[22] R. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. 16 (1967), 1031–1060.

[23] H. Triebel, *Theory of Function Spaces*, Monogr. Math. 78, Birkhäuser, Basel, 1983.

[24] —, *Theory of Function Spaces II*, Monogr. Math. 84, Birkhäuser, Basel, 1992.

Mathematisches Institut
Friedrich-Schiller-Universität Jena
Germany

Current address:
Mathematics Department
Princeton University
Princeton, NJ 08544, U.S.A.
E-mail: rytchkov@math.princeton.edu

Received February 8, 1999
Revised version January 24, 2000 (4257)

STUDIA MATHEMATICA 140 (2) (2000)

Restriction of an operator to the range of its powers

by

M. BERKANI (Oujda)

Abstract. Let T be a bounded linear operator acting on a Banach space X. For each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In [1] and [2] we have characterized operators T such that for a given integer n, the operator T_n is a Fredholm or a semi-Fredholm operator. We continue those investigations and we study the cases where T_n belongs to a given regularity in the sense defined by Kordula and Müller in [10]. We also consider the regularity of operators with topological uniform descent.

1. Introduction. Let $L(X)$ be the Banach algebra of bounded linear operators acting on a Banach space X and let $T \in L(X)$. We denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) *operator*. A *semi-Fredholm operator* is an upper or a lower semi-Fredholm operator. We let $\Phi_+(X)$ (resp. $\Phi_-(X)$) denote the set of upper (resp. lower) semi-Fredholm operators. If both $\alpha(T)$ and $\beta(T)$ are finite then T is called a *Fredholm operator* and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

For each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm (resp. semi-Fredholm) operator, then T is called a *B-Fredholm operator* (resp. a *semi-B-Fredholm*) operator. In [1] the author has studied this class of operators and proved [1, Theorem 2.7] that $T \in L(X)$ is a B-Fredholm operator if and only if $T = Q \oplus F$, where Q is a nilpotent operator and F is a Fredholm operator. In [2] we have proved the same result for semi-B-Fredholm operators acting on Hilbert spaces.

Recall that an operator $T \in L(X)$ has a *generalized inverse* if there is an $S \in L(X)$ such that $TST = T$. In this case T is also called a *relatively*

2000 *Mathematics Subject Classification*: 47A53, 47A55.

regular operator and S is called a *generalized inverse* of T. It is well known that T has a generalized inverse if and only if $R(T)$ and $N(T)$ are closed and complemented subspaces of X. In [3], S. R. Caradus has defined the following class of operators:

DEFINITION 1.1. $T \in L(X)$ is called a *generalized Fredholm operator* if T is relatively regular and there is a generalized inverse S of T such that $I - ST - TS$ is a Fredholm operator.

Let $\Phi_g(X)$ be the class of all generalized Fredholm operators on the Banach space X. In [14], [15], C. Schmoeger has studied this class of operators and proved [16, Theorem 1.1] that $T \in L(X)$ is a generalized Fredholm operator if and only if $T = Q \oplus F$, where Q is a finite-dimensional nilpotent operator and F is a Fredholm operator. Hence a generalized Fredholm operator is a B-Fredholm operator but the converse is not true, for example a nilpotent operator with an infinite-dimensional non-closed range is a B-Fredholm operator but not a generalized Fredholm operator. Moreover the class $\mathrm{BF}(X)$ of B-Fredholm operators satisfies the spectral mapping theorem while the class $\Phi_g(X)$ does not.

In [10] V. Kordula and V. Müller defined the concept of regularity follows:

DEFINITION 1.2. A non-empty subset $\mathbf{R} \subset L(X)$ is called a *regularity* if it satisfies the following conditions:

(i) If $A \in L(X)$ and $n \geq 1$ then $A \in \mathbf{R}$ if and only if $A^n \in \mathbf{R}$.

(ii) If $A, B, C, D \in L(X)$ are mutually commuting operators satisfying $AC + BD = I$ then $AB \in \mathbf{R}$ if and only if $A, B \in \mathbf{R}$.

A regularity $\mathbf{R}$ defines in a natural way a spectrum by $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{R}\}$ for every $T \in L(X)$. Moreover the spectrum $\sigma_{\mathbf{R}}$ satisfies the spectral mapping theorem.

In [12], M. Mbekhta and V. Müller, using the concept of regularity, studied various classes of operators defined by means of kernels and ranges. They considered regularities $\mathbf{R}_i$ for $1 \leq i \leq 15$ and studied their properties. The definitions of those regularities are given in the next section but herein they are indexed differently.

The aim of this paper is to answer the following question: Let $T \in L(X)$; under which conditions does there exist an integer $n \in \mathbb{N}$ such that the operator T_n belongs to a given regularity $\mathbf{R}_i, 1 \leq i \leq 15$? In order to answer this question we introduce the concept of B-regularity. If $T \in L(X)$ and if there exists an integer n for which T_n belongs to the regularity $\mathbf{R}_i$, we will say that T belongs to the B-regularity $\mathbf{BR}_i$. Then in Theorem 3.6 we summarize the relations between the regularities $\mathbf{R}_i$, $1 \leq i \leq 15$, and the corresponding B-regularities in the following table:

$\mathbf{BR}_1 = \mathbf{BR}_2 = \mathbf{BR}_3 = \mathbf{R}_3$	$\mathbf{BR}_4 = \mathbf{BR}_5 = \mathbf{R}_5$
$\mathbf{BR}_6 = \mathbf{BR}_7 = \mathbf{BR}_8 = \mathbf{R}_8$	$\mathbf{BR}_9 = \mathbf{BR}_{10} = \mathbf{R}_{10}$
$\mathbf{BR}_{11} = \mathbf{BR}_{12} = \mathbf{BR}_{13} = \mathbf{R}_{13}$	$\mathbf{BR}_{14} = \mathbf{BR}_{15} = \mathbf{R}_{15}$

This table gives several links between the regularities $\mathbf{R}_i, 1 \leq i \leq 15$. In particular it permits us to understand better the class $\mathrm{BF}(X)$ of B-Fredholm operators. First it shows that $BF(X) = \mathbf{R}_5 \cap \mathbf{R}_{10}$ and secondly it establishes that $\mathrm{BF}(X)$ is a regularity.

In the case of Hilbert spaces we extend the characterization obtained for semi-B-Fredholm operators and prove that if H is a Hilbert space, $T \in L(H)$ and $1 \leq i \leq 13$, then $T \in \mathbf{BR}_i$ if and only if $T = Q \oplus F$ where Q is a nilpotent operator and $F \in \mathbf{R}_i$.

We also consider the set $\mathbf{R}_{16}$ of operators with topological uniform descent defined by S. Grabiner [4], and prove that $\mathbf{R}_{16}$ is a regularity containing the regularity $\mathbf{R}_{13}$ of quasi-Fredholm operators. Using an example due to M. Mbekhta and V. Müller we show that $\mathbf{R}_{13}$ is a proper subset of $\mathbf{R}_{16}$, and we prove that $\mathbf{BR}_{16} = \mathbf{R}_{16}$. We mention that it has already been proved by P. W. Poon [13, Theorem 5.2.14] that $\mathbf{R}_{16}$ is a regularity, but the argument was different. Using a theorem due to S. Grabiner [4, Theorem 4.7] we formulate a general punctured neighborhood theorem for operators in B-regularities.

Henceforth, if E and F are two vector spaces, the notation $E \simeq F$ means that E and F are isomorphic. If E, F are vector subspaces of the same vector space X we write $E \subset_{\mathrm{e}} F$ if there exists a finite-dimensional vector subspace G of X such that $E \subset F + G$. We write $E =_{\mathrm{e}} F$ if $E \subset_{\mathrm{e}} F$ and $F \subset_{\mathrm{e}} E$. We also define the infimum of the empty set to be ∞.

2. Preliminaries

DEFINITION 2.1. Let $T \in L(X)$, $n \in \mathbb{N}$ and let

$$c_n(T) = \dim R(T^n)/R(T^{n+1}).$$

The *descent* of T is defined by $\delta(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$, and the *essential descent* by $\delta_{\mathrm{e}}(T) = \inf\{n : c_n(T) < \infty\} = \inf\{n : R(T^n) =_{\mathrm{e}} R(T^{n+1})\}$.

DEFINITION 2.2. Let $T \in L(X)$, $n \in \mathbb{N}$ and let

$$c'_n(T) = \dim N(T^{n+1})/N(T^n).$$

The *ascent* of T is defined by $a(T) = \inf\{n : c'_n(T) = 0\} = \inf\{n : N(T^n) = N(T^{n+1})\}$, and the *essential ascent* by $a_{\mathrm{e}}(T) = \inf\{n : c'_n(T) < \infty\} = \inf\{n : N(T^{n+1}) =_{\mathrm{e}} N(T^n)\}$.

PROPOSITION 2.3 [12]. *Let $T \in L(X)$. The sequence $(k_n(T))$ defined by*

$$k_n(T) = \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))$$

satisfies the following relations:

(i) *If $c_n(T) < \infty$ then $k_n(T) = c_n(T) - c_{n+1}(T)$.*
(ii) *If $c'_n(T) < \infty$ then $k_n(T) = c'_n(T) - c'_{n+1}(T)$.*
(iii) *If $A, B, C, D \in L(X)$ are mutually commuting operators satisfying $AC + BD = I$ then $\max(k_n(A), k_n(B)) \le k_n(AB) \le k_n(A) + k_n(B)$.*

We now give the definitions of the regularities $\mathbf{R}_i$, $1 \le i \le 15$.

DEFINITION 2.4 [12].

$\mathbf{R}_1 = \{T \in L(X) : T \text{ is onto}\}$,

$\mathbf{R}_2 = \{T \in L(X) : T \in \Phi_-(X) \text{ and } \delta(T) < \infty\}$,

$\mathbf{R}_3 = \{T \in L(X) : \delta(T) < \infty \text{ and } R(T^{\delta(T)}) \text{ is closed}\}$,

$\mathbf{R}_4 = \Phi_-(X)$,

$\mathbf{R}_5 = \{T \in L(X) : \delta_e(T) < \infty \text{ and } R(T^{\delta_e(T)}) \text{ is closed}\}$,

$\mathbf{R}_6 = \{T \in L(X) : T \text{ is bounded below}\}$,

$\mathbf{R}_7 = \{T \in L(X) : T \in \Phi_+(X) \text{ and } a(T) < \infty\}$,

$\mathbf{R}_8 = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}$,

$\mathbf{R}_9 = \Phi_+(X)$,

$\mathbf{R}_{10} = \{T \in L(X) : a_e(T) < \infty \text{ and } R(T^{a_e(T)+1}) \text{ is closed}\}$,

$\mathbf{R}_{11} = \{T \in L(X) : N(T) \subset R^\infty(T) \text{ and } R(T) \text{ is closed}\}$,

$\mathbf{R}_{12} = \{T \in L(X) : N(T) \subset_e R^\infty(T) \text{ and } R(T) \text{ is closed}\}$,

$\mathbf{R}_{13} = \{T \in L(X) : \exists p \in \mathbb{N} : R(T) + N(T^p) = R(T) + N^\infty(T) \text{ and } R(T^{p+1}) \text{ is closed}\}$,

$\mathbf{R}_{14} = \{T \in L(X) : k_n(T) < \infty \text{ for every } n \in \mathbb{N} \text{ and } R(T) \text{ is closed}\}$,

$\mathbf{R}_{15} = \{T \in L(X) : \exists p \in \mathbb{N} : k_n(T) < \infty\ (n \ge p) \text{ and } R(T^{p+1}) \text{ is closed}\}$.

We have $\mathbf{R}_1 \subset \mathbf{R}_2 \subset \mathbf{R}_3 \subset \mathbf{R}_3 \cup \mathbf{R}_4 \subset \mathbf{R}_5 \subset \mathbf{R}_{13}$, $\mathbf{R}_6 \subset \mathbf{R}_7 \subset \mathbf{R}_8 \subset \mathbf{R}_8 \cup \mathbf{R}_9 \subset \mathbf{R}_{10} \subset \mathbf{R}_{13}$, $\mathbf{R}_{11} \subset \mathbf{R}_{12} \subset \mathbf{R}_{13} \subset \mathbf{R}_{13} \cup \mathbf{R}_{14} \subset \mathbf{R}_{15}$. The operators of $\mathbf{R}_{11}$ and $\mathbf{R}_{12}$ are called *semi-regular* and *essentially semi-regular operators*. The operators of $\mathbf{R}_{13}$ are called *quasi-Fredholm operators*. The inclusion $\mathbf{R}_5 \cup \mathbf{R}_{10} \subset \mathbf{R}_{13}$ will be seen in Proposition 3.4.

REMARK. It is easily seen that the definition of quasi-Fredholm operators given here is equivalent to the definition given in the case of Hilbert spaces by Labrousse in [11, Définition 3.1.2].

DEFINITION 2.5 [11]. Let $T \in L(X)$ and set $\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N} : m \ge n \Rightarrow R(T^n) \cap N(T) \subset R(T^m) \cap N(T)\}$. Then the *degree of stable iteration* of T is defined as $\operatorname{dis}(T) = \inf \Delta(T)$.

PROPOSITION 2.6. *Let $T \in L(X)$. Then $T \in \mathbf{R}_{13}$ if and only if* $\operatorname{dis}(T) = d$ *and*:

(a) *$R(T^n)$ is a closed subspace of X for each integer $n \ge d$.*
(b) *$R(T) + N(T^d)$ is a closed subspace of X.*

Proof. Let $T \in \mathbf{R}_{13}$. Then there exists $n \in \mathbb{N}$ such that for all $m \ge n$, $R(T) + N(T^m) = R(T) + N(T^n)$ and $R(T^{n+1})$ is closed. From [7, Lemma 3.5] we have

$$\frac{N(T) \cap R(T^n)}{N(T) \cap R(T^{n+1})} \simeq \frac{N(T^{n+1}) + R(T)}{N(T^n) + R(T)}.$$

Hence $\operatorname{dis}(T) = d \in \mathbb{N}$ and for every $n \ge d$, $k_n(T) = 0$. Using [12, Lemma 12] we see that $R(T^m)$ is closed for all $m \ge d$. Moreover $R(T) + N(T^d) = T^{-d}(R(T^{d+1}))$ is closed.

Conversely, if $\operatorname{dis}(T) = d$ then for all $m \ge n$, $R(T) + N(T^d) = R(T) + N(T^m) = R(T) + N^\infty(T)$. So $T \in \mathbf{R}_{13}$.

From now on, $\mathrm{QF}(d)$ will denote the set of quasi-Fredholm operators T with $\operatorname{dis}(T) = d$.

REMARK. The regularities $\mathbf{R}_i, 1 \le i \le 15$, and also the regularity $\mathbf{R}_{16}$ (see Section 4) are defined independently of the Banach space X considered. So when we say that an operator T belongs to one of those regularities we mean the regularity on the Banach space where T is defined. This leads to the following definition.

DEFINITION 2.7. Let $\mathbf{R}$ be one of the regularities $\mathbf{R}_i, 1 \le i \le 16$, defined on the Banach space X. We define the *associated B-regularity* $\mathbf{BR}$ as the set $\mathbf{BR} = \{T \in L(X) : \exists n \in \mathbb{N} : R(T^n) \text{ is closed and } T_n \in \mathbf{R}\}$.

3. Properties of B-regularities.

The following important technical lemma is easily checked:

LEMMA 3.1. *Let $T \in L(X)$ and $p, n \in \mathbb{N}$. Then*:

(i) *$c_p(T_n) = c_{n+p}(T)$. In particular $c_0(T_n) = \alpha(T_n) = c_n(T)$.*
(ii) *$c'_p(T_n) = c'_{n+p}(T)$. In particular $c'_0(T_n) = \alpha(T_n) = c'_n(T)$.*
(iii) *$k_p(T_n) = k_{n+p}(T)$.*

PROPOSITION 3.2. *Let $T \in L(X)$. Then T is a quasi-Fredholm operator if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is semi-regular.*

Proof. Suppose that T is a quasi-Fredholm operator and let $d = \operatorname{dis}(T)$. We know that $R(T^d)$ is closed. Consider the operator $T_d : R(T^d) \to R(T^d)$. Then $R(T_d) = R(T^{d+1})$ is closed and $N(T_d) = N(T) \cap R(T^d) = N(T) \cap R(T^m) \subset R(T_d^m)$ for all $m \ge d$. So T_d is a semi-regular operator.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is semi-regular. As $\mathbf{R}_{11}$ is a regularity, for each integer $m \ge 1$, T_n^m is semi-regular. Then $R(T^p)$ is closed for each $p \ge n$. Since T_n is semi-regular, we have $N(T_n) = N(T) \cap R(T^n) = N(T) \cap R(T^m)$ for all $m \ge n$. Hence $d = \operatorname{dis}(T) \in \mathbb{N}$ and $R(T) + N(T^m) = R(T) + N(T^d)$ for all $m \ge d$. Moreover for $m \ge d$, $k_m(T) = 0$. Since $R(T^m)$ is closed for each $m \ge n$, using [12, Lemma 12] we see that $R(T^m)$ is closed for each $m \ge d$. So T is a quasi-Fredholm operator.

From this proposition we immediately get the following corollary:

COROLLARY 3.3. *Let $T \in L(X)$ and $d \in \mathbb{N}$. Then $T \in \mathrm{QF}(d)$ if and only if $\operatorname{dis}(T) = d$ and $R(T^{d+1})$ is closed.*

Proof. If $T \in \mathrm{QF}(d)$, then $\operatorname{dis}(T) = d$ and $R(T^{d+1})$ is closed. Conversely, if $\operatorname{dis}(T) = d$ and $R(T^{d+1})$ is closed then $k_n(T) = 0$ for $n \ge d$. Using [12, Lemma 11] we see that $R(T^d)$ is closed. Since $k_0(T_d) = k_d(T)$ we have $k_0(T_d) = 0$. As $R(T^{d+1})$ is closed, T_d is semi-regular and so $T \in \mathrm{QF}(d)$.

REMARK. Let $T \in L(X)$. If for some $n \in \mathbb{N}$, $c_n(T) < \infty$ (resp. $c_n'(T) < \infty$) and $R(T^{n+1})$ is closed, then $k_p(T) < \infty$ for $p \ge n$. So from [12, Lemma 12], $R(T^p)$ is closed for $p \ge n$. Since the sequence $(c_p(T))_p$ (resp. $(c_p'(T))_p$) is decreasing, it is constant for p large enough. So T is a quasi-Fredholm operator.

Using the numbers $c_n(T)$, $c_n'(T)$ and $k_n(T)$ we formulate in another way the definitions of the regularities $\mathbf{R}_i$, $1 \le i \le 13$.

PROPOSITION 3.4. *For $T \in L(X)$, let $d = \operatorname{dis}(T)$. Then:*

$$\mathbf{R}_1 = \{T \in \mathbf{R}_{13} : c_0(T) = 0\},$$

$$\mathbf{R}_2 = \{T \in \mathbf{R}_{13} : c_0(T) < \infty \text{ and } c_d(T) = 0\},$$

$$\mathbf{R}_3 = \{T \in \mathbf{R}_{13} : c_d(T) = 0\},$$

$$\mathbf{R}_4 = \{T \in \mathbf{R}_{13} : c_0(T) < \infty\},$$

$$\mathbf{R}_5 = \{T \in \mathbf{R}_{13} : c_d(T) < \infty\},$$

$$\mathbf{R}_6 = \{T \in \mathbf{R}_{13} : c_0'(T) = 0\},$$

$$\mathbf{R}_7 = \{T \in \mathbf{R}_{13} : c_0'(T) < \infty \text{ and } c_d'(T) = 0\},$$

$$\mathbf{R}_8 = \{T \in \mathbf{R}_{13} : c_d'(T) = 0\},$$

$$\mathbf{R}_9 = \{T \in \mathbf{R}_{13} : c_0'(T) < \infty\},$$

$$\mathbf{R}_{10} = \{T \in \mathbf{R}_{13} : c_d'(T) < \infty\},$$

$$\mathbf{R}_{11} = \Big\{T \in \mathbf{R}_{13} : \sum_{i=0}^{\infty} k_i(T) = 0\Big\},$$

$$\mathbf{R}_{12} = \Big\{T \in \mathbf{R}_{13} : \sum_{i=0}^{\infty} k_i(T) < \infty\Big\},$$

$$\mathbf{R}_{13} = \{T \in L(X) : d \in \mathbb{N} \text{ and } R(T^{d+1}) \text{ is closed}\}.$$

Proof. The formula for $\mathbf{R}_{13}$ already appears in Corollary 3.3. Let $T \in L(X)$ and let $p \in \mathbb{N}$. If $k_n(T) < \infty$ $(n \ge p)$ and if $R(T^m)$ is closed for some $m > p$ then by [12, Lemma 12], $R(T^n)$ is closed for all $n \ge p$. Hence the new definitions of the regularities $\mathbf{R}_{11}$ and $\mathbf{R}_{12}$ agree with the previous ones.

Now if $T \in \mathbf{R}_{13}$ and if there exists $p \in \mathbb{N}$ such that $c_p(T) < \infty$ or $c_p'(T) < \infty$ then $k_n(T) < \infty$ $(n \ge p)$. Since $R(T^m)$ is closed for m large enough, $R(T^n)$ is closed for $n \ge p$. It follows that the new definitions of the regularities $\mathbf{R}_i$, $1 \le i \le 4$, or $6 \le i \le 9$ agree with those given before.

By the remark following Corollary 3.3 we easily obtain the formulas for $\mathbf{R}_{10}$ and $\mathbf{R}_5$.

From this proposition and using Lemma 3.1 we immediately get the following corollary.

COROLLARY 3.5. *Let $T \in L(X)$ and $\operatorname{dis}(T) = d$. Then:*

(i) $T \in \mathbf{R}_3$ *if and only if* $T \in \mathbf{R}_{13}$ *and* $T_d \in \mathbf{R}_1$.
(ii) $T \in \mathbf{R}_5$ *if and only if* $T \in \mathbf{R}_{13}$ *and* $T_d \in \mathbf{R}_4$.
(iii) $T \in \mathbf{R}_8$ *if and only if* $T \in \mathbf{R}_{13}$ *and* $T_d \in \mathbf{R}_6$.
(iv) $T \in \mathbf{R}_{10}$ *if and only if* $T \in \mathbf{R}_{13}$ *and* $T_d \in \mathbf{R}_9$.

In the following theorem we give a complete description of the B-regularities $\mathbf{BR}_i$, $1 \le i \le 15$.

THEOREM 3.6. *We have the following relations between the regularities $\mathbf{R}_i$, $1 \le i \le 15$, and the corresponding B-regularities:*

$\mathbf{BR}_1 = \mathbf{BR}_2 = \mathbf{BR}_3 = \mathbf{R}_3$	$\mathbf{BR}_4 = \mathbf{BR}_5 = \mathbf{R}_5$
$\mathbf{BR}_6 = \mathbf{BR}_7 = \mathbf{BR}_8 = \mathbf{R}_8$	$\mathbf{BR}_9 = \mathbf{BR}_{10} = \mathbf{R}_{10}$
$\mathbf{BR}_{11} = \mathbf{BR}_{12} = \mathbf{BR}_{13} = \mathbf{R}_{13}$	$\mathbf{BR}_{14} = \mathbf{BR}_{15} = \mathbf{R}_{15}$

Proof. For $T \in L(X)$, let $d = \operatorname{dis}(T)$. Since $\mathbf{R}_1 \subset \mathbf{R}_2 \subset \mathbf{R}_3$, we have $\mathbf{BR}_1 \subset \mathbf{BR}_2 \subset \mathbf{BR}_3$. Moreover from Corollary 3.5 if $T \in \mathbf{R}_3$, then $T_d \in \mathbf{R}_1$. Hence $T \in \mathbf{BR}_1$ and $\mathbf{R}_3 \subset \mathbf{BR}_1$.

Let now $T \in \mathbf{BR}_3$. Then there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n \in \mathbf{R}_3$. Hence $T_n \in \mathbf{R}_{13}$ and there exists $p \in \mathbb{N}$ such that $c_p(T_n) = 0$. By Lemma 3.1 it follows that $c_{p+n}(T) = 0$. Thus $d = \operatorname{dis}(T) \in \mathbb{N}$, $c_d(T) = 0$ and $R(T^{d+1})$ is closed. So $T \in \mathbf{R}_3$. Consequently, $\mathbf{BR}_1 = \mathbf{BR}_2 = \mathbf{BR}_3 = \mathbf{R}_3$.

Using Lemma 3.1 and following the same method, we easily obtain the other relations stated in this theorem.

COROLLARY 3.7. *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if $T \in \mathbf{BR}_5 \cap \mathbf{BR}_{10}$.*

Proof. From [1, Proposition 2.6] if $T \in L(X)$ then T is a B-Fredholm operator if and only $T \in \mathbf{BR}_{13}$, $c_d(T) < \infty$ and $c'_d(T) < \infty$. Hence T is a B-Fredholm operator if and only if $T \in \mathbf{BR}_5 \cap \mathbf{BR}_{10}$.

Let $T \in \mathbf{BR}_i$, let $\varrho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in \mathbf{BR}_i\}$ be the *$\mathbf{BR}_i$-resolvent* of T and let $\sigma_{\mathbf{BR}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{BR}_i\}$ be the *$\mathbf{BR}_i$-spectrum* of T.

COROLLARY 3.8. *Let $T \in L(X)$ and $1 \le i \le 13$. Then $\sigma_{\mathbf{BR}_i}(T)$ is a closed subset of the usual spectrum $\sigma(T)$.*

Proof. If $\lambda \notin \sigma(T)$, then $T - \lambda I$ is invertible and $\lambda \notin \sigma_{\mathbf{BR}_i}(T)$. So $\sigma_{\mathbf{BR}_i}(T) \subset \sigma(T)$. Using the properties of the $\mathbf{R}_i$-spectrum, $1 \le i \le 13$, established in [12], we see that $\sigma_{\mathbf{BR}_i}(T)$ is closed in $\sigma(T)$ for $1 \le i \le 13$.

COROLLARY 3.9. *Let $T \in L(X)$ and let f be an analytic function in a neighborhood of $\sigma(T)$ which is non-constant on any connected component of $\sigma(T)$. Then $f(\sigma_{\mathbf{BR}_i}(T)) = \sigma_{\mathbf{BR}_i}(f(T))$.*

Proof. By the preceding theorem, the B-regularities $\mathbf{BR}_i$, $1 \le i \le 15$, are regularities. Hence the corollary is a direct consequence of [10, Theorem 1.4].

COROLLARY 3.10. *Let $T \in \mathbf{BR}_i$, let $i \in \{4, 5, 9, 10, 11, 12, 13, 14, 15\}$, and let $F \in L(X)$ be a finite-dimensional operator. Then $T + F \in \mathbf{BR}_i$.*

Proof. This is a direct consequence of the properties of the regularities $\mathbf{R}_i$, $1 \le i \le 15$, established in [12] and [9].

PROPOSITION 3.11. *Let $T \in L(X)$ and let $i \in \{1, 2, 4, 6, 7, 9, 11, 12, 14\}$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n \in \mathbf{R}_i$ then $R(T^m)$ is closed and $T_m \in \mathbf{R}_i$ for each $m \ge n$. Moreover if T_n is a Fredholm operator then T_m is a Fredholm operator and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for each $m \ge n$.*

Proof. From the assumption it follows that for each $p \ge n$ the operator $T_n^{p-n} : R(T^n) \to R(T^n)$ is in $\mathbf{R}_i$. Hence $R(T_n^{p-n}) = R(T^p)$ is closed in

$R(T^n)$. Since $R(T^n)$ is closed in X we infer that for each $p \ge n$, $R(T^p)$ is closed in X. Let now $m > n$. Then

$$(*)\qquad \begin{gathered} c_p(T_m) = c_{p+m-n}(T_n), \quad c'_p(T_m) = c'_{p+m-n}(T_n), \\ k_p(T_m) = k_{p+m-n}(T_n). \end{gathered}$$

If $T_n \in \mathbf{R}_{14}$ then $k_p(T_m) = k_{p+m-n}(T_n) < \infty$ for all p. Since $R(T_m) = R(T^{m+1})$ is closed we get $T_m \in \mathbf{R}_{14}$.

If $i \in \{1, 2, 4, 6, 7, 9, 11, 12\}$ then $T_n \in \mathbf{R}_{13}$. Let $d_n = \operatorname{dis}(T_n)$. Then $k_p(T_n) = 0$ for $p \ge d_n$ and $k_p(T_m) = k_{p+m-n}(T_n) = 0$ if $p + m - n \ge d_n$. Thus $\operatorname{dis}(T_m) = 0$ if $m - n \ge d_n$ and $\operatorname{dis}(T_m) = d_n - (m - n)$ if $m - n < d_n$. Since $R(T_m)$ is closed we obtain $T_m \in \mathbf{R}_{13}$. Using $(*)$ and Proposition 3.4 we see that $T_m \in \mathbf{R}_i$.

Suppose now that T_n is an upper semi-Fredholm operator. Then $\alpha(T_n) < \infty$. As

$$N(T_m) = N(T) \cap R(T^m) \subset N(T) \cap R(T^n) = N(T_n),$$

we have $\alpha(T_m) < \infty$. Hence T_m is an upper semi-Fredholm operator. In the same way if T_n is a lower semi-Fredholm operator then $\beta(T_n) < \infty$. As $R(T_n) = R(T^{n+1})$, there exists a finite-dimensional subspace F of $R(T^n)$ such that $R(T^n) = F + R(T^{n+1})$. Then $R(T^m) = T^{m-n}(F) + R(T^{m+1})$ and $R(T_m) = R(T^{m+1})$ is of finite codimension in $R(T^m)$. Consequently, T_m is a lower semi-Fredholm operator. Moreover if T_n is a Fredholm operator, then T_m is a Fredholm operator. From [7, Lemma 3.5] we have

$$\frac{N(T) \cap R(T^n)}{N(T) \cap R(T^{n+1})} \simeq \frac{N(T^{n+1}) + R(T)}{N(T^n) + R(T)}$$

and from [7, Lemma 3.2] we get

$$\frac{R(T^n)}{R(T^{n+1})} \simeq \frac{X}{R(T) + N(T^n)} \quad \text{and} \quad \frac{R(T^{n+1})}{R(T^{n+2})} \simeq \frac{X}{R(T) + N(T^{n+1})}.$$

Hence $\alpha(T_n) - \alpha(T_{n+1}) = \beta(T_n) - \beta(T_{n+1})$ and so $\operatorname{ind}(T_{n+1}) = \operatorname{ind}(T_n)$. It follows that $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for each $m \ge n$.

THEOREM 3.12. *Let H be a Hilbert space, $T \in L(H)$, and $1 \le i \le 13$. Then $T \in \mathbf{BR}_i$ if and only if there exist two closed subspaces M and N of H such that $H = M \oplus N$ and:*

(i) *$T(N) \subset N$ and $T_{|N}$ is a nilpotent operator,*
(ii) *$T(M) \subset M$ and $T_{|M} \in \mathbf{R}_i$.*

Proof. Suppose that $T \in \mathbf{BR}_i$. Then T is a quasi-Fredholm operator. Hence from [9, Théorème 3.2.1] there exist two closed subspaces M, N of H and an integer $d \in \mathbb{N}$ such that $H = M \oplus N$ and:

(i) $T(N) \subset N$ and $T_{|N}$ is a nilpotent operator of degree d,
(ii′) $T(M) \subset M$ and $T_{|M} \in \mathbf{R}_{11}$.

Moreover $R(T^d_{|M}) = R(T^d)$, $(T_{|M})_d = T_d$ and $\mathrm{dis}(T_{|M}) = 0$. We also have

$$c_0(T_{|M}) = c_d(T_{|M}) = c_0((T_{|M})_d) = c_d(T),$$
$$c'_0(T_{|M}) = c'_d(T_{|M}) = c'_0((T_{|M})_d) = c'_d(T), \quad k_p(T_{|M}) = 0, \quad p \geq d.$$

Since $R(T_{|M})$ is closed, using Proposition 3.4 we see that $T_{|M} \in \mathbf{R}_i$.

Conversely, suppose that there exist two closed subspaces M and N of H as in the statement of the theorem.

Let $r = \inf\{n \in \mathbb{N} : (T_{|N})^n = 0\}$, $s = \mathrm{dis}(T_{|M})$ and $d = \max(r, s)$. Then $R(T^d) = R((T_{|M})^d)$ is closed and $T_d = (T_{|M})_d$. Moreover we have

$$c_0(T_d) = c_d(T) = c_d(T_{|M}) = c_s(T_{|M}),$$
$$c'_0(T_d) = c'_d(T) = c'_d(T_{|M}) = c'_s(T_{|M}), \quad k_p(T_d) = k_{p+d}(T) = k_{p+d}(T_{|M}).$$

Using Proposition 3.4, we see that $T_d \in \mathbf{R}_i$ and $T \in \mathbf{BR}_i$.

Setting $T_{|N} = Q$ and $T_{|M} = F$, we have the following corollary:

COROLLARY 3.13. *Let H be a Hilbert space, $T \in L(H)$, and $1 \leq i \leq 13$. Then $T \in \mathbf{BR}_i$ if and only if $T = Q \oplus F$ where Q is a nilpotent operator and $F \in \mathbf{R}_i$.*

4. Regularities of operators with topological uniform descent. In this part we consider the set $\mathbf{R}_{16}$ of operators with topological uniform descent defined by Grabiner [4], and we prove that $\mathbf{R}_{16}$ is a regularity containing the regularity $\mathbf{R}_{13}$ of quasi-Fredholm operators as a proper subset. It has already been proved by P. W. Poon [13, Theorem 5.2.14] that $\mathbf{R}_{16}$ is a regularity, but our method of proof is rather direct and different. We also reformulate a theorem of Grabiner in terms of the numbers c_n, c'_n and k_n, which gives a general punctured neighborhood theorem useful for B-regularities. This theorem extends naturally the classical punctured neighborhood theorem for semi-Fredholm operators stated in [8, Theorems 3 and 5]. As proved in [2] it also extends some of its recent generalizations obtained by Schmoeger [16], Harte [5], Harte and Lee [6].

DEFINITION 4.1. Let $T \in L(X)$ and let $d \in \mathbb{N}$. Then T has a *uniform descent* for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geq d$, in other words if $k_n(T) = 0$ $(n \geq d)$. If in addition $R(T) + N(T^d)$ is closed then T is said to have a *topological uniform descent* for $n \geq d$.

From this definition we see easily that if T is a quasi-Fredholm operator of degree d then T is an operator of topological uniform descent for $n \geq d$. But the converse is not always true as shown by the following example.

EXAMPLE. Let H be a Hilbert space with an orthonormal basis $\{e_{ij}\}_{i,j=1}^{\infty}$ and let the operator T be defined by

$$Te_{i,j} = \begin{cases} 0 & \text{if } j = 1, \\ i^{-1}e_{i,1} & \text{if } j = 2, \\ e_{i,j-1} & \text{otherwise.} \end{cases}$$

In [12, Example 5] it is proved that $R(T) = R(T^2)$ and $R(T)$ is not closed. Hence $R(T^n)$ is not closed for all $n \geq d$ and so T is not a quasi-Fredholm operator. Since $R(T) = R(T^2)$, T is an operator of uniform descent for $n \geq 1$ and $N(T) + R(T) = X$. Hence $N(T) + R(T)$ is closed. Using [4, Theorem 3.2] we see that T is an operator of topological uniform descent for $n \geq 1$.

Let $T \in L(X)$. Using the isomorphism $X/N(T) \simeq R(T)$ and following [4], we define a topology on $R(T)$ as follows:

DEFINITION 4.2. Let $T \in L(X)$. The *operator range topology* on $R(T)$ is defined by the norm $\|\cdot\|_T$ such that for all $y \in R(T)$,

$$\|y\|_T = \inf\{\|x\| : x \in X, \ y = Tx\}.$$

Let $\mathbf{R}_{16} = \{T \in L(X) : \exists d \in \mathbb{N} : T$ is of topological uniform descent for $n \geq d\}$.

THEOREM 4.3. (i) *Let $A \in L(X)$ and $n \geq 1$. Then $A \in \mathbf{R}_{16}$ if and only if $A^n \in \mathbf{R}_{16}$.*

(ii) *If $A, B, C, D \in L(X)$ are mutually commuting operators satisfying $AC + BD = I$ then $AB \in \mathbf{R}_{16}$ if and only if $A, B \in \mathbf{R}_{16}$. Consequently, $\mathbf{R}_{16}$ is a regularity.*

Proof. (i) By [12, Lemma 9], $k_n(A^m) = 0$ for $n \geq p$ if and only if $k_n(A) = 0$ for $n \geq mp$. Hence A^m is of uniform descent for $n \geq p$ if and only if A is of uniform descent for $n \geq mp$. Moreover if A^m is of topological uniform descent for $n \geq p$, then $R(A^{m(p+1)})$ is closed in the operator range topology on $R(A^{mp})$. Hence A is of topological uniform descent for $n \geq mp$. Conversely, if A is of topological uniform descent for $n \geq p$ then it is also of topological uniform descent for $n \geq mp$. Hence $R(A^{m(p+1)})$ is closed in the operator range topology of $R(A^{mp})$. So A^m is of topological uniform descent for $n \geq p$. Thus $A \in \mathbf{R}_{16}$ if and only if $A^n \in \mathbf{R}_{16}$.

(ii) By [12, Lemma 1], $R((AB)^n) = R(A^n) \cap R(B^n)$. Suppose that $R((AB)^{n+1})$ is closed in the operator range topology of $R((AB)^n)$. Let $y \in R(A^n)$ be such that $y \in \overline{R(A^{n+1})}$. So there is a sequence $(y_m) \subset R(A^{n+1})$ such that $\|y_m - y\|_{A^n} \to 0$ as $m \to \infty$. We have $\|y_m - y\|_{A^n} = \inf\{\|x_m - x\| : y_m = A^n x_m, \ y = A^n x\}$, and $\|B^{n+1}y_m - B^{n+1}y\|_{(AB)^n} \leq \|B\| \cdot \|y_m - y\|_{A^n}$. Using our hypothesis we see that $B^{n+1}y \in R(A^{n+1}B^{n+1})$. By [12, Lemma 1] we know that $N(B^{n+1}) \subset R(A^{n+1})$, so $y \in R(A^{n+1})$.

Consequently, $R(A^{n+1})$ is closed in the operator range topology of $R(A^n)$. Similarly, $R(B^{n+1})$ is closed in the operator range topology of $R(B^n)$.

Using the same method we can show that if $R(A^{n+1})$ is closed in the operator range topology of $R(A^n)$, and $R(B^{n+1})$ is closed in the operator range topology of $R(B^n)$, then $R((AB)^{n+1})$ is closed in the operator range topology of $R((AB)^n)$.

Moreover by [12, Lemma 8] we have

$$\max\{k_n(A), k_n(B)\} \le k_n(AB) \le k_n(A) + k_n(B).$$

Hence $AB \in \mathbf{R}_{16}$ if and only if $A, B \in \mathbf{R}_{16}$. Since $\mathbf{R}_{16}$ is a nonempty set, it is a regularity. Moreover for all $i, 1 \le i \le 15$, $\mathbf{R}_i \subset \mathbf{R}_{16}$.

Using the same methods as in the previous part we obtain the following:

PROPOSITION 4.4. $\mathbf{BR}_{16} = \mathbf{R}_{16}$.

We now give Grabiner's punctured neighborhood theorem [4, Theorem 4.7]:

THEOREM 4.5. *Suppose that T is a bounded operator with topological uniform descent for $n \ge d$ on the Banach space X, $n, d \in \mathbb{N}$, and that V is a bounded operator that commutes with T. If $V - T$ is sufficiently small and invertible, then:*

(a) *V has closed range and $k_p(V) = 0$ for each integer $p \ge 0$.*
(b) *$c_p(V) = c_d(T)$ for each integer $p \ge 0$.*
(c) *$c'_p(V) = c'_d(T)$ for each integer $p \ge 0$.*

References

[1] M. Berkani, *On a class of quasi-Fredholm operators*, Integral Equations Oper. Theory 34 (1999), 244–249.

[2] M. Berkani and M. Sarih, *On semi-B-Fredholm operators*, submitted.

[3] S. R. Caradus, *Operator Theory of the Pseudo-Inverse*, Queen's Papers in Pure and Appl. Math. 38 (1974), Queen's Univ., 1974.

[4] S. Grabiner, *Uniform ascent and descent of bounded operators*, J. Math. Soc. Japan 34 (1982), 317–337.

[5] R. Harte, *On Kato non-singularity*, Studia Math. 117 (1996), 107–114.

[6] R. Harte and W. Y. Lee, *A note on the punctured neighbourhood theorem*, Glasgow Math. J. 39 (1997), 269–273.

[7] M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor*, Math. Ann. 172 (1967), 105–115.

[8] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Anal. Math. 6 (1958), 261–322.

[9] J. J. Koliha, M. Mbekhta, V. Müller and P. W. Poon, *Corrigendum and addendum: "On the axiomatic theory of spectrum II"*, Studia Math. 130 (1998), 193–198.

[10] V. Kordula and V. Müller, *On the axiomatic theory of spectrum*, ibid. 119 (1996), 109–128.

[11] J. P. Labrousse, *Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm*, Rend. Circ. Mat. Palermo (2) 29 (1980), 161–258.

[12] M. Mbekhta and M. Müller, *On the axiomatic theory of spectrum II*, Studia Math. 119 (1996), 129–147.

[13] P. W. Poon, *Spectral properties and structure theorems for bounded linear operators*, thesis, Dept. of Math. and Statist., Univ. of Melbourne, 1997.

[14] C. Schmoeger, *On a class of generalized Fredholm operators, I*, Demonstratio Math. 30 (1997), 829–842.

[15] —, *On a class of generalized Fredholm operators, V*, ibid. 32 (1999), 595–604.

[16] —, *On a generalized punctured neighborhood theorem in $\mathcal{L}(X)$*, Proc. Amer. Math. Soc. 123 (1995), 1237–1240.

Faculté des Sciences
Département de Mathématiques
Université Mohammed I
Oujda, Maroc
E-mail: berkani@sciences.univ-oujda.ac.ma

Received April 28, 1999
Revised version October 1, 1999 (4314)