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Commutative, radical amenable Banach algebras

by

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Abstract. There has been a considerable search for radical, amenable Banach algebras. Noncommutative examples were finally found by Volker Runde [R]; here we present the first commutative examples. Centrally placed within the construction, the reader may be pleased to notice a reprise of the undergraduate argument that shows that a normed space with totally bounded unit ball is finite-dimensional; we use the same idea (approximate the norm 1 vector x within distance η by a "good" vector y_1; then approximate $(x-y_1)/\eta$ within distance η by a "good" vector y_2, thus approximating x within distance η^2 by $y_1+\eta y_2$, and so on) to go from $\eta = 9/10$ in Lemma 1.5 to arbitrarily small η in Lemma 2.1. This is not an arbitrary decision on the part of the author; it really is forced on him by the nature of the construction, see e.g. (6.1) for a place where η small at the start will not do.

0. Introduction. The father of the notion of amenability in Banach algebras is Barry Johnson [J], who in 1972 showed that a locally compact group G is amenable if and only if the group algebra $L^1(G)$ has a certain cohomology property, which he naturally christened "amenability" of the Banach algebra.

Since then the theory of amenable Banach algebras has had some beautiful theorems, such as Haagerup's theorem [H] that a C^* algebra is amenable if and only if it is nuclear; but has also had something of a dearth of good examples. As Grønbæk [G] remarked in 1991, "except in the C^* algebra case, no substantial enlargement of the class of amenable Banach algebras has been discovered since Johnson's original paper."

Certain "enlargements" have been achieved since; in [GJW], for example, a condition (related to the approximation property) is given that is sufficient for the amenability of the algebra $\mathcal{K}(X)$ of compact operators on a Banach space X. That paper includes (in passing) a slightly pessimistic remark about the chances of finding radical amenable Banach algebras, which were eventually found by Runde [R].

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This author had a "toe in the water" as regards matters amenable, when he collaborated [LRRW] with Loy, Runde and Willis to produce some singly generated, weakly amenable Banach algebras which are integral domains and have bounded approximate identities that are normalised powers of the generator. Those algebras are not amenable, however; and the algebras in this paper do not appear to be singly generated (and they certainly are not integral domains).

Now a Banach algebra $\mathcal{A}$ is called *amenable* if $H^1(\mathcal{A}, X^*) = \{0\}$ for all Banach $\mathcal{A}$-bimodules X, i.e. if every continuous derivation from $\mathcal{A}$ to a dual Banach $\mathcal{A}$-bimodule X^* is inner. But in a constructive paper like this, we work with the "nuts and bolts" characterisation of amenability, namely: $\mathcal{A}$ is amenable if and only if it has an *approximate diagonal*, that is, a bounded net (Δ_n) in the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that for every $a \in \mathcal{A}$, the commutators $[a, \Delta_n] \to 0$, and if $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ is the product map, then $\pi(\Delta_n)a \to a$. For the proof that this condition characterises amenability, I was going to refer the reader to [BD], Theorem 43.9; however, the referee wisely pointed out that this theorem deals only with *unital* Banach algebras, so a word of explanation about nonunital Banach algebras (such as radical ones) is in order. In fact $\mathcal{A}$ is amenable if and only if the unital algebra $\mathcal{A}^\sharp$ is amenable, if and only if $\mathcal{A}$ has an approximate diagonal, if and only if $\mathcal{A}^\sharp$ has an approximate diagonal. Let us prove this.

Whether $\mathcal{A}$ is unital or not, $H^1(\mathcal{A}, X^*)$ is the quotient of the Banach space of continuous derivations $\mathcal{A} \to X^*$ by the subspace of inner derivations $\delta_f : a \mapsto af - fa$ for $f \in X^*$. So $H^1(\mathcal{A}, X^*) = \{0\}$ if and only if every continuous derivation is inner.

Suppose $\mathcal{A}$ is not amenable, so for suitable X a continuous derivation $\delta : \mathcal{A} \to X^*$ exists that is not inner. One may extend δ continuously to the algebra $\mathcal{A}^\sharp$ with unit adjoined, by defining $\delta(1) = 0$; and one may make X into a Banach $\mathcal{A}^\sharp$-bimodule by defining $1x = x1 = x$ for all $x \in X$. Then $\delta : \mathcal{A}^\sharp \to X^*$ is a continuous derivation that is not inner, so $\mathcal{A}^\sharp$ is not amenable. By [BD], Theorem 43.9, $\mathcal{A}^\sharp$ has no approximate diagonal. But if (d_n) were an approximate diagonal for $\mathcal{A}$, one may check that

$$D_n = 1 \otimes 1 - 1 \otimes \pi(d_n) - \pi(d_n) \otimes 1 + \pi(d_n) \otimes \pi(d_n) + 2d_n - d_n^2$$

would be an approximate diagonal for $\mathcal{A}^\sharp$; so $\mathcal{A}$ has no approximate diagonal either.

Conversely, if $\mathcal{A}$ is amenable, consider any derivation $\delta : \mathcal{A}^\sharp \to X^*$, where X^* is a dual Banach $\mathcal{A}$-bimodule that is *unit-linked*, i.e. $1x = x1 = x$ for all $x \in X$. One must then have $\delta(1) = 0$, and $\delta|_{\mathcal{A}}$ is a derivation on $\mathcal{A}$, so for some $x \in X$ we have $\delta(a) = ax - xa$ for all $a \in \mathcal{A}$. The same is true when $a = 1$, so $\delta : \mathcal{A}^\sharp \to X^*$ is inner. By [BD], Lemma 43.6, the unit-linked case is enough to establish that $\mathcal{A}^\sharp$ is amenable, and therefore has an approximate diagonal (D_n) which may be written

$$D_n = \lambda_n 1 \otimes 1 - 1 \otimes \alpha_n - \beta_n \otimes 1 + \Delta_n$$

with $\alpha_n, \beta_n \in \mathcal{A}$ and $\Delta_n \in \mathcal{A} \hat{\otimes} \mathcal{A}$. One may then check that necessarily $\lambda_n \to 1$, (α_n), (β_n) are bounded approximate identities for $\mathcal{A}$, and

$$d_n = \Delta_n - \beta_n \otimes \alpha_n$$

is an approximate diagonal for $\mathcal{A}$.

Let us now make some related definitions of our own.

1. Preliminary definitions. Within this paper we are invariably using diagonal elements Δ_n of norm at most 1. So, we say $\mathcal{A}$ has a *metric approximate diagonal* if it has an approximate diagonal whose elements are norm bounded by 1. The "metric" concept is useful because one can of course multiply two tensors together, getting $a \otimes b \cdot c \otimes d = ac \otimes db$; this multiplication extends linearly and continuously throughout the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$, making that space a Banach algebra; and it is very useful to us to be able to multiply several "diagonal elements" together without fear that they may become unbounded. We do a good deal of multiplying of elements in $\mathcal{A} \hat{\otimes} \mathcal{A}$ when proving Lemma 2.1.

We are aiming throughout to construct commutative, radical, amenable Banach algebras, or "CRAB" algebras. The main building block from which they are constructed is the "FDNC" algebra (*Finite Dimensions, Nilpotent, Commutative*), by which we mean a finite-dimensional commutative Banach algebra, every element of which is nilpotent. Obviously for such an algebra $\mathcal{A}$ there is a d such that $x^d = 0$ for every $x \in \mathcal{A}$; the least such d is called the *degree* of $\mathcal{A}$.

Now an FDNC algebra is basically just a collection of matrices; but the variety of algebra norms one can put on such a collection is truly fascinating and will herein be used to advantage. Here are some possibilities.

Definition 1.1. Let $\mathcal{B}$ be a Banach algebra and $\mathcal{A} \subset \mathcal{B}$ a subalgebra. A *metric approximate unit* for $\mathcal{A}$ with *constant* δ is an element $u \in \mathcal{B}$ such that $\|u\| \le 1$, and $\|ua - a\| \le \delta\|a\|$ for every $a \in \mathcal{A}$.

As an easy introduction to the methods of proof to be used in the absolutely fundamental Lemma 1.5, we shall shortly prove the following lemma (or I suspect one can deduce it as a corollary of [DW]):

Lemma 1.2. *For every FDNC algebra $\mathcal{A}$ and every $\delta > 0$ there is an extension $\mathcal{B} \supset \mathcal{A}$, also an FDNC algebra, containing a metric approximate unit for $\mathcal{A}$ with constant δ.*

We have an idea analogous to "metric approximate unit", namely the "metric diagonal element", that helps with making approximate diagonals.

DEFINITION 1.3. Let $\mathcal{A}$ be an FDNC algebra and $u, a \in \mathcal{A}$ with $\|u\| \le 1$. The element $\Delta \in \mathcal{A} \hat{\otimes} \mathcal{A}$ is a (strong) *metric approximate commutant* for a in $\mathcal{A}$ (with *image* u and *constant* ζ) if we have $\pi(\Delta) = u$ (where π is the natural multiplication map $\mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$), $\|\Delta\| \le 1$, and

$$\|[a, \Delta]\| \le \zeta \|a\|. \tag{1.1}$$

DEFINITION 1.4. Also, if $\eta > 0$ is relatively large, and $\zeta > 0$ is small, it is of great interest to us to know if a slightly weaker condition holds: if $\pi(\Delta) = u$ and $\|\Delta\| \le 1$ as above, and for some $a \in \mathcal{A}$ we can find $y \in \mathcal{A}$ such that $\|y - a\| \le \eta \|a\|$ and $\|[y, \Delta]\| \le \zeta \|a\|$, then we say Δ is a *weak metric approximate commutant* for a, with image u and constants η and ζ.

As we mentioned in the abstract, we are forced by the nature of the construction to start off with $\eta \ge 9/10$. Here is the lemma involved.

LEMMA 1.5. *Let $\mathcal{A}$ be an FDNC algebra and $u \in \mathcal{A}$, $\|u\| \le 1$. Then for any $a \in \mathcal{A}$ and $\zeta > 0$, $\eta \in [9/10, 1]$ there is an FDNC algebra $\mathcal{B}$ containing $\mathcal{A}$ such that $\mathcal{B} \hat{\otimes} \mathcal{B}$ contains a weak metric approximate commutant Δ for a with image u, and constants η and ζ.*

Now Lemma 1.5 is the heart of this paper; everything else fits in around it. Before attempting to prove Lemmas 1.2 and 1.5, let us do the "soft" part of this paper, where we take the result of Lemma 1.5 and "stretch" it until we get a CRAB algebra.

2. Constructing CRAB algebras using Lemma 1.2 and 1.5.

Throughout this section we shall assume that Lemmas 1.2 and 1.5 are true, facts we shall not begin to prove until later. Given these facts, we may construct CRAB algebras as follows.

A useful advantage of having everything commutative is that metric approximate commutants can be multiplied together: if $a \in \mathcal{A}$ and $\Delta_1, \Delta_2 \in \mathcal{A} \hat{\otimes} \mathcal{A}$ then we have $[a, \Delta_1\Delta_2] = [a, \Delta_1]\Delta_2 = [a, \Delta_2]\Delta_1$; hence if for $i = 1, 2$ each Δ_i is a weak metric approximate commutant for a_i with image u_i and constants η_i, ζ_i, then $\Delta_1\Delta_2$ is a weak metric approximate commutant for both a_1 and a_2 with image u_1u_2 and constants no worse than $\max(\eta_1, \eta_2)$, $\max(\zeta_1, \zeta_2)$. This idea is used in proving the following lemma:

LEMMA 2.1. *For each FDNC algebra $\mathcal{A}$, each $x, u \in \mathcal{A}$ with $\|u\| = \|x\| = 1$, and each $\zeta > 0$, $\eta \in [9/10, 1]$ and $n \in \mathbb{N}$, there is an FDNC algebra $\mathcal{B}_n \supset \mathcal{A}$ containing a weak metric approximate commutant Δ_n for x with image u^n and constants η^n and $n\zeta$.*

Proof. By induction on n. The result $n = 1$ is Lemma 1.5. Our induction hypothesis is that we have an extension $\mathcal{B}_n$ satisfying the conditions (with the metric approximate commutant used being $\Delta_n \in \mathcal{B}_n \hat{\otimes} \mathcal{B}_n$), and we seek

a suitable extension $\mathcal{B}_{n+1} \supset \mathcal{B}_n$, containing a suitable metric approximate commutant Δ_{n+1}. By induction hypothesis there is a $y \in \mathcal{B}_n$ with $\|x - y\| \le \eta^n$ such that $\|[y, \Delta_n]\| \le n\zeta$. Now the vector $x' = (x - y)/\eta^n$ has norm at most 1, so by Lemma 1.5, there is an extension $\mathcal{B}_{n+1} \supset \mathcal{B}_n$ that contains an element z with $\|z - (x - y)\eta^{-n}\| \le \eta$, and a metric approximate commutant Δ such that $\|\Delta\| \le 1$, $\pi(\Delta) = u$ and $\|[z, \Delta]\| \le \zeta$. Write $\Delta_{n+1} = \Delta \cdot \Delta_n \in \mathcal{B}_{n+1} \hat{\otimes} \mathcal{B}_{n+1}$. Then we have $\pi(\Delta_{n+1}) = u^{n+1}$, obviously $\|\Delta_{n+1}\| \le 1$, and $\|[z, \Delta_{n+1}]\| \le \zeta$. Writing $y' = y + \eta^n z$, we have

$$\|y' - x\| = \|y - x - \eta^n z\| \le \eta^{n+1}.$$

Also

$$\|[y', \Delta_{n+1}]\| = \|[y', \Delta \cdot \Delta_n]\| \le \|\Delta\| \cdot \|[y, \Delta_n]\| + \|\Delta_n\| \cdot \|[\Delta, \eta^n z]\| \le (n+1)\zeta.$$

So Δ_{n+1} is our required metric approximate commutant, and the lemma is proved.

Having proved Lemma 2.1, it is a short step to proving the following natural sequel:

LEMMA 2.2. *If $\mathcal{A}$ is an FDNC algebra and $\delta > 0$, there is an FDNC extension $\mathcal{B} \supset \mathcal{A}$ containing a metric approximate unit u for $\mathcal{A}$ with constant δ, and an element $\Delta \in \mathcal{B} \hat{\otimes} \mathcal{B}$ such that $\pi(\Delta) = u$, and for every $a \in \mathcal{A}$, Δ is a strong metric approximate commutant for a with constant δ.*

Proof. Take a basis of $\mathcal{A}$, say $a_1, \ldots, a_n$, and let us also find a constant K so that each $x \in \mathcal{A}$ is written $\sum_{i=1}^n \lambda_i a_i$ with

$$\sum_{i=1}^n |\lambda_i| \le K\|x\|. \tag{2.1}$$

Next, pick any $\eta \in [9/10, 1)$ and choose n so large that

$$2\eta^n K < \delta/2. \tag{2.2}$$

Then choose ζ so small that

$$n\zeta K < \delta/2. \tag{2.3}$$

Use Lemma 1.2 to find an extension $\mathcal{B}_0 \supset \mathcal{A}$ containing a metric approximate unit u_0 for $\mathcal{A}$ with constant $\delta' = \delta/n^2$. Then use Lemma 2.1 to find an extension $\mathcal{B}_1 \supset \mathcal{B}_0$ having a weak metric approximate commutant Δ_1 for a_1 with image u_0^n and constants $\eta^n, n\zeta$. Likewise find $\mathcal{B}_2 \supset \mathcal{B}_1$ having a weak metric approximate commutant Δ_2 for a_2 with image u_0^n and the same constants $\eta^n, n\zeta$. Continue like this until we have an extension $\mathcal{B}_n \supset \mathcal{B}_1$ with weak metric approximate commutants Δ_i for each a_i, $i = 1, \ldots, n$, with the same image u_0^n and the same constants $\eta^n, n\zeta$.

Consider now the product $\Delta = \prod_{i=1}^n \Delta_i$. Then $\|\Delta\| \le 1$, and $\pi(\Delta) = u_0^{n^2} = u$, where since u_0 is a metric approximate unit for $\mathcal{A}$ with constant

$\delta' = \delta/n^2$, u is a metric approximate unit for $\mathcal{A}$ with constant no worse than δ. Further, if $x = \sum_{i=1}^n \lambda_i a_i \in \mathcal{A}$, we have

$$[x, \Delta] = \sum_{i=1}^n \lambda_i [a_i, \Delta_i] \prod_{j \neq i} \Delta_j,$$

hence

$$\|[x, \Delta]\| \le \sum_{i=1}^n |\lambda_i| \max \|[a_i, \Delta_i]\| \le K\|x\| \max \|[a_i, \Delta_i]\|.$$

But for each i there is a $y_i \in \mathcal{B}_i$ such that $\|a_i - y_i\| \le \eta^n$ and $\|[y_i, \Delta_i]\| \le n\zeta$. Clearly, $\|[a_i - y_i, \Delta_i]\| \le 2\|a_i - y_i\| \cdot \|\Delta_i\| \le 2\eta^n$, so

$$\max \|[a_i, \Delta_i]\| \le 2\eta^n + n\zeta$$

and

$$\|[x, \Delta]\| \le K\|x\|(2\eta^n + n\zeta) \le \delta\|x\|,$$

by (2.2) and (2.3). Thus Δ is a strong metric diagonal element for every $x \in \mathcal{A}$ relative to $\mathcal{A}$, with image u and constants δ, δ; and our lemma is proved.

So, if $\mathcal{A}$ is an FDNC algebra and $\mathcal{B} \supset \mathcal{A}$ an FDNC extension satisfying (for some $\delta > 0$) the conditions of Lemma 2.2 above, we say that $\mathcal{B}$ is a *diagonal extension* of $\mathcal{A}$ with constant δ.

THEOREM 2.3. *Let $(\mathcal{A}_i)_{i=1}^\infty$ be a nested sequence of FDNC algebras such that for each i, $\mathcal{A}_{i+1}$ is a diagonal extension of $\mathcal{A}_i$ with constant δ_i; and suppose that $\delta_i \to 0$. Let $\mathcal{A}$ be the completion of the union $\bigcup_{i=1}^\infty \mathcal{A}_i$. Then $\mathcal{A}$ is a CRAB algebra.*

Proof. Let $\Delta_{n+1} \in \mathcal{A}_{n+1} \hat{\otimes} \mathcal{A}_{n+1}$ be a strong metric diagonal element for all $x \in \mathcal{A}_n$ relative to $\mathcal{A}_n$, with constants δ_n, δ_n. We claim that the sequence $\Delta_n \in \mathcal{A} \hat{\otimes} \mathcal{A}$ is an *approximate diagonal* for $\mathcal{A}$, and so, $\mathcal{A}$ is amenable. For if $x \in \mathcal{A}_m$ for some m, it is plain that $[x, \Delta_n] \to 0$ and $\pi(\Delta_n)x \to x$ as $n \to \infty$; but such elements are dense in $\mathcal{A}$, and the sequence Δ_n is uniformly bounded, so the result follows, and $\mathcal{A}$ is indeed amenable.

Finally $\mathcal{A}$ has a dense set of nilpotent elements and is commutative, so it is plainly radical. Therefore, it is a commutative, radical, amenable Banach algebra.

3. Proof of Lemma 1.2. This proof is somewhat like parts of [LRRW], only simpler. We may assume that $\delta < 1$. Let $\mathcal{A}$ be the given FDNC algebra, and let $\mathcal{A}_1$ be the extension obtained by adjoining a unit to $\mathcal{A}$, and imposing the norm

$$\|\lambda 1 + a\| = |\lambda| + \|a\|_{\mathcal{A}}$$

($\lambda \in \mathbb{C}$, $a \in \mathcal{A}$). We consider the algebra $\mathcal{A}[y]$, consisting of polynomials in the symbol y with coefficients in $\mathcal{A}_1$, such that the constant coefficient lies in $\mathcal{A}$. If we adjoin the unit, allowing the constant coefficient to lie in all of $\mathcal{A}_1$, we call the algebra $\mathcal{A}_1[y]$. These are infinite-dimensional spaces of polynomials, which we reduce to finite dimensions by taking the quotient spaces $\mathcal{B} = \mathcal{A}[y]/\langle y^N \rangle$ and $\mathcal{B}_1 = \mathcal{A}_1[y]/\langle y^N \rangle$ for some large N, where $\langle y^N \rangle$ is the ideal $\{y^N q(y) : q \in \mathcal{A}_1[y]\}$. By demanding that $y^N = 0$ in this way, we ensure that $\mathcal{B}$ is another FDNC algebra, with degree at most $N + d - 1$, where d is the degree of $\mathcal{A}$ (for any $x \in \mathcal{B}$ can be written $a + yx_1$ where $a \in \mathcal{A}$; and each term in the binomial expansion of $(a + yx_1)^{N+d-1}$ has either a factor a^d or a factor y^N).

If we impose a norm on $\mathcal{B}_1$ such that $\mathcal{A}$ is embedded isometrically as the constants, then $\mathcal{B} \subset \mathcal{B}_1$ is an FDNC algebra extending $\mathcal{A}$. The simplest example of such a norm is

$$\Big|\Big|\Big| \sum_{i=0}^{N-1} a_i y^i \Big|\Big|\Big| = \sum_{i=0}^{N-1} \|a_i\|_{\mathcal{A}_1}, \tag{3.1}$$

where evidently we have $|||y||| = 1$. Of course, we need something a little more complicated; given the constant $\delta > 0$, choose an N, and let $\|\cdot\|$ be the largest algebra (semi-)norm on $\mathcal{B}$ such that $\|x\| \le |||x|||$ for all x (the norm $|||\cdot|||$ as in (3.1)), and

$$\|ay - a\| \le \delta\|a\| \tag{3.2}$$

for every $a \in \mathcal{A}$. Plainly such a (semi-)norm exists, it is the restriction to $\mathcal{B}$ of the seminorm on $\mathcal{B}_1$ given by

$$\|x\| = \inf \Big\{ \sum_{i=0}^M |||x_i||| : x = \sum_{i=0}^M x_i s_i,\ s_i \in \mathcal{S} \Big\}, \tag{3.3}$$

where

$$\mathcal{S} = \Big\{ \delta^{-k} \prod_{i=1}^k (a_i y - a_i) : k \ge 0,\ a_i \in \mathcal{A},\ \|a_i\| = 1\ (i = 1, \ldots, k) \Big\}. \tag{3.4}$$

Now if $\dim \mathcal{A} = n$, and $a^d = 0$ for every $a \in \mathcal{A}$, then any product of nd vectors in $\mathcal{A}$ is zero. So the nonzero elements of $\mathcal{S}$ involve values $k \le nd - 1$; and hence, $|||s||| \le (2/\delta)^k \le (2/\delta)^{nd-1}$ (since $\delta < 1$) for every $s \in \mathcal{S}$. Therefore $\|\cdot\|$ really is a norm, not just a seminorm; we know $\|x\| \ge (\delta/2)^{nd-1}|||x|||$ for every $x \in \mathcal{B}_1$. We claim that if N is chosen large enough, then $\mathcal{A}$ is isometrically embedded in $(\mathcal{B}, \|\cdot\|)$—and in view of (3.2), $y \in (\mathcal{B}, \|\cdot\|)$ is the desired metric approximate unit for $\mathcal{A}$. We prove this by dualising—let $a \in \mathcal{A}$ have norm 1, and let $a^* : \mathcal{A} \to \mathbb{C}$ be a support functional for a, having $\|a^*\|_{\mathcal{A}^*} = 1$ and $a^*(a) = 1$. If we can extend a^* to a functional $\bar{a}^* : \mathcal{B}_1 \to \mathbb{C}$ so that $\bar{a}^*$ has norm 1 with respect to the new norm on $\mathcal{B}_1$, then $\|a\|_{\mathcal{B}} \ge \|a\|_{\mathcal{A}}$ as required.

The extension we use is as follows. First, extend a^* to $\mathcal{A}_1$ by defining $a^*(1) = 0$. Then let $\bar{a}^*$ be the unique linear map $\mathcal{B}_1 \to \mathbb{C}$ such that

$$\bar{a}^*(y^k b) = (1 - k/N)a^*(b) \tag{3.5}$$

for each $k = 0, \ldots, N-1$ and each $b \in \mathcal{A}_1$. To show that $\|\bar{a}^*\| = 1$ it is sufficient, in view of (3.3), that $|\bar{a}^*(sz_1)| \le 1$ for all z_1, s such that $|||z_1||| = 1$, $s \in \mathcal{S}$. Because of the nature (see (3.1)) of the norm $|||\cdot|||$, we can assume that z_1 is of form $y^l b$ for $b \in \mathcal{A}_1$ with $\|b\|_{\mathcal{A}_1} = 1$, and $l \le N-1$. Then the general element $z = sz_1$ is of form

$$z = \delta^{-k} \prod_{i=1}^{k} (a_i y - a_i) \cdot y^l b, \tag{3.6}$$

where $\|a_i\|_{\mathcal{A}}, \|b\|_{\mathcal{A}_1} = 1$. If $k = 0$ then it is obvious that $|\bar{a}^*(y^l b)| = (1 - l/N)|\bar{a}^*(b)| \le 1$. If $0 < k < dn$, where $n = \dim \mathcal{A}$, we proceed as follows.

CASE 1: If $k + l \le N$, then the highest power of y involved is at most N, so for all relevant indices r we get $\bar{a}^*(y^r\alpha) = (1 - r/N)\bar{a}^*(\alpha)$ for all $\alpha \in \mathcal{A}_1$. The "relevant" α is always $\alpha = b\prod_{i=1}^k a_i$, hence

$$\bar{a}^*(z) = \delta^{-k} a^*(\alpha) \sum_{s=0}^{k} \binom{k}{s} (-1)^s \Big(1 - \frac{s+l}{N}\Big) = \begin{cases} \delta^{-1} a^*(\alpha) \cdot 1/N & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

since for $k > 1$ both $\sum_{s=0}^k \binom{k}{s}(-1)^s$ and $\sum_{s=0}^k s\binom{k}{s}(-1)^s$ are zero. In the nonzero case when $k = 1$, since $|a^*(\alpha)| \le 1$ we certainly have $|\bar{a}^*(z)| \le 1/(\delta N)$, which is less than 1 for large $N > 1/\delta$.

CASE 2: If $k + l > N$, then since $k < dn$, for each $0 \le r \le k$ the value of $\bar{a}^*(y^{r+l}\alpha)$ (as involved when computing $\bar{a}^*(z)$ from (3.6)) is either zero (when $r + l \ge N$) or a value $(1 - (r+l)/N)a^*(\alpha)$, where the coefficient $1 - (r+l)/N \le nd/N$. Hence,

$$|\bar{a}^*(z)| \le \delta^{-k} \sum_{s=0}^{k} \binom{k}{s} \frac{nd}{N} \le 2^k \frac{nd}{N\delta^k} \le 2^{nd} \frac{nd}{N\delta^{nd}}$$

(we can assume that $\delta < 1$), which is again less than 1 for large enough $N > 2^{nd} nd/\delta^{nd}$.

Thus, for large enough N, the extension $\bar{a}^* : \mathcal{B}_1 \to \mathbb{C}$ defined in (3.5) will always have norm 1, and $\mathcal{A}$ is indeed embedded isometrically in $(\mathcal{B}, \|\cdot\|)$, $\mathcal{B}$ being an FDNC algebra containing a metric approximate unit y for $\mathcal{A}$ with constant δ. And so our lemma is proved.

4. Beginning the proof of Lemma 1.5. The conditions involving a in Definition 1.4 are homogeneous in a, so we may assume $\|a\| = 1$.

Our proof starts out like the proof of Lemma 1.2, in that we take an N much larger than n, the dimension of the given algebra $\mathcal{A}$, or d, the degree of $\mathcal{A}$. But instead of adding one extra generator y, we add N extra generators $(y_i)_{i=1}^N$, considering the algebra $\mathcal{A}_1[y_1, \ldots, y_N]$ (hereafter abbreviated to $\mathcal{A}_1[\mathbf{y}]$) of polynomials in the y_i with coefficients in $\mathcal{A}_1$. Of course we define $\mathcal{A}[\mathbf{y}]$ to be the ideal of polynomials whose constant coefficient lies in $\mathcal{A}$; and our final algebra $\mathcal{B}$ will be a suitable quotient of $\mathcal{A}[\mathbf{y}]$.

A general product $\prod_{i=1}^N y_i^{r_i}$ will be referred to as $\mathbf{y}^{\mathbf{r}}$, and we will write $|\mathbf{r}|$ for the sum $\sum_{i=1}^N r_i$. Our choice of quotient will of course have the effect of ensuring that $\mathbf{y}^{\mathbf{r}} = 0$ for large enough $|\mathbf{r}|$, in fact this will be true for $|\mathbf{r}| \ge N^2 d$. More interestingly, if $u \in \mathcal{A}$ is our given image vector (such that $\pi(\Delta)$ must be equal to u), we shall demand that $\prod_{i=1}^N y_i^N = u$, that is, if $\mathbf{1}$ denotes the vector $(1, \ldots, 1) \in (\mathbb{Z}^+)^N$, then $\mathbf{y}^{N\mathbf{1}} = u$.

So we consider the quotient space $\mathcal{B}_1 = \mathcal{A}_1[\mathbf{y}]/I$, where $I = I_0 + I_1$ is the sum of the ideal I_0 generated by $\{\mathbf{y}^{\mathbf{r}} : |\mathbf{r}| \ge N^2 d\}$, and the ideal I_1 generated by $\mathbf{y}^{N\mathbf{1}} - u$. Our algebra $\mathcal{B}$ will (as usual) be the ideal of polynomials $q(\mathbf{y}) + I$ such that the constant coefficient of q lies in $\mathcal{A}$.

As when proving Lemma 1.2, we have a simple "initial norm" on $\mathcal{A}_1[\mathbf{y}]$, the norm $|||\cdot|||$ such that

$$\Big|\Big|\Big|\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{y}^{\mathbf{r}}\Big|\Big|\Big| = \sum \|a_{\mathbf{r}}\|_{\mathcal{A}_1}. \tag{4.1}$$

Given $\eta \in [9/10, 1]$ and $a \in \mathcal{A}$, we define a (semi-)norm $\|\cdot\|$ on $\mathcal{B}_1$ to be the largest seminorm on the quotient algebra $\mathcal{A}_1[\mathbf{y}]/I$ such that for all $x \in \mathcal{A}_1[\mathbf{y}]$, $\|x + I\| \le |||x|||$, and such that

$$\Big\|\Big(\frac{1}{2N}\sum_{i=1}^N y_i\Big) - a + I\Big\| \le \eta. \tag{4.2}$$

From now on we shall slightly abuse notation by dropping the "$+I$". We claim that $\mathcal{A}$ is, for large enough N, isometrically embedded in $(\mathcal{B}, \|\cdot\|)$ as the constants; and of course $\mathcal{B}$ is an FDNC algebra. Let $y = \frac{1}{2N}\sum_{i=1}^N y_i$ so that by (4.2), $\|y - a\| \le \eta$. Let Δ be the element

$$(N-1)^{-N} \sum_{\mathbf{1} \le \mathbf{r} \le (N-1)\mathbf{1}} \mathbf{y}^{\mathbf{r}} \otimes \mathbf{y}^{N\mathbf{1}-\mathbf{r}} \in \mathcal{B} \mathbin{\hat{\otimes}} \mathcal{B}. \tag{4.3}$$

Now $\Delta = \prod_{i=1}^N \Delta_i$, where $\Delta_i = (N-1)^{-1}\sum_{j=1}^{N-1} y_i^j \otimes y_i^{N-j}$; hence, since $\|y_i\| \le 1$, we have $\|\Delta_i\| \le 1$, and $\|\Delta\| \le 1$; and since $\|[y_i, \Delta_i]\| \le 2/(N-1)$, we find that $\|[y_i, \Delta]\| \le 2/(N-1)$ for all i, and so $\|[y, \Delta]\| \le 1/(N-1)$. This is less than the given constant $\zeta > 0$, provided we choose $N > 1 + 1/\zeta$.

So Δ is a weak metric approximate commutant for the given $a \in \mathcal{A}$, with image $\pi(\Delta) = \mathbf{y}^{N\mathbf{1}} = u \pmod I$, and constants η and ζ. But to complete the proof of Lemma 1.5, we face the tricky task of proving that $\mathcal{A}$ is embedded isometrically in $\mathcal{B}$.

5. Starting to prove that $\mathcal{A}$ is embedded isometrically in $\mathcal{B}$. As before, we dualise. Since we know $\|x\|_{\mathcal{B}} \le \|x\|_{\mathcal{A}}$ for $x \in \mathcal{A}$, it is enough if an arbitrary norm 1 linear functional $a^* \in \mathcal{A}^*$ can be extended, still with norm 1, to $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$.

As before we extend a^* to $\mathcal{A}_1$ by defining $a^*(1) = 0$; then we define an extension $\bar{a}^* : \mathcal{B}_1 \to \mathbb{C}$ as follows. For each set of indices $\mathbf{r} \in (\mathbb{Z}^+)^N$ and each $x \in \mathcal{A}_1$ we define

$$\bar{a}^*(\mathbf{y}^{\mathbf{r}}x) = \begin{cases} a^*(u^k a^l x) & \text{if } \mathbf{r} = kN\mathbf{1} + \mathbf{s},\ k \in [0,d),\ |\mathbf{s}| = l \in [0,d), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Note that provided $N^2 \ge d$, the cases where $\bar{a}^*$ is nonzero above do not overlap for different values of k, since each value of k involves only indices $\mathbf{r}$ with $|\mathbf{r}| \in [kN^2, kN^2 + d)$. So $\bar{a}^*$ is well defined as a map $\mathcal{A}_1[\mathbf{y}] \to \mathbb{C}$. To see that it is well defined on $\mathcal{B}_1 = \mathcal{A}_1[\mathbf{y}]/I$ we need a lemma asserting that $\bar{a}^*$, as defined in (5.1) above, annihilates I; which we prove immediately.

LEMMA 5.1. *The functional $\bar{a}^*$ defined in* (5.1) *annihilates the ideal I, provided $N > d$.*

Proof. We show separately that $\bar{a}^*$ annihilates I_0 and I_1.

(a) $\bar{a}^*$ annihilates I_0 if $\bar{a}^*(\mathbf{y}^{\mathbf{r}}x) = 0$ for every $x \in \mathcal{A}_1$ and $|\mathbf{r}| \ge N^2 d$. But $\bar{a}^*(\mathbf{y}^{\mathbf{r}}x) = 0$ unless $|\mathbf{r}| \in [kN^2, kN^2 + d)$ for some $k < d$. Since we chose $N > d$, that implies $|\mathbf{r}| < (d-1)N^2 + d - 1 < dN^2$.

(b) $\bar{a}^*$ annihilates I_1 if $\bar{a}^*((\mathbf{y}^{N\mathbf{1}} - u)\mathbf{y}^{\mathbf{r}}x) = 0$ for every $\mathbf{r}$ and x. If $\mathbf{r} = kN\mathbf{1} + \mathbf{s}$ and $0 \le |\mathbf{s}| < d$ for some $k < d$, then (5.1) gives

$$\bar{a}^*(u\mathbf{y}^{\mathbf{r}}x) = a^*(a^{|\mathbf{s}|}u^{k+1}x). \tag{5.2}$$

Otherwise, we have $\bar{a}^*(u\mathbf{y}^{\mathbf{r}}x) = 0$. Now if $\mathbf{r} = kN\mathbf{1} + \mathbf{s}$ with $0 \le |\mathbf{s}| < d$ and $k < d - 1$ then (5.1) also gives

$$\bar{a}^*(\mathbf{y}^{N\mathbf{1}+\mathbf{r}}x) = a^*(a^{|\mathbf{s}|}u^{k+1}x). \tag{5.3}$$

Otherwise, we have $\bar{a}^*(\mathbf{y}^{N\mathbf{1}+\mathbf{r}}x) = 0$. In the case $\mathbf{r} = kN\mathbf{1} + \mathbf{s}$ with $0 \le |\mathbf{s}| < d$ and $k = d - 1$, when we appear to get different results for $\bar{a}^*(u\mathbf{y}^{\mathbf{r}}x)$ and $\bar{a}^*(\mathbf{y}^{N\mathbf{1}+\mathbf{r}}x)$, this is appearance only because $u^{k+1} = u^d = 0$ for this value of k. So $\bar{a}^*$ does indeed annihilate the ideal I_1.

Having got Lemma 5.1 out of the way, we must prove that $\|\bar{a}^*\| \le 1$, given $\|a^*\| \le 1$. Now by (4.2), the norm $\|\cdot\|$ on $\mathcal{B}_1$ is the greatest (semi-)norm on the quotient algebra such that $\|\cdot\| \le |||\cdot|||$, and $\|\frac{1}{2N}\sum_{i=1}^N y_i - a\| \le \eta$. Since the norm $|||\cdot|||$ is given by (4.1), plainly we have

$$\|x\| = \inf\Big\{ \sum_{k,\mathbf{r}} \|b_{k,\mathbf{r}}\|_{\mathcal{A}_1} : x = \sum_{k,\mathbf{r}} \eta^{-k}\Big(\frac{1}{2N}\sum_{i=1}^N y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b_{k,\mathbf{r}} \bmod I \Big\}. \tag{5.4}$$

At this point we pause to note that this expression is a norm, not merely a seminorm, on $\mathcal{B}_1$, because values $k \ge N^2 d + d - 1$ give

$$\Big(\frac{1}{2N}\sum_{i=1}^N y_i - a\Big)^k = 0 \bmod I$$

in any case, hence $\|x\| \ge \eta^{N^2 d + d - 1}|||x|||$ for all $x \in \mathcal{B}_1$.

To show that $\|\bar{a}^*\| \le 1$, knowing already that it annihilates I, it is enough to obtain the estimate

$$\Big|\bar{a}^*\Big(\Big(\frac{1}{2N}\sum_{i=1}^N y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b\Big)\Big| \le \eta^k \tag{5.5}$$

for every $k \ge 0$, $b \in \mathcal{A}_1$ of norm 1, and $\mathbf{r} \in (\mathbb{Z}^+)^N$. To establish this we consider separately "large" and "small" sizes of k. The easiest to deal with is when k is large.

6. Proving (5.5) for "large" $k \ge d(\log 4)/(\log 6/5)$. Write the expression $\big(\frac{1}{2N}\sum_{i=1}^N y_i - a\big)^k$ as a sum

$$\sum_{0 \le |\mathbf{s}| \le k} \mathbf{y}^{\mathbf{s}}(2N)^{-|\mathbf{s}|}\binom{k}{\mathbf{s}}(-a)^{k-|\mathbf{s}|},$$

where $\binom{k}{\mathbf{s}}$ denotes the multinomial coefficient $\binom{k}{s_1,\ldots,s_N}$. We note that since $a^d = 0$, the $\mathcal{A}_1$ norm of the coefficient of $\mathbf{y}^{\mathbf{s}}$ in the sum is no greater than 4^d times the norm of the corresponding coefficient in the multinomial expansion of $\big(\frac{1}{2N}\sum_{i=1}^N y_i - \frac{1}{4}a\big)^k$. Since $\|a\| \le 1$, the sum of all those norms is at most $(3/4)^k$; so the sum of the norms of coefficients in the original expansion is at most $4^d \cdot (3/4)^k$. Since $\|b\|$ is also less than or equal to 1, we may therefore write $\big(\frac{1}{2N}\sum_{i=1}^N y_i - a\big)^k \mathbf{y}^{\mathbf{r}} b$ as a sum $\sum_{\mathbf{s}} c_{\mathbf{s}}\mathbf{y}^{\mathbf{s}}$ with $\sum_{\mathbf{s}} \|c_{\mathbf{s}}\|_{\mathcal{A}_1} \le 4^d \cdot (3/4)^k$. Now $\bar{a}^*$ acts on each $c_{\mathbf{s}}\mathbf{y}^{\mathbf{s}}$ as either zero or a linear map $a^*(u^j a^l c_{\mathbf{s}})$. At any rate, $|\bar{a}^*(c_{\mathbf{s}}\mathbf{y}^{\mathbf{s}})| \le \|c_{\mathbf{s}}\|_{\mathcal{A}_1}$. But then we can sum this estimate over $\mathbf{s}$, use the triangle inequality, and obtain

$$\Big\|\bar{a}^*\Big(\Big(\frac{1}{2N}\sum_{i=1}^N y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b\Big)\Big\| \le 4^d\Big(\frac{3}{4}\Big)^k \le \eta^k, \tag{6.1}$$

as required by (5.5), provided $k \ge d(\log 4)/(\log 4\eta/3)$. Since $\eta \ge 9/10$, $k \ge d(\log 4)/(\log 6/5)$ will suffice, bringing the discussion of this case to a close.

7. Proving (5.5) when $k \le d(\log 4)/(\log 6/5)$, for certain values of r. The values of $\mathbf{r}$ with which we shall deal in this section are when $\mathbf{r} = Nt\mathbf{1} + \mathbf{r}'$ for some $0 \le t < d$ and $\mathbf{r}' \ge \mathbf{z}$, $|\mathbf{r}'| < d$. Once again we use a multinomial expansion to write

$$(7.1)\quad \Big(\frac{1}{2N}\sum_{i=1}^{N} y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b = \sum_{\mathbf{z}\le\mathbf{s},\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|}(-a)^{k-|\mathbf{s}|} b\mathbf{y}^{\mathbf{r}+\mathbf{s}}$$
$$= \sum_{\mathbf{z}\le\mathbf{s},\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|}(-a)^{k-|\mathbf{s}|} b\mathbf{y}^{Nt\mathbf{1}+\mathbf{r}'+\mathbf{s}}.$$

Now if $|\mathbf{r}'| + |\mathbf{s}| < d$, certainly (5.1) gives us

$$\bar{a}^*(a^{k-|\mathbf{s}|} b\mathbf{y}^{Nt\mathbf{1}+\mathbf{r}'+\mathbf{s}}) = a^*(u^t a^{k+|\mathbf{r}'|} b).$$

In fact this is also the case for $d \le |\mathbf{r}'| + |\mathbf{s}| < N^2$, because in such cases (5.1) gives $\bar{a}^*(a^{k-|\mathbf{s}|} b\mathbf{y}^{Nt\mathbf{1}+\mathbf{r}'+\mathbf{s}}) = 0$, which (since $a^d = 0$ and $k + |\mathbf{r}'| \ge |\mathbf{s}| + |\mathbf{r}'| \ge d$) is also the value of the expression $a^*(u^t a^{k+|\mathbf{r}'|} b)$. Provided N is reasonably large compared to d ($N^2 > d(1 + (\log 4)/(\log 6/5))$ will do), values $|\mathbf{r}'| + |\mathbf{s}| \ge N^2$ can never occur; so applying $\bar{a}^*$ to (7.1) we get

$$\bar{a}^*\Big(\Big(\frac{1}{2N}\sum_{i=1}^{N} y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b\Big) = \sum_{\mathbf{z}\le\mathbf{s},\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|}(-1)^{k-|\mathbf{s}|} a^*(u^t a^{k+|\mathbf{r}'|} b)$$
$$= (1/2 - 1)^k a^*(u^t a^{k+|\mathbf{r}'|} b),$$

by the "multinomial theorem"; which has absolute value at most $(1/2)^k$ since $|a^*(u^t a^{k+|\mathbf{r}'|} b)| \le 1$. And that is less than η^k because $\eta > 1/2$. Thus (5.5) is established, for the values of k and $\mathbf{r}$ described at the beginning of this section.

8. Proving (5.5) when $k \le d(\log 4)/(\log 6/5)$, for all other values of r. We now assume that the index $\mathbf{r}$ is **not** of form $Nt\mathbf{1} + \mathbf{r}'$ for any $0 \le t < d$ and $\mathbf{r}' \ge \mathbf{z}$, $|\mathbf{r}'| < d$. Nevertheless, if every term in the multinomial expansion of

$$(8.1)\qquad \bar{a}^*\Big(\Big(\frac{1}{2N}\sum_{i=1}^{N} y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b\Big)$$

is not simply zero, (5.1) tells us that for at least one index $\mathbf{s}$, $\mathbf{z} \le \mathbf{s}$, $|\mathbf{s}| \le k$, we must have $\mathbf{r} + \mathbf{s} = Nt\mathbf{1} + \mathbf{r}'$ for some such t and $\mathbf{r}'$. Because the possible values of $|\mathbf{r} + \mathbf{s}|$ cannot span an interval of width as big as $N^2 - d$ (provided $N^2 > d(1 + (\log 4)/(\log 6/5))$), only one value of t is ever involved as $\mathbf{s}$ varies. In fact, there is a least $\mathbf{s} \ge \mathbf{z}$ for which $\mathbf{r} + \mathbf{s} \ge Nt\mathbf{1}$, namely $\mathbf{s} = \mathbf{s}_0 = (Nt\mathbf{1} - \mathbf{r})_+$. The other values of $\mathbf{s}$ for which $\mathbf{r} + \mathbf{s}$ has the correct

form all satisfy $\mathbf{s} \ge \mathbf{s}_0$. Now, the multinomial expansion for (8.1) is

$$\sum_{\mathbf{z}\le\mathbf{s},\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|}(-1)^{k-|\mathbf{s}|}\bar{a}^*(a^{k-|\mathbf{s}|} b\mathbf{y}^{\mathbf{r}+\mathbf{s}}),$$

and in the cases when it does not just give zero, (5.1) gives $\bar{a}^*(a^{k-|\mathbf{s}|} b\mathbf{y}^{\mathbf{r}+\mathbf{s}})$ as a number of absolute value at most 1. Therefore

$$\Big|\bar{a}^*\Big(\Big(\frac{1}{2N}\sum_{i=1}^{N} y_i - a\Big)^k \mathbf{y}^{\mathbf{r}} b\Big)\Big| \le \sum_{\mathbf{s}\ge\mathbf{s}_0,\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|}.$$

Now for fixed p, the number of $\mathbf{s}$ with $\mathbf{s} \ge \mathbf{s}_0$ and $|\mathbf{s} - \mathbf{s}_0| = p$ is at most N^p (we must add p unit vectors to $\mathbf{s}_0$ and there are N to choose from at each step). The largest conceivable multinomial coefficient $\binom{k}{\mathbf{s}}$ is $k!$. So very crudely, we have

$$(8.2)\qquad \sum_{\mathbf{s}\ge\mathbf{s}_0,\,|\mathbf{s}|\le k} \binom{k}{\mathbf{s}}\Big(\frac{1}{2N}\Big)^{|\mathbf{s}|} \le \sum_{p=0}^{k-|\mathbf{s}_0|}\Big(\frac{1}{2N}\Big)^{p+|\mathbf{s}_0|} N^p \cdot k! \le \frac{(k+1)!}{2N},$$

as we do at least have $|\mathbf{s}_0| \ge 1$. But k is bounded above by $d(\log 4)/(\log 6/5)$; so we bring our proof to a close by observing that (8.2) will, as required, be at most η^k, provided N is so large that

$$N > \eta^{-k_1}(k_1 + 1)!/2,$$

where $k_1 = [d(\log 4)/(\log 6/5)]$, the integer part of that multiple of d. Thus, having imposed another moderately demanding condition on how large N must be, Lemma 1.5 is proved, and we know by Theorem 2.3 that nonzero CRAB algebras exist.

9. Conclusion. In fact we have proved slightly more than the existence of CRAB algebras, namely

THEOREM 9.1. *Every FDNC algebra can be isometrically embedded in a CRAB algebra $\mathcal{B}$ such that $\mathcal{B}$ has a metric approximate diagonal, and the nilpotent elements of $\mathcal{B}$ are dense in $\mathcal{B}$.*

I would conjecture that CRAB algebras are in fact "fairly common" among radical commutative Banach algebras; something like the following may be true:

CONJECTURE 9.2. *Every commutative radical Banach algebra $\mathcal{A}$ can be isometrically embedded in a CRAB algebra $\mathcal{B}$ with a metric approximate diagonal. If $\mathcal{A}$ is an integral domain, $\mathcal{B}$ can be an integral domain. If $\mathcal{A}$ has compact multiplication, $\mathcal{B}$ can have compact multiplication.*

(We recall, however, that by Corollary 3.5 of [LRRW], an amenable Banach algebra which has the approximation property cannot both be an integral domain and have compact multiplication.) In other words, we conjecture that there is no "forbidden subalgebra" for a CRAB algebra, beyond the requirement that subalgebras must be radical and commutative; and that the use of nilpotency, which simplified things in this paper, can be "gotten around" to produce integral domains; and that compact multiplication is also a possibility (though again, not I believe with our present construction).

I hope that this paper sheds some light on the area of amenable Banach algebras in general and CRAB algebras in particular; and that having found some examples, we will come to understand better the nature of these unusual algebras.

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On the size of approximately convex sets in normed spaces

by

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Abstract. Let X be a normed space. A set $A \subseteq X$ is *approximately convex* if $d(ta + (1-t)b, A) \le 1$ for all $a, b \in A$ and $t \in [0,1]$. We prove that every n-dimensional normed space contains approximately convex sets A with $\mathcal{H}(A, \mathrm{Co}(A)) \ge \log_2 n - 1$ and $\mathrm{diam}(A) \le C\sqrt{n}(\ln n)^2$, where $\mathcal{H}$ denotes the Hausdorff distance. These estimates are reasonably sharp. For every $D > 0$, we construct worst possible approximately convex sets in $C[0,1]$ such that $\mathcal{H}(A, \mathrm{Co}(A)) = \mathrm{diam}(A) = D$. Several results pertaining to the Hyers–Ulam stability theorem are also proved.

1. Introduction. Let $(X, \|\cdot\|)$ be a normed space. In the following definition $d(x, A) = \inf\{\|x - a\| : a \in A\}$ denotes the distance from x to the set A.

DEFINITION 1.1. A set $A \subseteq X$ is *approximately convex* if

$$d(tx + (1-t)y, A) \le 1 \quad \text{for all } x, y \in A \text{ and } t \in [0,1].$$

Recall that the *Hausdorff distance* between subsets A and B of X is defined by

$$\mathcal{H}(A, B) = \sup\{d(x, B), d(y, A) : x \in A,\ y \in B\}.$$

Thus, A is approximately convex if and only if

$$\sup_{t \in [0,1]} \mathcal{H}(tA + (1-t)A, A) \le 1.$$

The aim of this article is to study the relationship between the size of an approximately convex set, as measured by its *diameter*

$$\mathrm{diam}(A) = \sup\{\|x - y\| : x, y \in A\},$$

and the extent to which A fails to be convex, as measured by the Hausdorff distance $\mathcal{H}(A, \mathrm{Co}(A))$ from A to its *convex hull* $\mathrm{Co}(A)$.

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