda_j| \lesssim \|u\|_{H^1(\mathbb{R})}$ and u_j are $(1, 2, 0)$-atoms. Recall that a function v is called a $(1, 2, 0)$-*atom* provided there exist $a \in \mathbb{R}$ and $r > 0$ such that

(i) $\operatorname{supp} v \subset I(a, r)$;
(ii) $\|v\|_2 < |I(a, r)|^{-1/2}$;
(iii) $\int_{\mathbb{R}^d} v(x)\,dx = 0$.

Therefore it is enough to show that $\|h_\lambda * v\|_1 \le C$ for some $C > 0$ and every 2-atom v. We have

$$\|h_\lambda * v\|_1 = \int_{I^*} |h_\lambda * v| + \int_{\mathbb{R}^n\setminus I^*} |h_\lambda * v|.$$

The second integral on the right hand side satisfies the required estimate by (7). To estimate the first one, notice that since $\widehat{h}_\lambda$ is a bounded function,

$\|h_\lambda * v\|_2 \lesssim \|v\|_2$. Thus, by the Cauchy–Schwarz inequality

$$\int_{I^*} |h_\lambda * v|\,dx \le \Big(\int_{I^*} dx\Big)^{1/2}\|h_\lambda * v\|_2 \lesssim |I(a, 2r)|^{1/2}\|v\|_2 \le 2^{1/2}. \ \blacksquare$$

LEMMA 2. *Suppose ψ satisfies conditions* (2) *and* $\operatorname{supp}\psi \subset (3/4, 3/2)$. *Let ψ_n be given by $\widehat{\psi}_n(t) = \widehat{\psi}(1 + (t - 2^n)2^n)$ for $n = 1, 2, \ldots,$ and $w = \sum_{n=0}^{\infty}\psi_n$. Then $\widehat{w}$ is a Fourier multiplier on $H^1(\mathbb{R})$.*

Proof. Since H^1 Fourier multipliers are closed under bounded pointwise convergence, it is enough to show that $w_k = \sum_{n=0}^{k}\psi_n$ is an H^1 Fourier multiplier with norm bounded independently of $k = 1, 2, \ldots$ Let $\phi_0, \phi_1, \phi_2, \ldots \in \mathcal{S}'(\mathbb{R})$ be such that $1 = \widehat{\phi}_0 + \sum_{n=1}^{\infty}\widehat{\phi}_n$ is a smooth partition of unity such that

$$\widehat{\phi}_1(x) = 1 \quad \text{for } x \in \operatorname{supp}\widehat{\psi}, \qquad \widehat{\phi}_n(t) = \widehat{\phi}_1(2^{-n}t).$$

By Littlewood–Paley theory we have

$$\|f\|_{H^1} \approx \Big\|\Big(\sum_{n=0}^{\infty} |f * \phi_n|^2\Big)^{1/2}\Big\|_1. \tag{8}$$

Let $\theta, \theta_1, \theta_2, \ldots \in \mathcal{S}'(\mathbb{R})$ be such that $\widehat{\theta} : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\widehat{\phi}_1(x) = 1$ for $x \in \operatorname{supp}\widehat{\theta}$ and $\widehat{\theta}(x) = 1$ for $x \in \operatorname{supp}\widehat{\psi}$ and $\widehat{\theta}_n(t) = \widehat{\theta}(t - 2^n + 1)$ for $n = 1, 2, \ldots$ By (8) and the Marcinkiewicz–Zygmund theorem (cf. [MZ, Theorem 1], or [GR, Theorem V.2.7]),

$$\Big\|\Big(\sum_{n=0}^{\infty} |f * \phi_n * \theta_n|^2\Big)^{1/2}\Big\|_1 \lesssim \|f\|_{H^1}. \tag{9}$$

Hence, since $\theta_n * \phi_n = \theta_n$, $n = 1, 2, \ldots,$ from (8) and (9) we get

$$\Big\|\sum_{n=0}^{k} f * \theta_n\Big\|_{H^1} \lesssim \|f\|_{H^1}. \tag{10}$$

To simplify the formulas we set $\chi(t) = e^{2\pi i t}$. Thus $\chi^\alpha(t) = e^{2\pi i\alpha t}$. Notice now that

$$\chi^{2^{k-n}-2^n}\cdot\psi_n = \psi_{2^k, k-n}, \tag{11}$$

$$\chi^{2^{k-n}-2^n}\cdot\theta_n = \theta_{k-n}. \tag{12}$$

Since $\theta_j * \phi_j = \theta_j$ and $\theta_j * \phi_i = 0$ for $j \ne i$, we have

$$\begin{aligned}\phi_j * \sum_{n=0}^{k} (f * \theta_n)\chi^{2^{k-n}-2^n} &= \phi_j * \sum_{n=0}^{k} f\chi^{2^{k-n}-2^n} * \theta_{k-n} \\ &= f\chi^{2^{k-j}-2^j} * \theta_{k-j} = (f * \theta_j)\chi^{2^{k-j}-2^j}.\end{aligned}$$

Therefore, by Littlewood–Paley theory (we put $f_n = f * \theta_n$),

$$\Big\|\sum_{n=0}^{k} f_n\chi^{2^{k-n}-2^n}\Big\|_{H^1} = \Big\|\sum_{j=0}^{k}\phi_j * \sum_{n=0}^{k}(f*\theta_n)\chi^{2^{k-n}-2^n}\Big\|_{H^1} \tag{13}$$
$$\approx \Big\|\Big(\sum_{n=0}^{k}|f_n\chi^{2^{k-n}-2^n}|^2\Big)^{1/2}\Big\|_1$$
$$= \Big\|\Big(\sum_{n=0}^{k}|f_n|^2\Big)^{1/2}\Big\|_1 \approx \Big\|\sum_{n=0}^{k} f_n\Big\|_{H^1}.$$

Notice that (13) holds for every collection of functions $f_0, f_1, \ldots, f_k$ such that $\operatorname{supp} f_j \subset \operatorname{supp}\theta_j$, in particular for the collection $f_j * \psi_j$, $j = 0, 1, 2, \ldots$ Therefore

$$\|w_k * f\|_{H^1} = \Big\|\sum_{n=0}^{k} f_n * \psi_n\Big\|_{H^1} \approx \Big\|\sum_{n=0}^{k}(f_n*\psi_n)\chi^{2^{k-n}-2^n}\Big\|_{H^1}$$
$$= \Big\|\sum_{n=0}^{k} f_n\chi^{2^{k-n}-2^n} * \psi_n\chi^{2^{k-n}-2^n}\Big\|_{H^1}$$
$$= \Big\|\sum_{n=0}^{k} f_n\chi^{2^{k-n}-2^n} * \psi_{2^k,k-n}\Big\|_{H^1}.$$

Applying now consecutively Lemma 1, (13) and (10), we get

$$\|w_k * f\|_{H^1} \lesssim \|T_{\widehat{h}_{2^k}}\| \cdot \Big\|\sum_{n=0}^{k} f_n\chi^{2^{k-n}-2^n}\Big\|_{H^1}$$
$$\approx \|T_{\widehat{h}_{2^k}}\| \cdot \Big\|\sum_{n=0}^{k} f_n\Big\|_{H^1} \lesssim \|T_{\widehat{h}_{2^k}}\| \cdot \|f\|_{H^1},$$

and the lemma follows from Lemma 1. ■

In the next lemma we apply Lemma 2 to construct some specific Fourier multiplier on $H^1(\mathbb{R}^2)$. We now introduce some notation to be used below. For every measure g on $\mathbb{R}$ we define a measure $\widetilde{g}$ (resp. $\overline{g}$) on $\mathbb{R}^2$ by $\widetilde{g}(x) = g(x_1)\otimes\delta_0(x_2)$ (resp. $\overline{g}(x) = \delta_0(x_1)\otimes g(x_2)$). Clearly $\widetilde{g}^\wedge(\xi) = \widehat{g}(\xi_1)$ and $\overline{g}^\wedge(\xi) = \widehat{g}(\xi_2)$. Also we put $\widetilde{\chi}(\xi) = e^{2\pi i\xi_1}$ and $\overline{\chi}(\xi) = e^{2\pi i \xi_2}$.

LEMMA 3. *Suppose that $\eta, \eta_1, \eta_2, \ldots \in \mathcal{S}'(\mathbb{R})$ are such that $\widehat{\eta} : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\operatorname{supp}\widehat{\eta} \subset [a,b] \subset (3/4, 3/2)$ and $\widehat{\eta}_n(t) = \widehat{\eta}(2^{-n}t)$. Let $\psi, \gamma_1, \gamma_2, \ldots \in \mathcal{S}'(\mathbb{R})$ be such that $\widehat{\psi}$ is a smooth function with support in $[-1/2, 1/2]$, ψ satisfies conditions (2) and $\widehat{\gamma}_n(t) = \widehat{\psi}(2^n t)$. Let*

$$T_{\widehat{m}} f = \sum_{n=0}^{\infty} \widetilde{\eta}_n * \overline{\gamma}_n * f,$$

i.e. $m \in \mathcal{S}'(\mathbb{R}^2)$ is such that

$$\widehat{m}(\xi) = \sum_{n=0}^{\infty}\widehat{\eta}_n(\xi_1)\widehat{\gamma}_n(\xi_2).$$

Then $\widehat{m}$ is a Fourier multiplier on $H^1(\mathbb{R}^2)$.

Proof. Let $\widehat{\mu}$ be a smooth function which is a Fourier transform of a bounded measure μ such that $\operatorname{supp}\widehat{\mu}$ is bounded and $\widehat{\mu}(t) = 1$ for $-1 \le t \le 1$. Let $\beta_n \in \mathcal{S}'(\mathbb{R})$, $n = 0, 1, 2, \ldots$, be such that $\widehat{\beta}_n(t) = \widehat{\mu}(2^{-n}t)$, and $v \in \mathcal{S}'(\mathbb{R})$ be such that

$$T_{\widehat{v}} f = \sum_{n=1}^{\infty} f * \widetilde{\eta}_n * \overline{\beta}_n,$$

i.e. $\widehat{v}(\xi) = \sum_{n=0}^{\infty}\widehat{\eta}_n(\xi_1)\widehat{\beta}_n(\xi_2)$. By the Stein theorem (cf. [S, Chap VII, Theorem 9]), $\widehat{v}$ is a Fourier multiplier on $H^1(\mathbb{R}^2)$. Also convolution with a bounded measure is a bounded operator on $H^1(\mathbb{R}^2)$. Hence the formula

$$T_{\widehat{s}} f = \overline{\mu} * \sum_{n=0}^{\infty} f * \widetilde{\eta}_n = \overline{\mu} * \sum_{n=0}^{\infty} f * \widetilde{\eta}_n * \overline{\beta}_n$$

defines a bounded operator on $H^1(\mathbb{R}^2)$. Let $\varrho, \varrho_1, \varrho_2, \ldots \in \mathcal{S}'(\mathbb{R}^2)$ be such that $\widehat{\varrho} : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function satisfying $\operatorname{supp}\widehat{\varrho} \subset [-3/2, 3/2] \times [-3/2, 3/2] \setminus [-3/4, 3/4] \times [-3/4, 3/4]$ and $\widehat{\varrho}(x) = 1$ for $x \in [-b, b] \times [-b, b] \setminus [-a, a] \times [-a, a]$ and $\widehat{\varrho}_n(x) = \widehat{\varrho}(2^{-n}x)$ for $n = 1, 2, \ldots$ Let $f \in H^1(\mathbb{R}^2)$. Since $\widetilde{\eta}_n * \overline{\mu} * \varrho_n = \widetilde{\eta}_n * \overline{\mu}$, by Littlewood–Paley theory we get (setting $f_n = f * (\widetilde{\eta}_n * \overline{\mu})$)

$$\Big\|\Big(\sum_{n=0}^{\infty}|f_n|^2\Big)^{1/2}\Big\|_1 \lesssim \Big\|\sum_{n=0}^{\infty} f_n\Big\|_{H^1} = \|T_{\widehat{s}}\| \cdot \|f\|_{H^1} \lesssim \|f\|_{H^1}.$$

Therefore

$$\|T_{\widehat{m}} f\|_{H^1} = \Big\|\sum_{n=0}^{\infty} f * (\widetilde{\eta}_n * \overline{\gamma}_n)\Big\|_{H^1} = \Big\|\sum_{n=0}^{\infty} f_n * \overline{\gamma}_n\Big\|_{H^1}$$
$$\approx \Big\|\Big(\sum_{n=0}^{\infty}|f_n * \overline{\gamma}_n|^2\Big)^{1/2}\Big\|_1 = \Big\|\Big(\sum_{n=0}^{\infty}|(f_n * \overline{\gamma}_n)\overline{\chi}^{2^n}|^2\Big)^{1/2}\Big\|_1$$
$$= \Big\|\Big(\sum_{n=0}^{\infty}|f_n\overline{\chi}^{2^n} * \overline{\gamma}_n\overline{\chi}^{2^n}|^2\Big)^{1/2}\Big\|_1.$$

Notice that $\overline{\gamma}_n\overline{\chi}^{2^n} = \overline{\psi}_n$, where ψ_n are the functions from Lemma 2, and $\overline{w} = \sum_{n=0}^{\infty}\overline{\psi}_n$. Moreover, the Fourier transform of the function $\sum_{n=0}^{\infty} f_n\overline{\chi}^{2^n}$

is supported on the halfplane $\{\xi : \xi_2 > 0\}$. Therefore, by Lemma 2,

$$\|T_{\widehat{m}} f\|_{H^1} \approx \Big\| \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^n} * \overline{\psi}_n \Big\|_{H^1} = \Big\| \overline{w} * \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^n} \Big\|_{H_1} \lesssim \Big\| \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^n} \Big\|_{H^1}.$$

Since $f_n \overline{\chi}^{2^n} * \varrho_n = f_n \overline{\chi}^{2^n}$ we can again use Littlewood–Paley theory to get

$$\Big\| \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^n} \Big\|_{H^1} \approx \Big\| \Big(\sum_{n=0}^{\infty} |f_n \overline{\chi}^{2^n}|^2 \Big)^{1/2} \Big\|_1 = \Big\| \Big(\sum_{n=0}^{\infty} |f_n|^2 \Big)^{1/2} \Big\|_1 \lesssim \|f\|_{H^1}. \ \blacksquare$$

Proof of Theorem 1. It is enough to show the theorem for multipliers $m = \sum_{n=0}^{\infty} m_n$ with $\operatorname{supp} \widehat{m}_n \subset K_n = [a2^n, b2^n] \times [-2^{-n-2}, 2^{-n-2}]$ where $3/4 < a < 1 < b < 3/2$ are fixed. Indeed, first observe that since the composition with a homothety does not change the multiplier norm, one can assume that $C = 1/2$. Then notice that one can easily construct nine smooth functions $g_0, g_1, \ldots, g_8$ such that $g_0 + g_1 + \ldots + g_8 = 1$ on $\{\xi : |\xi_1 \xi_2| < 1/2\}$, g_0 has a bounded support, g_1 is supported on $A = \bigcup_{n=0}^{\infty} K_n$, and for $j = 2, 3, \ldots, 8$, g_j is a composition of g_1 with some similarity, and $g_1 \widehat{m}$ satisfies condition (1) as also does $\widehat{m}$ (with, maybe, another constant). Clearly the proof for $g_2 \widehat{m}, g_3 \widehat{m}, \ldots, g_8 \widehat{m}$ is similar to that for $g_1 \widehat{m}$ and the last function has the required property. The remaining summand $g = g_0 \widehat{m}$ has bounded support and $\partial^{\alpha+\beta} g / \partial \xi_1^{\alpha} \partial \xi_2^{\beta} \in L^1(\mathbb{R}^2)$ for $\alpha, \beta \le 2$. Therefore $x_1^{\alpha} x_2^{\beta} \widehat{g}$ are bounded functions for $\alpha, \beta \le 2$. Hence $|\widehat{g}(x)| \lesssim (1 + x_1^2)^{-1} (1 + x_2^2)^{-1}$ and thus $\widehat{g} \in L^1(\mathbb{R}^2)$, which means that g is a Fourier multiplier.

Let $\eta, \eta_1, \eta_2, \ldots \in \mathcal{S}'$ be such that $\widehat{\eta} : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\widehat{\eta}(t) = 1$ for $t \in [a, b]$ and $\operatorname{supp} \widehat{\eta} \subset [3/4, 3/2]$, and $\widehat{\eta}_n(t) = \widehat{\eta}(2^{-n} t)$. Let $\gamma, \gamma_1, \gamma_2, \ldots \in \mathcal{S}'(\mathbb{R})$ be such that $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying (2) and $\widehat{\gamma}(t) = 1$ for $t \in [-1/4, 1/4]$ and $\operatorname{supp} \widehat{\gamma} \subset [-1/2, 1/2]$, and $\widehat{\gamma}_n(t) = \widehat{\gamma}(2^n t)$. Clearly we have $\widehat{\eta}_n(\xi_1) \widehat{\gamma}_n(\xi_2) = 1$ for $\xi \in K_n$. We set $f_n = f * \widetilde{\eta}_n * \widetilde{\gamma}_n$. Applying Littlewood–Paley theory twice (in both cases summands are supported by pairwise disjoint dyadic frames), we get

$$\begin{aligned} \|T_{\widehat{m}} f\|_1 &= \Big\| \sum_{n=1}^{\infty} f_n * m_n \Big\|_1 \approx \Big\| \Big(\sum_{n=1}^{\infty} |f_n * m_n|^2 \Big)^{1/2} \Big\|_1 \\ &= \Big\| \Big(\sum_{n=1}^{\infty} |(f_n * m_n) \overline{\chi}^{2^{-n}}|^2 \Big)^{1/2} \Big\|_1 \\ &= \Big\| \Big(\sum_{n=0}^{\infty} |f_n \overline{\chi}^{2^{-n}} * m_n \overline{\chi}^{2^{-n}}|^2 \Big)^{1/2} \Big\|_1 \\ &\approx \Big\| \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^{-n}} * m_n \overline{\chi}^{2^{-n}} \Big\|_1. \end{aligned}$$

Notice now that $\sum_{n=0}^{\infty} f_n \overline{\chi}^{2^{-n}} \in H^1(\mathbb{R} \times \mathbb{R})$ (because its Fourier transform is supported by $\mathbb{R}_+ \times \mathbb{R}_+$). On the other hand $w \in \mathcal{S}'(\mathbb{R}^2)$ given by $\widehat{w} = \sum_{n=0}^{\infty} \widehat{m}_n * \delta_{(0, 2^{-n})}$ satisfies the assumptions of Theorem 1 of [W1]. Indeed, we have

$$\begin{aligned} 2^{n(\alpha-1)} 2^{-n(\beta-1)} & \int\limits_{K_n + (0, 2^{-n})} \Big| \frac{\partial^{\alpha+\beta}}{\partial \xi_1^{\alpha} \partial \xi_2^{\beta}} (\widehat{m} * \delta_{(0,2^{-n})})(\xi) \Big| \, d\xi_1 \, d\xi_2 \\ &= 2^{n(\alpha-\beta)} \int\limits_{K_n} \Big| \frac{\partial^{\alpha+\beta}}{\partial \xi_1^{\alpha} \partial \xi_2^{\beta}} \widehat{m}(\xi) \Big| \, d\xi_1 \, d\xi_2 = I_{\alpha,\beta}(2^n) < C. \end{aligned}$$

Therefore, by Theorem 1 of [W1], it is a Fourier multiplier on $H^1(\mathbb{R} \times \mathbb{R})$. Hence we derive that

$$\|m * f\|_1 \lesssim \Big\| \sum_{n=0}^{\infty} f_n \overline{\chi}^{2^{-n}} \Big\|_1.$$

Using Littlewood–Paley theory again we get

$$\begin{aligned} \|m * f\|_1 &\lesssim \Big\| \Big(\sum_{n=0}^{\infty} |f_n \overline{\chi}^{2^{-n}}|^2 \Big)^{1/2} \Big\|_1 = \Big\| \Big(\sum_{n=0}^{\infty} |f_n|^2 \Big)^{1/2} \Big\|_1 \\ &\approx \Big\| \sum_{n=0}^{\infty} f_n \Big\|_{H^1}. \end{aligned}$$

By Lemma 3 the last expression is bounded by $\|f\|_{H^1}$ and we are done.

Proof of Corollary 1. We have

$$\frac{\partial^{\alpha+\beta}}{\partial \xi_1^{\alpha} \partial \xi_2^{\beta}} \widehat{m}(\xi) = \sum_j \binom{\alpha}{j} \binom{\beta}{j} j! \cdot \xi_1^{\beta-j} \xi_2^{\alpha-j} f^{(\alpha+\beta-j)}(\xi_1 \xi_2).$$

Substituting $s = r^{-1} \xi_1$, $t = r \xi_2$, we get

$$\begin{aligned} I_{\alpha,\beta}(r) &\lesssim r^{\alpha-\beta} \sum_j \int\limits_{\substack{r < \xi_1 < 2r \\ \xi_1 > \xi_2}} |\xi_1^{\beta-j} \xi_2^{\alpha-j} f^{(\alpha+\beta-j)}(\xi_1 \xi_2)| \, d\xi \\ &\lesssim \sum_j \iint\limits_{1 < s < 2} |f^{(\alpha+\beta-j)}(st)| \, ds \, dt \lesssim \sum_{k \le \alpha+\beta} \|f^{(k)}\|_1. \ \blacksquare \end{aligned}$$

Proof of Corollary 2. Let $\phi_0, \phi_1, \phi_2, \ldots \in \mathcal{S}'(\mathbb{R})$ be such that $1 = \widehat{\phi}_0 + \sum_{n=1}^{\infty} \widehat{\phi}_n$ is a smooth partition of unity satisfying $\operatorname{supp} \phi_1 \subset [1/2, 2]$ and $\widehat{\phi}_n(t) = \widehat{\phi}_1(2^{-n} t)$. Denote by $R(g)$ the operator given by the Fourier multiplier $\widehat{m}(\xi) = g(\xi_1 \xi_2)$. Let $g_n(t) = f \phi_n(2^n t)$. Then

$$\|R(f)\| \le \sum_{n=1}^{\infty} \|R(f \phi_n)\| = \sum_{n=1}^{\infty} \|R(g_n)\|$$

and the functions g_n have uniformly bounded supports. So, by Corollary 1,

$$\|R(f)\| \lesssim \sum_{n=0}^{\infty}\sum_{k=0}^{4}\|g_n^{(k)}\|_1.$$

Clearly we have

$$\|g_n^{(k)}\|_1 \lesssim 2^{(k-1)n}\|f^{(k)}_{|\operatorname{supp}\phi_n}\|_1.$$

Therefore,

$$\|R(f)\| \lesssim \sum_{n=0}^{\infty}\sum_{k=0}^{4}2^{(k-1)n}\|f^{(k)}_{|\operatorname{supp}\phi_n}\|_1 \lesssim \sum_{n=0}^{\infty}\int_{\mathbb{R}}(1+|t|)^{k-1}|f^{(k)}(t)|\,dt.$$

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