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On the bundle convergence of double orthogonal series in noncommutative L_2-spaces

by

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Abstract. The notion of bundle convergence in von Neumann algebras and their L_2-spaces for single (ordinary) sequences was introduced by Hensz, Jajte, and Paszkiewicz in 1996. Bundle convergence is stronger than almost sure convergence in von Neumann algebras. Our main result is the extension of the two-parameter Rademacher–Men'shov theorem from the classical commutative case to the noncommutative case. To our best knowledge, this is the first attempt to adopt the notion of bundle convergence to multiple series. Our method of proof is different from the classical one, because of the lack of the triangle inequality in a noncommutative von Neumann algebra.

In this context, bundle convergence resembles the regular convergence introduced by Hardy in the classical case. The noncommutative counterpart of convergence in Pringsheim's sense remains to be found.

1. von Neumann algebras and bundle convergence. As a background, we shall give a brief account, without proofs, of the basic notions and results in von Neumann algebras. The reader interested in details may consult the books by Dixmier [3] and Jajte [7, 8].

Let $\mathfrak{A}$ be a σ-finite von Neumann algebra with a faithful and normal state ϕ. Then the Cauchy–Schwarz inequality holds true:

$$|\phi(B^*A)|^2 \le \phi(B^*B)\phi(A^*A), \qquad A, B \in \mathfrak{A}. \tag{1.1}$$

Furthermore,

$$|\phi(A)| \le \|A\|_\infty, \qquad A \in \mathfrak{A}, \tag{1.2}$$

where $\|\cdot\|_\infty$ denotes the operator norm in $\mathfrak{A}$.

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One may introduce a scalar product on $\mathfrak{A}$ by setting

$$\langle A,B\rangle := \phi(B^*A), \quad A,B\in\mathfrak{A}. \tag{1.3}$$

By (1.1), it is easy to see that $(\mathfrak{A}, \langle\cdot,\cdot\rangle)$ is a prehilbert space over the field $\mathbb{C}$ of complex numbers. Denote by $L_2 = L_2(\mathfrak{A},\phi)$ its completion, by $(\cdot,\cdot)$ the scalar product, and by $\|\cdot\|$ the norm in L_2.

The celebrated Gelfand–Naimark–Segal representation theorem states that there exists a one-to-one $*$-homeomorphism π of $\mathfrak{A}$ into the algebra of all bounded linear operators acting on L_2 and there is a cyclic vector ω in L_2 such that

$$\phi(A) = (\pi(A)\omega,\omega), \quad A\in\mathfrak{A}. \tag{1.4}$$

Relations (1.3) and (1.4) can be combined into the following one:

$$\langle A,B\rangle = (\pi(A)\omega,\pi(B)\omega), \quad A,B\in\mathfrak{A}.$$

From (1.2) and the equality

$$\|A^*A\|_\infty = \|A\|_\infty^2, \quad A\in\mathfrak{A}, \tag{1.5}$$

which holds in any von Neumann algebra $\mathfrak{A}$, it follows immediately that

$$\|\pi(A)\omega\| = \{\phi(A^*A)\}^{1/2} \le \|A^*A\|_\infty^{1/2} = \|A\|_\infty, \quad A\in\mathfrak{A}. \tag{1.6}$$

Consequently, $\mathfrak{A}$ endowed with the scalar product $\langle\cdot,\cdot\rangle$ defined in (1.3) can be identified with the norm dense subset $\pi(\mathfrak{A})\omega$ of L_2.

The notion of bundle convergence in L_2 (as well as in the von Neumann algebra $\mathfrak{A}$) for single (ordinary) sequences was recently introduced by Hensz, Jajte, and Paszkiewicz [6]. One can adapt it to double sequences as follows.

With any double sequence $(D_{kl} : k,l = 1,2,\ldots)$ of operators in $\mathfrak{A}_+$, the cone of positive (selfadjoint) operators in $\mathfrak{A}$, for which

$$\sum_{k=1}^\infty\sum_{l=1}^\infty \phi(D_{kl}) < \infty, \tag{1.7}$$

one can associate a double bundle

$$\mathcal{P} = \mathcal{P}(D_{kl}) := \Big\{P\in\operatorname{Proj}\mathfrak{A} : \sup_{m,n\ge1}\Big\|P\Big(\sum_{k=1}^m\sum_{l=1}^n D_{kl}\Big)P\Big\|_\infty \text{ is finite and } \|PD_{kl}P\|_\infty\to 0 \text{ as } k+l\to\infty\Big\}, \tag{1.8}$$

where $\operatorname{Proj}\mathfrak{A}$ denotes the class of all projections in $\mathfrak{A}$.

The crucial fact is that the bundle $\mathcal{P}$ is "abundant" enough to contain projections arbitrarily close to the identity operator 1 in $\mathfrak{A}$: for every $\varepsilon > 0$ there exists a projection P in $\mathcal{P}(D_{kl})$ such that $\phi(1-P) < \varepsilon$.

Now, a double sequence $(\varrho_{mn} : m,n = 1,2,\ldots)$ of vectors in L_2 is said to be *bundle convergent* to some ϱ in L_2, in symbols,

$$\varrho_{mn} \xrightarrow{\text{b}} \varrho \quad \text{as } m+n\to\infty,$$

if there exists a sequence (R_{mn}) of operators in $\mathfrak{A}$ such that

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \|\varrho_{mn} - \varrho - \pi(R_{mn})\omega\|^2 < \infty, \tag{1.9}$$

and there exists a bundle $\mathcal{P} = \mathcal{P}(D_{kl})$ such that for each projection P in $\mathcal{P}$,

$$\|R_{mn}P\|_\infty \to 0 \quad \text{as } m+n\to\infty. \tag{1.10}$$

Since the intersection of two (or even a countable number of) bundles is a bundle again, bundle convergence enjoys the property of additivity. Furthermore, bundle convergence implies almost sure convergence (but not conversely), and the limit of a bundle convergent sequence is unique in the selfadjoint part of L_2. For details, we refer the reader to [5] (almost sure convergence) and [6] (bundle convergence).

Remark. We point out that the above definition of bundle convergence of double sequences can be actually reduced to the case of the classical bundle convergence of single sequences introduced by Hensz, Jajte, and Paszkiewicz [6] as follows.

Let $(D_j : j = 1,2,\ldots)$ be a single sequence of operators in $\mathfrak{A}_+$ such that

$$\sum_{j=1}^\infty \phi(D_j) < \infty. \tag{1.11}$$

The classical (single) bundle $\mathcal{P} = \mathcal{P}(D_j)$ associated with (D_j) is defined by

$$\mathcal{P} := \Big\{P\in\operatorname{Proj}\mathfrak{A} : \sup_{n\ge1}\Big\|P\Big(\sum_{j=1}^n D_j\Big)P\Big\|_\infty \text{ is finite and } \|PD_jP\|_\infty\to 0 \text{ as } j\to\infty\Big\}. \tag{1.12}$$

Then a single sequence $(\zeta_j : j = 1,2,\ldots)$ of vectors in L_2 is said to be *bundle convergent* to some ζ in L_2 if there exists a single sequence $(A_j : j = 1,2,\ldots)$ of operators in $\mathfrak{A}$ such that

$$\sum_{j=1}^\infty \|\zeta_j - \zeta - \pi(A_j)\omega\|^2 < \infty \tag{1.13}$$

and there exists a classical (single) bundle $\mathcal{P}$ such that for each P in $\mathcal{P}$ we have

$$\|A_jP\|_\infty \to 0 \quad \text{as } j\to\infty. \tag{1.14}$$

Now, our definitions (1.7)–(1.10) are simple reformulations of (1.11)–(1.14) if the terms of the double sequence (ϱ_{kl}) are reindexed into a single sequence (ζ_j), for example, by making use of the familiar diagonalization process due to Cantor. That is, set

$$\zeta_1 := \varrho_{11}, \quad \zeta_2 := \varrho_{12}, \quad \zeta_3 := \varrho_{21}, \quad \zeta_4 := \varrho_{13},$$
$$\zeta_5 := \varrho_{22}, \quad \zeta_6 := \varrho_{31}, \quad \zeta_7 := \varrho_{14}, \quad \text{etc.}$$

(1.8) is the only condition which requires an explanation. According to the above diagonalization process, we have to require that

$$\sup\Big\{\Big\|P\Big(\sum_{(k,l):k+l\le m} D_{kl} + \sum_{l=1}^{n} D_{m+1-l,l}\Big)P\Big\|_\infty : \\ n = 1,2,\ldots,m \text{ and } m = 1,2,\ldots\Big\}$$

is finite, instead of the first requirement in (1.8). However, these two requirements are equivalent, due to the fact that the D_{kl} are positive (selfadjoint) operators in $\mathfrak{A}$.

We note that it is immaterial how the double sequence (ϱ_{kl}) is rearranged into a single sequence.

For simplicity, however, we adhere to the notations in (1.7)–(1.10) in what follows.

2. Noncommutative Rademacher–Men'shov theorem. We recall that a double sequence $(\xi_{kl} : k,l = 2,3,\ldots)$ of vectors in L_2 is called *orthogonal* if

$$(\xi_{kl}, \xi_{k_1 l_1}) = 0 \quad \text{whenever} \quad (k,l) \ne (k_1, l_1).$$

By the completeness of L_2, if

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty} \|\xi_{kl}\|^2 < \infty,$$

then the sum

$$\sigma := \sum_{k=2}^{\infty}\sum_{l=2}^{\infty} \xi_{kl} \tag{2.1}$$

as well as the remainder sums

$$\varrho_{mn} := \sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} \xi_{kl}$$

exist in the norm of L_2 for all $m,n \ge 1$. Denote by

$$\sigma_{mn} := \sum_{k=2}^{m}\sum_{l=2}^{n} \xi_{kl}, \qquad m,n \ge 2,$$

the rectangular partial sums of the double series in (2.1). It is plain that

$$\sigma - \sigma_{mn} = \varrho_{m1} + \varrho_{1n} - \varrho_{mn}, \qquad m,n \ge 1, \tag{2.2}$$

with the agreement that $\sigma_{mn} := o$ in case $\min(m,n) = 1$, where o is the zero vector in L_2.

Our main result reads as follows.

THEOREM 1. *If $(\xi_{kl} : k,l = 2,3,\ldots)$ is an orthogonal double sequence in $L_2 = L_2(\mathfrak{A},\phi)$ such that*

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty} \|\xi_{kl}\|^2 (\log k)^2 (\log l)^2 < \infty, \tag{2.3}$$

then

$$\varrho_{mn} \xrightarrow{\text{b}} o \quad \text{as } m+n \to \infty. \tag{2.4}$$

The logarithms are to base 2 in this paper.

PROBLEM. Theorem 1 can be viewed as the two-parameter Rademacher–Men'shov theorem in noncommutative L_2-spaces. By virtue of (2.2), we may interpret the conclusion of Theorem 1 in the form

$$\sigma_{mn} - \sigma \to o \quad \text{as } m,n \to \infty,$$

where σ is defined in (2.1). This kind of interpretation resembles the notion of regular convergence of double complex series, introduced by Hardy [4] and rediscovered by the first named author [10, 11] in an equivalent form.

However, the problem of how to attribute a precise meaning to the limit relation above in the sense that the rectangular partial sum σ_{mn} is arbitrarily "close" to its sum σ as both indices m and n are large enough, is still open. In this context, the limit relation (2.4) says that the remainder sum ϱ_{mn} of the series in (2.1) is arbitrarily small if at least one of the indices m and n is large enough.

On the other hand, the definition of the limit relation

$$\varrho_{mn} \xrightarrow{\text{b}} o \quad \text{as } m,n \to \infty, \tag{2.5}$$

which would be the noncommutative counterpart of convergence in Pringsheim's sense, has not been made clear yet. We emphasize that $\max(m,n) \to \infty$ in (2.4), while $\min(m,n) \to \infty$ in (2.5).

By definition, (2.4) guarantees the existence of a sequence (R_{mn}) of operators in $\mathfrak{A}$ satisfying (1.9) with $\varrho := o$ and the existence of a bundle $\mathcal{P}$ such that (1.10) is satisfied for each $P \in \mathcal{P}$. Motivated by (2.2), set

$$\widetilde{R}_{mn} := R_{m1} + R_{1n} - R_{mn}, \qquad m,n = 2,3,\ldots$$

It is plain that $\widetilde{R}_{mn} \in \mathfrak{A}$, and for each $P \in \mathcal{P}$,

$$\|\widetilde{R}_{mn} P\|_\infty \to 0 \quad \text{as } m,n \to \infty$$

(cf. (1.10)), but we are in trouble as to the fulfillment of an analogue of (1.9). Since

$$\sigma - \sigma_{mn} - \pi(\widetilde{R}_{mn})\omega$$
$$= (\varrho_{m1} - \pi(R_{m1})\omega) + (\varrho_{1n} - \pi(R_{1n})\omega) - (\varrho_{mn} - \pi(R_{mn})\omega),$$

we can only state, via the triangle inequality in L_2, that

$$\|\sigma - \sigma_{mn} - \pi(\widetilde{R}_{mn})\omega\| \to 0 \quad \text{as } m, n \to \infty,$$

without involving any summation with respect to m and/or n.

To sum up, in any reasonable definition of the limit relation (2.5) one has to require the fulfillment of (1.10) with $m, n \to \infty$ instead of $m + n \to \infty$. However, it is not clear what kind of substitute for (1.9) expressing a certain rate of approximation makes sense.

REMARK. Finally, we make a historical remark. In the classical commutative case, the almost sure (everywhere) convergence of double orthogonal series, under condition (2.3), was first proved by Agnew [1], while the regular convergence of the same double series was proved by the first named author [10]. The reader interested in the classical theory of orthogonal series may consult the monograph by Alexits [2].

3. Auxiliary inequalities.

We recall the definition

$$|A| := (A^*A)^{1/2}, \qquad A \in \mathfrak{A}.$$

The square root makes sense, since $A^*A \in \mathfrak{A}_+$. Unfortunately, the traditional triangle inequality

$$|A_1 + A_2| \le |A_1| + |A_2|, \qquad A_1, A_2 \in \mathfrak{A},$$

does not hold in general. However, the following weaker substitute is available in any von Neumann algebra $\mathfrak{A}$.

LEMMA 1 (see, e.g., [8, p. 4]). *If $c_j \in \mathbb{C}$ and $A_j \in \mathfrak{A}$ for $1 \le j \le n$, then*

$$\Big|\sum_{j=1}^{n} c_j A_j\Big|^2 \le \sum_{j=1}^{n} |c_j|^2 \sum_{j=1}^{n} |A_j|^2.$$

The next lemma gives a simple sufficient condition for bundle convergence.

LEMMA 2 (see [6, pp. 30–31]). *If $(\varrho_{mn}) \subset L_2$ and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \|\varrho_{mn}\|^2 < \infty,$$

then $\varrho_{mn} \xrightarrow{\text{b}} o$ as $m + n \to \infty$.

Actually, Lemma 2 is proved in [6] for a single sequence $(\varrho_n) \subset L_2$, but any diagonalization process reduces the two-parameter case to the one-parameter case in an obvious way. It does not matter how the double sequence (ϱ_{mn}) is rearranged into a single sequence.

The noncommutative version of the famous Rademacher–Men'shov inequality reads as follows.

LEMMA 3 (see [6] and also [8, Lemma 5.5.2 on p. 65]). *Given any finite orthogonal single sequence $(\xi_j : 1 \le j \le N)$ in $L_2(\mathfrak{A}, \phi)$ and a number $\delta > 0$, there exist a sequence $(A_j : 1 \le j \le N)$ of operators in $\mathfrak{A}$ and an operator D in $\mathfrak{A}_+$ such that*

$$\|\xi_j - \pi(A_j)\omega\| < \delta, \qquad 1 \le j \le N,$$
$$\Big|\sum_{j=1}^{n} A_k\Big|^2 \le D, \qquad 1 \le n \le N,$$

and

$$\phi(D) \le (\log 2N)^2 \sum_{j=1}^{N} \|\xi_j\|^2 + \delta.$$

The next lemma formulates the noncommutative version of the two-parameter Rademacher–Men'shov inequality.

LEMMA 4 (see [9]). *Given a finite orthogonal double sequence $(\xi_{ij} : 1 \le i \le M,\ 1 \le j \le N)$ in $L_2(\mathfrak{A}, \phi)$ and a number $\delta > 0$, there exist a double sequence $(A_{ij} : 1 \le i \le M,\ 1 \le j \le N)$ of operators in $\mathfrak{A}$ and an operator D in $\mathfrak{A}_+$ such that*

$$\|\xi_{ij} - \pi(A_{ij})\omega\| < \delta, \qquad 1 \le i \le M,\ 1 \le j \le N,$$
$$\Big|\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\Big|^2 \le D, \qquad 1 \le m \le M,\ 1 \le n \le N,$$

and

$$\phi(D) \le (\log 2M)^2 (\log 2N)^2 \sum_{i=1}^{M}\sum_{j=1}^{N} \|\xi_{ij}\|^2 + \delta.$$

4. Proof of the main result.

We begin with proving two lemmas, which are interesting in themselves.

LEMMA 5. *If (ξ_{kl}) is an orthogonal double sequence in L_2 such that*

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty} \|\xi_{kl}\|^2 (\log k)(\log l) < \infty, \tag{4.1}$$

then

$$\varrho_{2^p,2^q} \xrightarrow{\text{b}} o \quad \text{as } p + q \to \infty. \tag{4.2}$$

Proof. By orthogonality and (4.1), we have

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\|\varrho_{2^p,2^q}\|^2=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=2^p+1}^{\infty}\sum_{l=2^q+1}^{\infty}\|\xi_{kl}\|^2$$
$$=\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2\sum_{p:2^p<k}1\sum_{q:2^q<l}1$$
$$\le\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2(\log k)(\log l)<\infty.$$

Now, Lemma 2 yields (4.2).

LEMMA 6. *If* (ξ_{kl}) *is an orthogonal double sequence in* $L_2(\mathfrak{A},\phi)$ *such that*

$$(4.3)\qquad \sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2(\log k)^2(\log l)<\infty,$$

then

$$(4.4)\qquad \varrho_{m,2^q}-\varrho_{2^p,2^q}\xrightarrow{\rm b} o\quad \textit{as } p+q\to\infty,\ \textit{while } m\in I_p.$$

Proof. We have to construct an appropriate bundle. To this end, first set

$$\widehat{\Xi}_{kq}:=\sum_{l=2^q+1}^{\infty}\xi_{kl},\qquad k=2,3,\ldots;\ q=0,1,\ldots$$

Clearly, for each fixed $q\ge 0$, the single sequence $(\widehat{\Xi}_{kq}:k=2,3,\ldots)$ is orthogonal in L_2 and

$$\|\widehat{\Xi}_{kq}\|^2=\sum_{l=2^q+1}^{\infty}\|\xi_{kl}\|^2.$$

Second, let $(\delta_p:p=0,1,\ldots)$ be a sequence of positive numbers such that

$$(4.5)\qquad \sum_{p=0}^{\infty}2^{3p}\delta_p^2<\infty,$$

and apply Lemma 3 separately for each dyadic group

$$(\widehat{\Xi}_{kq}:k\in I_p:=\{2^p,2^p+1,\ldots,2^{p+1}-1\})$$

with the product $\delta_p\delta_q$, where $p\ge 1$ and $q\ge 0$. As a result, we obtain a double sequence

$$(\widehat{A}_{kq}:k\in I_p\text{ for }p=1,2,\ldots;q=0,1,\ldots)$$
$$\equiv(\widehat{A}_{kq}:k=2,3,\ldots;q=0,1,\ldots)$$

of operators in $\mathfrak{A}$ and a double sequence $(\widehat{D}_{pq})$ of operators in $\mathfrak{A}_+$ with the following properties:

$$(4.6)\qquad \|\widehat{\Xi}_{kq}-\pi(\widehat{A}_{kq})\omega\|<\delta_p\delta_q,\quad k\in I_p,$$
$$(4.7)\qquad |\widehat{S}_{mq}-\widehat{S}_{2^p,q}|^2\le\widehat{D}_{pq},\quad m\in I_p,$$

where

$$(4.8)\qquad \widehat{S}_{mq}:=\sum_{k=2}^{m}\widehat{A}_{kq},$$

and

$$(4.9)\qquad \phi(\widehat{D}_{pq})\le(p+1)^2\sum_{k\in I_p}\|\widehat{\Xi}_{kq}\|^2+\delta_p\delta_q,\quad p=1,2,\ldots;\ q=0,1,\ldots$$

By (4.3), (4.5), and (4.9), we have

$$\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\phi(\widehat{D}_{pq})\le\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\Big\{(p+1)^2\sum_{k\in I_p}\sum_{l=2^q+1}^{\infty}\|\xi_{kl}\|^2+\delta_p\delta_q\Big\}$$
$$\le\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2(\log 2k)^2\sum_{q:2^q<l}1+\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\delta_p\delta_q<\infty,$$

where we used the fact that $\sum\delta_p<\infty$, thanks to the familiar Cauchy inequality for real numbers. This means that the double sequence $(\widehat{D}_{pq}:p\ge 1,\ q\ge 0)$ of operators in $\mathfrak{A}_+$ determines a bundle, say $\widehat{\mathcal{P}}$.

Next, we prove that

$$(4.10)\qquad \varrho_{m,2^q}-\varrho_{2^p,2^q}-\pi(\widehat{S}_{mq}-\widehat{S}_{2^p,q})\omega\xrightarrow{\rm b} o\quad\text{as } p+q\to\infty,\ m\in I_p.$$

From (4.5), (4.6), and (4.8) it follows that

$$\sum_{p=1}^{\infty}\sum_{m\in I_p}\sum_{q=0}^{\infty}\|(\varrho_{2^p,2^q}-\varrho_{m,2^q})-\pi(\widehat{S}_{2^p,q}-\widehat{S}_{mq})\omega\|^2$$
$$=\sum_{p=1}^{\infty}\sum_{m\in I_p}\sum_{q=0}^{\infty}\Big\|\sum_{k=2^p+1}^{m}(\widehat{\Xi}_{kq}-\pi(\widehat{A}_{kq})\omega)\Big\|^2$$
$$\le\sum_{p=1}^{\infty}\sum_{m\in I_p}\sum_{q=0}^{\infty}(m-2^p)^2\delta_p^2\delta_q^2\le\frac{1}{3}\sum_{p=1}^{\infty}2^{3p}\delta_p^2\sum_{q=0}^{\infty}\delta_q^2<\infty.$$

Again, applying Lemma 2 gives (4.10).

In order to prove (4.4), it remains to show that

$$(4.11)\qquad \pi(\widehat{S}_{mq}-\widehat{S}_{2^p,q})\omega\xrightarrow{\rm b} o\quad\text{as } p+q\to\infty,\ m\in I_p.$$

Since $\widehat{S}_{mq} - \widehat{S}_{2^p,q}$ itself belongs to $\mathfrak{A}$, the fulfillment of condition (1.9) is trivial. Let us check (1.10). Given any projection P in $\widehat{\mathcal{P}}$, by (4.7), (1.5), and the second property of a bundle expressed in (1.8), we conclude that

$$\|(\widehat{S}_{mq} - \widehat{S}_{2^p,q})P\|_\infty^2 = \|P|\widehat{S}_{mq} - \widehat{S}_{2^p,q}|^2 P\|_\infty \le \|P\widehat{D}_{pq}P\|_\infty \to 0 \quad \text{as } p+q\to\infty,\ m\in I_p.$$

This justifies (4.11) and completes the proof of Lemma 6.

The symmetric counterpart of Lemma 6 is the following.

LEMMA 7. *If* (ξ_{kl}) *is an orthogonal double sequence in* L_2 *such that*

$$\sum_{k=2}^\infty \sum_{l=2}^\infty \|\xi_{kl}\|^2 (\log k)(\log l)^2 < \infty,$$

then

$$\varrho_{2^p,n} - \varrho_{2^p,2^q} \xrightarrow{\text{b}} o \quad \text{as } p+q\to\infty,\ n\in I_q. \tag{4.12}$$

REMARK. It is interesting to note that after having proved Theorem 1, Lemmas 5–7 can be essentially improved. (See Corollaries 1 and 2 in Section 5.)

Proof of Theorem 1. We start with the representation

$$\varrho_{mn} = \varrho_{2^p,2^q} + (\varrho_{m,2^q} - \varrho_{2^p,2^q}) + (\varrho_{2^p,n} - \varrho_{2^p,2^q}) + (\varrho_{mn} - \varrho_{m,2^q} - \varrho_{2^p,n} + \varrho_{2^p,2^q}), \quad m\in I_p \text{ and } n\in I_q. \tag{4.13}$$

Taking into account (4.2), (4.4), (4.12), (4.13), and the additivity property of bundle convergence, it is enough to prove that

$$\tau_{mn} := \varrho_{mn} - \varrho_{m,2^q} - \varrho_{2^p,n} + \varrho_{2^p,2^q} \xrightarrow{\text{b}} o \quad \text{as } p+q\to\infty,\ m\in I_p \text{ and } n\in I_q, \tag{4.14}$$

in order to conclude (2.4).

Let (δ_p) be a sequence of positive numbers satisfying (4.5). We apply Lemma 4 separately for each dyadic group $(\xi_{kl} : k\in I_p \text{ and } l\in I_q)$ with $\delta_p\delta_q$, where p and q run over the positive integers. As a result, we obtain a double sequence

$$(A_{kl} : k\in I_p \text{ and } l\in I_q \text{ for } p,q=1,2,\dots) \equiv (A_{kl} : k,l=2,3,\dots)$$

of operators in $\mathfrak{A}$ and a double sequence (D_{pq}) of operators in $\mathfrak{A}_+$ with the following properties:

$$\|\xi_{kl} - \pi(A_{kl})\omega\| < \delta_p\delta_q, \quad k\in I_p \text{ and } l\in I_q; \tag{4.15}$$

$$|T_{mn}|^2 \le D_{pq}, \quad m\in I_p \text{ and } n\in I_q; \tag{4.16}$$

where

$$T_{mn} := S_{mn} - S_{m,2^q} - S_{2^p,n} + S_{2^p,2^q} \quad\text{with}\quad S_{mn} := \sum_{k=2}^m \sum_{l=2}^n A_{kl}, \tag{4.17}$$

and

$$\phi(D_{pq}) \le (p+1)^2(q+1)^2 \sum_{k\in I_p}\sum_{l\in I_q} \|\xi_{kl}\|^2 + \delta_p\delta_q, \quad p,q=1,2,\dots \tag{4.18}$$

From (2.3), (4.5), and (4.18) it follows that

$$\sum_{p=1}^\infty\sum_{q=1}^\infty \phi(D_{pq}) \le \sum_{p=1}^\infty\sum_{q=1}^\infty \Big\{(p+1)^2(q+1)^2 \sum_{k\in I_p}\sum_{l\in I_q}\|\xi_{kl}\|^2 + \delta_p\delta_q\Big\} \le \sum_{k=2}^\infty\sum_{l=2}^\infty \|\xi_{kl}\|^2(\log 2k)^2(\log 2l)^2 + \Big(\sum_{p=1}^\infty \delta_p\Big)^2 < \infty.$$

Consequently, the double sequence $(D_{pq} : p,q=1,2,\dots)$ of operators in $\mathfrak{A}_+$ determines a bundle, say $\mathcal{P}$.

Next, we prove that

$$\tau_{mn} - \pi(T_{mn})\omega = \varrho_{mn} - \varrho_{m,2^q} - \varrho_{2^p,n} + \varrho_{2^p,2^q} - \pi(S_{mn} - S_{m,2^q} - S_{2^p,n} + S_{2^p,2^q})\omega \xrightarrow{\text{b}} 0 \tag{4.19}$$

as $p+q\to\infty$, $m\in I_p$ and $n\in I_q$ (cf. (4.14) and (4.16)). By (4.5), (4.15), and (4.17), we have

$$\sum_{m=2}^\infty\sum_{n=2}^\infty \|\tau_{mn} - \pi(T_{mn})\omega\|^2 = \sum_{p=1}^\infty\sum_{q=1}^\infty\sum_{m\in I_p}\sum_{n\in I_q}\Big\|\sum_{k=2^p+1}^m\sum_{l=2^q+1}^n (\xi_{kl} - \pi(A_{kl})\omega)\Big\|^2 \le \sum_{p=1}^\infty\sum_{q=1}^\infty\sum_{m\in I_p}\sum_{n\in I_q}(m-2^p)^2(n-2^q)^2\delta_p^2\delta_q^2 \le \frac{1}{9}\Big(\sum_{p=1}^\infty 2^{3p}\delta_p^2\Big)^2 < \infty.$$

Again, applying Lemma 2 gives (4.19).

Finally, we show that

$$\pi(T_{mn})\omega = \pi(S_{mn} - S_{m,2^q} - S_{2^p,n} + S_{2^p,2^q})\omega \xrightarrow{\text{b}} o \tag{4.20}$$

as $p+q\to\infty$, $m\in I_p$ and $n\in I_q$. By (4.16) and (4.17), T_{mn} itself belongs to $\mathfrak{A}$, so (1.9) is automatically satisfied. Given any projection P in $\mathcal{P}$, by (4.16), (1.5), and (1.8), we conclude that (1.10) is also satisfied:

$$\|T_{mn}P\|_\infty^2 = \|P|T_{mn}|^2P\|_\infty \le \|PD_{pq}P\|_\infty \to 0$$

as $p+q\to\infty$, $m\in I_p$ and $n\in I_q$. This justifies (4.20).

Combining (4.19) and (4.20) yields (4.14), and completes the proof of Theorem 1.

5. Bundle convergence of subsequences. Instead of requiring the bundle convergence of the whole double sequence $(\varrho_{mn} : m, n = 1, 2, \ldots)$ to $o \in L_2$, one may raise the question of the bundle convergence of a double subsequence $(\varrho_{m_p,n_q} : p, q = 1, 2, \ldots)$ defined in the next theorem.

THEOREM 2. *If (ξ_{kl}) is an orthogonal double sequence of vectors in L_2, $(m_p : p = 1, 2, \ldots)$ and $(n_q : q = 1, 2, \ldots)$ are strictly increasing sequences of positive integers such that $m_1 = n_1 = 1$, and*

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(\log 2p)^2(\log 2q)^2 \sum_{k=m_p+1}^{m_{p+1}} \sum_{l=n_q+1}^{n_{q+1}} \|\xi_{kl}\|^2 < \infty, \tag{5.1}$$

then

$$\varrho_{m_p,n_q} := \sum_{k=m_p+1}^{\infty} \sum_{l=n_q+1}^{\infty} \xi_{kl} \xrightarrow{\text{b}} o \quad \text{as } p+q \to \infty. \tag{5.2}$$

By (2.2), we may interpret the limit relation (5.2) in the following sense: the double subsequence $(\sigma_{m_p,n_q} : p, q = 1, 2, \ldots)$ of the rectangular partial sums of the series in (2.1) converges to its sum σ:

$$\sigma_{m_p,n_q} - \sigma \to o \quad \text{as } p, q \to \infty$$

under condition (5.1), which is weaker than (2.3).

Proof of Theorem 2. Set

$$\Xi_{pq} := \sum_{k=m_p+1}^{m_{p+1}} \sum_{l=n_q+1}^{n_{q+1}} \xi_{kl}, \quad p, q = 1, 2, \ldots$$

Clearly, $(\Xi_{pq} : p, q = 1, 2, \ldots)$ is also an orthogonal double sequence of vectors in L_2 and

$$\|\Xi_{pq}\|^2 = \sum_{k=m_p+1}^{m_{p+1}} \sum_{l=n_q+1}^{n_{q+1}} \|\xi_{kl}\|^2.$$

Since the consecutive remainder sums of the new series $\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\Xi_{pq}$ coincide with the subsequence (ϱ_{m_p,n_q}) of the remainder sums of the original series in (2.1):

$$\sum_{p=r}^{\infty}\sum_{q=s}^{\infty}\Xi_{pq} = \sum_{k=m_r+1}^{\infty} \sum_{l=n_s+1}^{\infty} \xi_{kl}, \quad r, s = 1, 2, \ldots,$$

Theorem 2 is an immediate consequence of Theorem 1.

It is instructive to consider Theorem 2 in the particular cases where

$$m_p := 2^{p-1} \quad \text{and/or} \quad n_q := 2^{q-1}, \quad p, q = 1, 2, \ldots$$

COROLLARY 1. *If (ξ_{kl}) is an orthogonal double sequence of vectors in L_2 and*

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2(\log\log 2k)^2(\log\log 2l)^2 < \infty, \tag{5.3}$$

then $\varrho_{2^p,2^q} \xrightarrow{\text{b}} o$ as $p+q \to \infty$.

COROLLARY 2. *If (ξ_{kl}) is an orthogonal double sequence of vectors in L_2 and*

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\|\xi_{kl}\|^2(\log k)^2(\log\log 2l)^2 < \infty, \tag{5.4}$$

then $\varrho_{m,2^q} \xrightarrow{\text{b}} o$ as $m+q \to \infty$.

Corollaries 1 and 2 improve Lemmas 5 and 6, since conditions (5.3) and (5.4) are essentially weaker than (4.1) and (4.3), respectively. Lemma 7 can be improved analogously.

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Composition operators and the Hilbert matrix

by

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Abstract. The Hilbert matrix acts on Hardy spaces by multiplication with Taylor coefficients. We find an upper bound for the norm of the induced operator.

1. Introduction. The classical Hilbert inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{1/p} \le \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=0}^{\infty}|a_k|^p\right)^{1/p} \tag{1.1}$$

is valid for sequences $a = \{a_n\}$ in the sequence spaces l^p for $1 < p < \infty$, and the constant $\pi/\sin(\pi/p)$ is best possible [HLP]. Thus the Hilbert matrix

$$H = \left(\frac{1}{i+j+1}\right)_{i,j=0,1,2,\dots},$$

acting by multiplication on sequences, induces a bounded linear operator

$$Ha = b, \quad b_n = \sum_{k=0}^{\infty}\frac{a_k}{n+k+1},$$

on the l^p spaces with norm $\|H\|_{l^p\to l^p} = \pi/\sin(\pi/p)$ for $1 < p < \infty$.

The Hilbert matrix also induces an operator $\mathcal{H}$ on Hardy spaces H^p, as explained below, by its action on Taylor coefficients. In this article we prove an analogue of the inequality (1.1) on Hardy spaces. More precisely we show

THEOREM 1.1. (i) *If* $2 \le p < \infty$ *then*

$$\|\mathcal{H}(f)\|_{H^p} \le \frac{\pi}{\sin(\pi/p)}\|f\|_{H^p}$$

for each $f \in H^p$.

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