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### Composition operators and the Hilbert matrix

by

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**Abstract.** The Hilbert matrix acts on Hardy spaces by multiplication with Taylor coefficients. We find an upper bound for the norm of the induced operator.

**1. Introduction.** The classical Hilbert inequality

$$(1.1) \quad \left( \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right|^p \right)^{1/p} \leq \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=0}^{\infty} |a_n|^p \right)^{1/p}$$

is valid for sequences  $a = \{a_n\}$  in the sequence spaces  $l^p$  for  $1 < p < \infty$ , and the constant  $\pi/\sin(\pi/p)$  is best possible [HLP]. Thus the Hilbert matrix

$$H = \left( \frac{1}{i+j+1} \right)_{i,j=0,1,2,\dots},$$

acting by multiplication on sequences, induces a bounded linear operator

$$Ha = b, \quad b_n = \sum_{k=0}^{\infty} \frac{a_k}{n+k+1},$$

on the  $l^p$  spaces with norm  $\|H\|_{l^p \rightarrow l^p} = \pi/\sin(\pi/p)$  for  $1 < p < \infty$ .

The Hilbert matrix also induces an operator  $\mathcal{H}$  on Hardy spaces  $H^p$ , as explained below, by its action on Taylor coefficients. In this article we prove an analogue of the inequality (1.1) on Hardy spaces. More precisely we show

**THEOREM 1.1.** (i) *If  $2 \leq p < \infty$  then*

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\pi/p)} \|f\|_{H^p}$$

for each  $f \in H^p$ .

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(ii) If  $1 < p < 2$  then

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\pi/p)} \|f\|_{H^p}$$

for each  $f \in H^p$  with  $f(0) = 0$ .

The proof will be given in Section 3 and involves an expression of  $\mathcal{H}$  in terms of weighted composition operators of which we can estimate the Hardy space norms.

Recall that the *Hardy space*  $H^p$ ,  $1 \leq p \leq \infty$ , of the unit disc  $\mathbb{D}$  is the Banach space of analytic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  for which

$$(1.2) \quad \|f\|_{H^p} = \sup_{r < 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

for finite  $p$ , and  $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$ . For  $1 \leq p \leq q \leq \infty$  we have  $H^1 \supset H^p \supset H^q \supset H^\infty$  and  $H^p$  is embedded as a closed subspace in  $L^p(\mathbb{T})$ , the Lebesgue space on the unit circle, by identifying  $H^p$  with the closure of analytic polynomials in  $L^p(\mathbb{T})$ . Additional properties of Hardy spaces can be found in [DU].

To study the effect of the Hilbert matrix on Hardy spaces let  $f(z) = \sum_{n=0}^\infty a_n z^n$  belong to  $H^1$ . Hardy's inequality [DU, p. 48] says

$$\sum_{n \geq 0} \frac{|a_n|}{n+1} \leq \pi \|f\|_{H^1},$$

and it follows that the power series

$$F(z) = \sum_{n=0}^\infty \left( \sum_{k=0}^\infty \frac{a_k}{n+k+1} \right) z^n$$

has bounded coefficients, hence its radius of convergence is  $\geq 1$ . In this way we obtain a well defined analytic function  $F = \mathcal{H}(f)$  on the disc for each  $f \in H^1$ . A calculation shows that we can write

$$(1.3) \quad \mathcal{H}(f)(z) = \int_0^1 f(t) \frac{1}{1-tz} dt,$$

where the convergence of the integral is guaranteed by the the Fejér–Riesz inequality [DU, p. 46] and the fact that  $1/(1-tz)$  is bounded in  $t$  for each  $z \in \mathbb{D}$ .

The correspondence  $f \mapsto \mathcal{H}(f)$  is clearly linear and we consider the restriction of this mapping to the spaces  $H^p$  for  $p \geq 1$ . For  $p = 2$ , the isometric identification of  $H^2$  with  $l^2$  gives

$$\|\mathcal{H}\|_{H^2 \rightarrow H^2} = \pi.$$

On the other hand  $\mathcal{H}$  is not bounded on the spaces  $H^1$  and  $H^\infty$ . For  $H^\infty$  this is because the constant function 1 is mapped to

$$\mathcal{H}(1)(z) = \frac{1}{z} \log \frac{1}{1-z},$$

which is not a bounded function. For  $H^1$ , let  $\varepsilon > 0$  and let

$$f_\varepsilon(z) = \frac{1}{(1-z) \left( \frac{1}{z} \log \frac{1}{1-z} \right)^{1+\varepsilon}},$$

a function which belongs to  $H^1$  [DU, p. 13] and is positive on  $[0, 1]$ . We assert that the analytic function  $\mathcal{H}(f_\varepsilon)$  does not belong to  $H^1$  for small  $\varepsilon$ . Indeed using (1.3) we find

$$\mathcal{H}(f_\varepsilon)(z) = \sum_{n=0}^\infty \left( \int_0^1 t^n f_\varepsilon(t) dt \right) z^n$$

and if we assume  $\mathcal{H}(f_\varepsilon) \in H^1$  then Hardy's inequality implies that the quantity

$$\begin{aligned} \sum_{n=0}^\infty \frac{1}{n+1} \int_0^1 t^n f_\varepsilon(t) dt &= \int_0^1 f_\varepsilon(t) \sum_{n=0}^\infty \frac{t^n}{n+1} dt \\ &= \int_0^1 f_\varepsilon(t) \left( \frac{1}{t} \log \frac{1}{1-t} \right) dt \\ &= \int_0^1 \frac{1}{(1-t) \left( \frac{1}{t} \log \frac{1}{1-t} \right)^\varepsilon} dt \end{aligned}$$

is finite. For  $\varepsilon \leq 1$  this is a contradiction.

The operator  $\mathcal{H}$  is however bounded on  $H^p$  for all  $1 < p < \infty$ . This is known and a quick way to see it is to view  $\mathcal{H}$  as a Hankel operator. In fact  $\mathcal{H}$  is a prototype for Hankel operators (see [PA] for details). We will not pursue this aspect further except to note that a Hankel operator is bounded on  $H^2$  if and only if it is bounded on each  $H^p$  for  $1 < p < \infty$  (see [CS]). The results of [CS] also imply that  $\mathcal{H}$  is not bounded on  $H^1$ , a fact that we obtained by a direct argument above.

**2.  $\mathcal{H}$  in terms of composition operators.** In this section we indicate how  $\mathcal{H}$  can be written as an average of certain weighted composition operators.

Every analytic function  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  induces a bounded composition operator

$$C_\phi : f \mapsto f \circ \phi$$

on  $H^p$  for  $1 \leq p \leq \infty$  (see [DU, p. 29]). In addition if  $w(z)$  is a bounded analytic function then the weighted composition operator

$$C_{w,\phi}(f)(z) = w(z)f(\phi(z))$$

is bounded on each  $H^p$ . More information about these operators can be found in [CM] or [SH]. We will not need here any of their properties except the fact that they are bounded.

The connection of the Hilbert matrix with composition operators comes as follows. For  $f \in H^1$  the Fejér-Riesz theorem, which guarantees convergence, along with analyticity shows that the integral in (1.3) is independent of the path of integration. For  $z \in \mathbb{D}$  we can choose the path

$$(2.1) \quad \zeta(t) = \zeta_z(t) = \frac{t}{(t-1)z+1}, \quad 0 \leq t \leq 1,$$

i.e. a circular arc in  $\mathbb{D}$  joining 0 to 1. The change of variable in (1.3) gives

$$(2.2) \quad \mathcal{H}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} f\left(\frac{t}{(t-1)z+1}\right) dt.$$

This expression says that the transformation  $\mathcal{H}$  is an average

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z) dt$$

of the weighted composition operators

$$(2.3) \quad T_t(f)(z) = w_t(z)f(\phi_t(z)),$$

where

$$w_t(z) = \frac{1}{(t-1)z+1} \quad \text{and} \quad \phi_t(z) = \frac{t}{(t-1)z+1}.$$

It is easy to see that  $\phi_t$  is a self-map of the disc, hence  $f \mapsto f \circ \phi_t$  is bounded on  $H^p$ , and for each  $0 < t < 1$ ,  $w_t(z)$  is a bounded analytic function. Thus  $T_t : H^p \rightarrow H^p$ ,  $1 \leq p \leq \infty$ , is bounded for  $0 < t < 1$ .

**3. Proof of the Theorem.** We first obtain estimates for the norms of the weighted composition operator  $T_t$ . The estimates are achieved by transferring  $T_t$  to operators  $\tilde{T}_t$  acting on Hardy spaces of the right half plane, which are isometric to Hardy spaces of the disc. The form of  $\tilde{T}_t$  permits estimates of its norm, thereby the estimate for the norm of  $T_t$  follows.

LEMMA 3.1. *If  $p \geq 2$ , then*

$$(3.1) \quad \|T_t(f)\|_{H^p} \leq \frac{t^{1/p-1}}{(1-t)^{1/p}} \|f\|_{H^p}, \quad 0 < t < 1,$$

for each  $f \in H^p$ .

Proof. The Hardy space  $H^p(\Pi)$  of the right half plane  $\Pi = \{z : \Re(z) > 0\}$  consists of analytic functions  $f : \Pi \rightarrow \mathbb{C}$  such that

$$(3.2) \quad \|f\|_{H^p(\Pi)}^p = \sup_{0 < x < \infty} \int_{-\infty}^{\infty} |f(x+iy)|^p dy < \infty.$$

These are Banach spaces for  $1 \leq p < \infty$ .

Let  $\mu(z) = (1+z)/(1-z)$  be the conformal map of  $\mathbb{D}$  onto  $\Pi$  with inverse  $\mu^{-1}(z) = (z-1)/(z+1)$ , and let

$$V(f)(z) = \frac{(4\pi)^{1/p}}{(1-z)^{2/p}} f(\mu(z)), \quad f \in H^p(\Pi).$$

It can be checked that this map is a linear isometry from  $H^p(\Pi)$  onto  $H^p$  with inverse given by

$$V^{-1}(g)(z) = \frac{1}{\pi^{1/p}(1+z)^{2/p}} g(\mu^{-1}(z)), \quad g \in H^p.$$

Let  $\tilde{T}_t : H^p(\Pi) \rightarrow H^p(\Pi)$  be the operators defined by

$$\tilde{T}_t = V^{-1}T_tV$$

and suppose  $h \in H^p(\Pi)$ . A calculation shows that  $\tilde{T}_t$  are weighted composition operators given by

$$(3.3) \quad \tilde{T}_t(h)(z) = \frac{1}{(1-t)^{2/p}} \left( \frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{1-2/p} h(\Phi_t(z)), \quad 0 < t < 1,$$

where

$$\Phi_t(z) = \mu \circ \phi_t \circ \mu^{-1}(z) = \frac{t}{1-t}z + \frac{1}{1-t}$$

is an analytic function mapping  $\Pi$  into itself. By an elementary argument we see that if  $z \in \Pi$  then  $|(t-1)\mu^{-1}(z)+1| \geq t$  and since  $1-2/p \geq 0$  we have

$$|\tilde{T}_t(h)(z)| \leq \frac{t^{2/p-1}}{(1-t)^{2/p}} |h(\Phi_t(z))|.$$

Integrating for the norm we have

$$\begin{aligned} \|\tilde{T}_t(h)\|_{H^p(\Pi)} &= \sup_{0 < x < \infty} \left( \int_{-\infty}^{\infty} |\tilde{T}_t(h)(z)|^p dy \right)^{1/p} \\ &\leq \frac{t^{2/p-1}}{(1-t)^{2/p}} \sup_{0 < x < \infty} \left( \int_{-\infty}^{\infty} \left| h\left(\frac{t}{1-t}(x+iy) + \frac{1}{1-t}\right) \right|^p dy \right)^{1/p} \\ &= \frac{t^{2/p-1}}{(1-t)^{2/p}} \sup_{1/(1-t) < X < \infty} \left( \int_{-\infty}^{\infty} |h(X+iY)|^p \frac{1-t}{t} dY \right)^{1/p} \end{aligned}$$

where we have changed the variables  $X = \frac{t}{1-t}x + \frac{1}{1-t}$  and  $Y = \frac{t}{1-t}y$ , to obtain

$$\begin{aligned} &\leq \frac{t^{1/p-1}}{(1-t)^{1/p}} \sup_{0 < X < \infty} \left( \int_{-\infty}^{\infty} |h(X + iY)|^p dY \right)^{1/p} \\ &= \frac{t^{1/p-1}}{(1-t)^{1/p}} \|h\|_{H^p(\Pi)}. \end{aligned}$$

The conclusion follows. ■

For the final step of the proof we will need some classical identities about the Gamma and Beta functions (see for example [WW]). The Beta function is defined by

$$B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx$$

for each  $s, t$  with  $\Re(s) > 0, \Re(t) > 0$ . The value  $B(s, t)$  can be expressed in terms of the Gamma function as  $B(s, t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ . We are also going to use the functional equation for the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

which is valid for non-integer complex  $z$ .

Now suppose  $p \geq 2$  and  $f \in H^p$  with  $\|f\|_{H^p} = 1$ . Then

$$\begin{aligned} \|\mathcal{H}(f)\|_{H^p} &= \sup_{r < 1} \left( \int_0^{2\pi} |\mathcal{H}(f)(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} \\ &= \sup_{r < 1} \left( \int_0^{2\pi} \left| \int_0^1 T_t(f)(re^{i\theta}) dt \right|^p \frac{d\theta}{2\pi} \right)^{1/p} \\ &\leq \int_0^1 \sup_{r < 1} \left( \int_0^{2\pi} |T_t(f)(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} dt \end{aligned}$$

(by the continuous version of Minkowski's inequality)

$$\begin{aligned} &= \int_0^1 \|T_t(f)\|_{H^p} dt \leq \int_0^1 t^{1/p-1} (1-t)^{-1/p} dt \\ &= B(1/p, 1-1/p) = \Gamma(1/p)\Gamma(1-1/p) \\ &= \frac{\pi}{\sin(\pi/p)}, \end{aligned}$$

and this gives the assertion for  $p \geq 2$ .

Suppose now  $1 < p < 2$  and  $f \in H^p$  with  $f(0) = 0$ . Then  $f(z) = zf_0(z)$  with  $\|f\|_{H^p} = \|f_0\|_{H^p}$ . Writing  $\mathcal{H}$  in the integral form (2.2) we see that

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f_0)(z) dt$$

where  $T_t$  are the weighted composition operators

$$T_t(g)(z) = \frac{t}{((t-1)z+1)^2} g\left(\frac{t}{(t-1)z+1}\right).$$

We now follow the proof (with same notation) of Lemma 3.1 to estimate the norms of  $T_t$ . Letting  $\tilde{T}_t = V^{-1}T_tV : H^p(\Pi) \rightarrow H^p(\Pi)$  we find

$$(3.4) \quad \tilde{T}_t(h)(z) = \frac{t}{(1-t)^{2/p}} \left( \frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{2-2/p} h(\Phi_t(z)), \quad 0 < t < 1,$$

for each  $h \in H^p(\Pi)$ . Because  $2-2/p > 0$  for  $p > 1$ , the rest of the calculation in Lemma 3.1 goes through and we conclude

$$\|T_t(g)\|_{H^p} \leq \frac{t^{1/p-1}}{(1-t)^{1/p}} \|g\|_{H^p}, \quad 0 < t < 1,$$

for each  $g \in H^p$ . Using this norm estimate we can repeat the final step of the proof of the case  $p \geq 2$  to obtain

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\pi/p)} \|f_0\|_{H^p} = \frac{\pi}{\sin(\pi/p)} \|f\|_{H^p},$$

and this finishes the proof of the theorem.

REMARKS. We do not know if the inequalities in the theorem are sharp but it is expected they are. Also we have not been able to remove the restriction  $f(0) = 0$  in the case  $1 < p < 2$ . One can obtain an inequality which holds for all  $f \in H^p, 1 < p < 2$ , as follows: Write  $f(z) = f(0) + f_0(z)$  with  $f_0(0) = 0$ ; then

$$\begin{aligned} \mathcal{H}(f)(z) &= f(0)\mathcal{H}(1)(z) + \mathcal{H}(f_0)(z) \\ &= f(0) \frac{1}{z} \log \frac{1}{1-z} + \mathcal{H}(f_0)(z). \end{aligned}$$

Using the second part of Theorem 1.1 and the fact that  $\|f - f(0)\|_{H^p} \leq \|f\|_{H^p} + |f(0)| \leq 2\|f\|_{H^p}$  we obtain

$$\|\mathcal{H}(f)\|_{H^p} \leq \left( \frac{2\pi}{\sin(\pi/p)} + \left\| \log \frac{1}{1-z} \right\|_{H^p} \right) \|f\|_{H^p},$$

an inequality which is certainly not the best.

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