STUDIA MATHEMATICA 140 (2) (2000)

Linear extension operators for restrictions of function spaces to irregular open sets

by

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Abstract. Let an open set $\Omega \subset \mathbb{R}^n$ satisfy for some $0 \le d \le n$ and $\varepsilon > 0$ the condition: the d-Hausdorff content $H_d(\Omega \cap B) \ge \varepsilon |B|^{d/n}$ for any ball B centered in Ω of volume $|B| \le 1$. Let H_p^s denote the Bessel potential space on $\mathbb{R}^n$ ($1 < p < \infty$, $s > 0$), and let $H_p^s[\Omega]$ be the linear space of restrictions of elements of H_p^s to Ω endowed with the quotient space norm. We find sufficient conditions for the existence of a linear extension operator for $H_p^s[\Omega]$, i.e., a bounded linear operator $\operatorname{ext} : H_p^s[\Omega] \to H_p^s$ such that $\operatorname{ext} f|_\Omega = f$ for all f. The main result is that such an operator exists if (i) $d > n-1$ and $s > (n-d)/\min(p,2)$, or (ii) $d \le n-1$ and $s - [s] > (n-d)/\min(p,2)$. It is an open problem whether these assumptions are sharp.

1. Introduction. Let H_p^s ($1 < p < \infty$, $s > 0$) denote the Bessel potential space on $\mathbb{R}^n$, otherwise known as the fractional Sobolev space (it coincides with the usual Sobolev space if $s \in \mathbb{N}$). Let $\Omega \subset \mathbb{R}^n$ be an open set. Denote by $H_p^s[\Omega]$ the linear space whose elements f are the restrictions to Ω of functions $g \in H_p^s$. The norm in $H_p^s[\Omega]$ is given by

$$\|f\|_{H_p^s[\Omega]} = \inf\{\|g\|_{H_p^s} : g|_\Omega = f \text{ a.e.}\}.$$

With this norm $H_p^s[\Omega]$ becomes a Banach space, which is called the *restriction space* of H_p^s to Ω. The *restriction operator* $\operatorname{re}_\Omega : f \mapsto f|_\Omega$ is then a bounded linear operator from H_p^s to $H_p^s[\Omega]$. A bounded linear operator $\operatorname{ext} : H_p^s[\Omega] \to H_p^s$ is called a *linear extension operator* for $H_p^s[\Omega]$ if $\operatorname{re}_\Omega \circ \operatorname{ext} = \operatorname{id}$ in $H_p^s[\Omega]$. In this paper we are looking for sufficient conditions on Ω under which

(1.1) there exists a linear extension operator for $H_p^s[\Omega]$.

To get a better understanding and an additional motivation of the problem, consider the closed subspace $H_{p,\Omega}^s \subset H_p^s$ consisting of all functions

2000 *Mathematics Subject Classification*: Primary 46E35; Secondary 42B25.

Key words and phrases: Sobolev spaces, Besov–Triebel–Lizorkin spaces, restrictions, extension operators, irregular domains, Hausdorff content, local polynomial approximation, complemented subspaces.

$f \in H_p^s$ for which $f|_\Omega = 0$ a.e. Then (1.1) is equivalent to the geometrical fact that $H_{p,\Omega}^s$ is complemented in H_p^s. The proof of the equivalence is immediate if we note that $H_p^s[\Omega]$ is isometrically isomorphic to the quotient space $H_p^s/H_{p,\Omega}^s$. In particular, it follows that (1.1) is always true in the Hilbertian case $p = 2$.

As far as the case $p \neq 2$ is concerned, it follows from results of Seeger [19] that (1.1) is true for all $1 < p < \infty$, $s > 0$ if Ω satisfies the (ε, δ) condition of Jones [11]. The class of such Ω is strictly larger than the class of domains with Lipschitz boundary, for which (1.1) was obtained earlier by Strichartz [22].

Our approach to the problem (which leads to more general results) is based on the method of local polynomial approximation going back to Brudnyĭ (e.g. [3]) and developed further by several other authors [20], [13], [12], [5]. The idea is to find for a given $f \in H_p^s[\Omega]$ and every dyadic cube Q intersecting Ω a polynomial P approximating f "near" Q in a certain sense. Having a family of such polynomials, it is not difficult to construct an extension of f, as we show in Section 3. Thus, the problem essentially reduces to finding the polynomials. We propose to construct them by means of integration of f and its derivatives over some measures μ supported in $Q \cap \Omega$. For this method to work, it should be possible to choose a sufficiently regular μ, which relates directly to how massive $Q \cap \Omega$ is. We formulate the massiveness requirement in terms of Hausdorff contents.

Let $0 \le d \le n$. Recall that the *Hausdorff content* $H_d(X)$ of any set $X \subset \mathbb{R}^n$ is defined as

$$H_d(X) = \inf \sum_j r_j^d,$$

where the infimum is taken over all countable coverings of X by balls $B(x_j, r_j)$ with arbitrary centers x_j and radii r_j.

We say that an open set Ω is *d-thick* if there is an $\varepsilon > 0$ such that for all $x \in \Omega$ and $0 < r \le 1$,

$$H_d(B(x,r) \cap \Omega) \ge \varepsilon r^d. \tag{1.2}$$

Note that every Ω is 0-thick, and that the condition of n-thickness is equivalent to

$$|B(x,r) \cap \Omega| \ge \varepsilon r^n.$$

In particular, every (ε, δ) domain alluded to above, or every domain satisfying the cone condition is n-thick. It is also easily seen that every *connected* Ω is 1-thick.

In Section 4 we develop methods of constructing local polynomial approximations for functions defined in d-thick open sets. We propose two different methods. The first one works for all d and uses all derivatives up to order m of a given function f in order to construct a polynomial of degree m approximating f. The second method uses only values of f itself, but it is applicable only for $d > n-1$. The reason is that the set of zeros of a generic polynomial has Hausdorff dimension $n-1$.

By using the results of Sections 3 and 4, in Section 5 we prove the following theorem, which is essentially the main result of the paper.

THEOREM 1.1. *Let Ω be d-thick. Then* (1.1) *is true if either*

(a) $d > n-1$ *and* $s > (n-d)/\min(p,2)$, *or*

(b) $d \le n-1$ *and* $s - [s] > (n-d)/\min(p,2)$, *where* $[s]$ *is the integer part of* s.

In particular, if Ω is n-thick, then (1.1) is true for all s and p, which gives a generalization of Seeger's result.

As far as the sharpness of the theorem is concerned, one should distinguish the sharpness for each particular Ω and for the whole class of d-thick open sets. In the first sense the theorem is not sharp. E.g., Kalyabin (personal communication, see also [14]) has proved (1.1) for $\Omega \subset \mathbb{R}^2$ being a *cusp* of the form

$$\Omega = \{(x,y) : 0 < x < 1,\ 0 < y < x^\gamma\}, \quad \gamma > 1,$$

if $s - 1/p$ is noninteger. By means of our theorem, we could establish this result only for $s - [s] > 1/\min(p,2)$.

We do not know whether our theorem is sharp on the whole class of d-thick open sets. Moreover, we do not even have a single example of an open set Ω for which (1.1) would be false for some s and p. Finding such examples appears to be an interesting problem.

The paper is written in the framework of the theory of more general Besov spaces B_{pq}^s and Triebel–Lizorkin spaces F_{pq}^s. (Theorem 1.1 follows from Theorem 5.1 by noting that $H_p^s = F_{p,2}^s$.) This is done not only for the sake of generality, but also because the methods developed in the theory of those spaces (e.g., the technique of Peetre's maximal functions, and the atomic characterizations of Frazier–Jawerth and Netrusov) are well suited to our purposes. However, we have tried to make this paper comprehensible even to the reader who is not very familiar with the theory of B_{pq}^s and F_{pq}^s. Because of that, we have collected in Section 2 some definitions and facts that might not be widely known.

Finally, the following warning remark may be in order. If $s \in \mathbb{N}$, then the usual way to define the Sobolev space in an open set Ω is

$$H_p^s(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \sum_{|\alpha| \le s} \|D^\alpha f\|_{L_p(\Omega)} < \infty \Big\}.$$

Note that although $H_p^s[\Omega] = H_p^s(\Omega)$ for "nice" Ω (e.g., satisfying the (ε, δ) condition), in general only the inclusion $H_p^s[\Omega] \subset H_p^s(\Omega)$ is true.

Acknowledgments. The research of this paper was supported by a graduate fellowship of the Deutsche Forschungsgemeinschaft, and by the Russian Foundation for Basic Research grant 96–01–00243. The author is grateful to the referee for many suggestions on the improvement of the exposition.

2. Function spaces on $\mathbb{R}^n$. Let $\mathcal{S}$ and $\mathcal{S}'$ be the Schwartz spaces of (complex-valued) test functions and tempered distributions on $\mathbb{R}^n$. For $\varphi \in \mathcal{S}$ and $j \in \mathbb{Z}$ we will often use the notation $\varphi_j(x) = 2^{jn}\varphi(2^j x)$. We write $\widehat{\varphi}$ for the Fourier transform of φ.

We recall the definition of the Besov and Triebel–Lizorkin spaces (see, e.g., Triebel [23]). Let $\Phi \in \mathcal{S}$ be chosen so that $\operatorname{supp}\widehat{\Phi} \subset B(0,2)$ and $\widehat{\Phi}(\xi) \equiv 1$ on $B(0,1)$, and let $\varphi \in \mathcal{S}$ be given by $\widehat{\varphi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$. Note that in this case

$$\operatorname{supp}\widehat{\varphi} \subset B(0,2) \setminus B(0,1/2), \qquad \widehat{\Phi}(\xi) + \sum_{j=1}^{\infty} \widehat{\varphi}(2^{-j}\xi) \equiv 1 \quad \text{on } \mathbb{R}^n.$$

Let $\ell_q(L_p)$ and $L_p(\ell_q)$ be the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ with finite quasi-norms

$$\|\{g_j\}\|_{\ell_q(L_p)} = \|\{\|g_j\|_{L_p}\}\|_{\ell_q} \equiv \Big(\sum \|g_j\|_{L_p}^q\Big)^{1/q},$$

$$\|\{g_j\}\|_{L_p(\ell_q)} = \|\|\{g_j(\cdot)\}\|_{\ell_q}\|_{L_p} \equiv \Big\|\Big(\sum |g_j(\cdot)|^q\Big)^{1/q}\Big\|_{L_p}.$$

Definition 2.1. (a) Let $s \in \mathbb{R}$, $0 < p, q \le \infty$. Then (*Besov spaces*)

$$B_{pq}^s = \{f \in \mathcal{S}' : \|f\|_{B_{pq}^s} = \|\Phi * f\|_{L_p} + \|\{2^{js}\varphi_j * f\}_{j=1}^{\infty}\|_{\ell_q(L_p)} < \infty\}.$$

(b) Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. Then (*Triebel–Lizorkin spaces*)

$$F_{pq}^s = \{f \in \mathcal{S}' : \|f\|_{F_{pq}^s} = \|\Phi * f\|_{L_p} + \|\{2^{js}\varphi_j * f\}_{j=1}^{\infty}\|_{L_p(\ell_q)} < \infty\}.$$

It is well known that these definitions are independent of the choice of Φ, and that different choices lead to equivalent quasi-norms. We will also need a stronger variant of this assertion involving Peetre's maximal functions. For fixed $\Psi, \psi \in \mathcal{S}$, $\lambda > 0$, and any $f \in \mathcal{S}'$, $x \in \mathbb{R}^n$ these are given by

$$\Psi_\lambda^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Psi * f(y)|}{(1+|x-y|)^\lambda},$$
$$\psi_{j,\lambda}^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1+2^j|x-y|)^\lambda}, \qquad j \in \mathbb{N}. \tag{2.1}$$

Let $m \ge -1$ be the maximal integer such that

$$D^\beta \widehat{\psi}(0) = 0 \quad \text{for all} \quad |\beta| \le m. \tag{2.2}$$

Lemma 2.2. *Let $[s] \le m$, $0 < p, q \le \infty$, $\lambda > n/\min(p,q)$. Then for all $f \in \mathcal{S}'$,*

$$\|\Psi_\lambda^* f\|_{L_p} + \|\{2^{js}\psi_{j,\lambda}^* f\}_{j=1}^{\infty}\|_{\ell_q(L_p)} \le C\|f\|_{B_{pq}^s}, \tag{2.3a}$$

$$\|\Psi_\lambda^* f\|_{L_p} + \|\{2^{js}\psi_{j,\lambda}^* f\}_{j=1}^{\infty}\|_{L_p(\ell_q)} \le C\|f\|_{F_{pq}^s} \quad (p < \infty). \tag{2.3b}$$

Estimate (2.3b) is due to Peetre [16]; see also Bui, Paluszyński, and Taibleson [4], where a more explicit formulation is given. (2.3a) can be considered analogously. Under the additional restriction $n(1/p - 1) < m + 1$ both (2.3a) and (2.3b) are treated in Triebel [24]. A full proof can be found in Rychkov [18].

For later reference we note a simple but important property of Peetre's maximal functions which follows directly from (2.1): for all $x, y \in \mathbb{R}^n$,

$$\psi_{j,\lambda}^* f(x) \le \psi_{j,\lambda}^* f(y)(1 + 2^j|x-y|)^\lambda. \tag{2.4}$$

The next assertion supplies a very useful sufficient condition for the convergence of series in B_{pq}^s and F_{pq}^s. It is contained in well-known results of Frazier and Jawerth [8], [9] and Netrusov [15] on atomic characterizations of these spaces.

For $j \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ and $\mathbf{k} \in \mathbb{Z}^n$, let $Q_{j,\mathbf{k}}$ denote the cube given by

$$Q_{j,\mathbf{k}} = [2^{-j}k_1, 2^{-j}(k_1+1)] \times \ldots \times [2^{-j}k_n, 2^{-j}(k_n+1)]$$

(*dyadic cubes*). If $Q \subset \mathbb{R}^n$ is a cube with edges parallel to the coordinate axes (only such cubes are considered), and $a > 0$, then aQ denotes the cube with the same center and side length a times as large. Finally, let $\|\cdot\|_Q \equiv \|\cdot\|_{L_\infty(Q)}$.

Lemma 2.3. *For a given sequence $\{f^j\}_{j=0}^{\infty}$ of C^∞ functions on $\mathbb{R}^n$ and $S \in \mathbb{N}$ define*

$$d_{j,\mathbf{k}} = \max_{|\alpha| \le S} 2^{-j|\alpha|}\|D^\alpha f^j\|_{3Q_{j,\mathbf{k}}}.$$

Then for all $0 < p, q \le \infty$ and $(n/p - n)_+ < s < S$,

$$\Big\|\sum_{j=0}^{\infty} f^j\Big\|_{B_{pq}^s} \le C\Big\|\Big\{2^{js}\sum_{\mathbf{k} \in \mathbb{Z}^n} d_{j,\mathbf{k}}\chi_{j,\mathbf{k}}\Big\}_{j=0}^{\infty}\Big\|_{\ell_q(L_p)}, \tag{2.5a}$$

and for all $0 < p < \infty$, $0 < q \le \infty$ and $(n/\min(p,q) - n)_+ < s < S$,

$$\Big\|\sum_{j=0}^{\infty} f^j\Big\|_{F_{pq}^s} \le C\Big\|\Big\{2^{js}\sum_{\mathbf{k} \in \mathbb{Z}^n} d_{j,\mathbf{k}}\chi_{j,\mathbf{k}}\Big\}_{j=0}^{\infty}\Big\|_{L_p(\ell_q)} \tag{2.5b}$$

in the sense that if the right-hand side of (2.5a) *or* (2.5b) *is finite, then the series on the left-hand side converges in* S', *and the estimate is true. The constant* C *depends only on* s, p, q, n.

REMARK 2.4. The role of the assumptions

$$(2.6)\qquad s>\Big(\frac{n}{p}-n\Big)_+ \quad \text{for } B^s_{pq},\qquad s>\Big(\frac{n}{\min(p,q)}-n\Big)_+ \quad \text{for } F^s_{pq},$$

is that if s does not satisfy them, then one has to impose moment conditions on the atoms in the relevant atomic characterizations of B^s_{pq} and F^s_{pq} (see [8], [9], [15]). In other words, the convergence of the series $\sum_{j=0}^\infty f^j$ becomes then determined not only by smoothness and size conditions, but also by certain cancellations. Dealing with cancellation phenomena would require methods different from those developed in this paper, and we do not touch that case here.

We finally note that, as is well known, for $s>(n/p-n)_+$ one has $B^s_{pq},F^s_{pq}\subset L_{\max(1,p)}\subset L_1^{\mathrm{loc}}$. This follows from Definition 2.1, for $p\ge 1$ immediately, and for $p<1$ by means of a Nikol'skiĭ-type inequality (Triebel [23], 1.3.2, (1)).

3. A general extension theorem. The purpose of this section is to show how to construct an extension of a function f given in Ω, provided that we know its local polynomial approximations on dyadic cubes intersecting Ω. Thus, let Ω be a subset of $\mathbb{R}^n$ (in this section we do not assume it to be open). Define

$$\Gamma_j=\{\mathbf{k}\in\mathbb{Z}^n: 5Q_{j,\mathbf{k}}\cap\Omega\ne\emptyset\},\qquad j\in\mathbb{N}_0,$$
$$\Gamma=\{(j,\mathbf{k}): j\in\mathbb{N}_0,\ \mathbf{k}\in\Gamma_j\}.$$

We introduce the *local polynomial space* $P_m(\Omega)$, $m\in\mathbb{N}_0$. Its elements are indexed families $\mathcal{P}$ of polynomials,

$$(3.1)\quad \mathcal{P}=\{P_{j,\mathbf{k}}:(j,\mathbf{k})\in\Gamma\},\qquad P_{j,\mathbf{k}}\in\Pi_m\equiv\text{polynomials of degree }\le m.$$

Thus every cube $Q_{j,\mathbf{k}}$ with $\mathbf{k}\in\Gamma_j$ is assigned a polynomial $P_{j,\mathbf{k}}$. In applications (Section 5) these polynomials will approximate a function f given in Ω. In this section the function f does not appear explicitly, but still it is useful to think of $P_{j,\mathbf{k}}$ as approximants in order to understand what happens.

$P_m(\Omega)$ is a linear space with natural operations of scalar multiplication and sum (just add polynomials corresponding to the same cubes $Q_{j,\mathbf{k}}$). We are going to introduce suitable quasi-norms on $P_m(\Omega)$. Set $\varrho(\mathbf{u},\mathbf{k})=\max_{i=1,\dots,n}|u_i-k_i|$. For every $\mathcal{P}\in P_m(\Omega)$ define numbers $a_{j,\mathbf{k}}=a_{j,\mathbf{k}}(\mathcal{P})$, $(j,\mathbf{k})\in\Gamma$, by

$$(3.2)\quad \begin{aligned} a_{0,\mathbf{k}}&=\|P_{0,\mathbf{k}}\|_{Q_{0,\mathbf{k}}},\quad \mathbf{k}\in\Gamma_0,\\ a_{j,\mathbf{k}}&=\|P_{j,\mathbf{k}}-P_{j-1,\mathbf{k}'}\|_{Q_{j,\mathbf{k}}}\\ &\quad+\sum_{\substack{\mathbf{u}\in\Gamma_j\\ \varrho(\mathbf{u},\mathbf{k})=1}}\|P_{j,\mathbf{k}}-P_{j,\mathbf{u}}\|_{Q_{j,\mathbf{k}}},\quad \mathbf{k}\in\Gamma_j,\ j\ge1,\end{aligned}$$

where $\mathbf{k}'\in\mathbb{Z}^n$ is (uniquely) determined by the condition $Q_{j-1,\mathbf{k}'}\supset Q_{j,\mathbf{k}}$ (note that $\mathbf{k}'\in\Gamma_{j-1}$). Further put

$$(3.3)\qquad A_j(\mathcal{P})(x)=\sum_{\mathbf{k}\in\Gamma_j}a_{j,\mathbf{k}}\chi_{j,\mathbf{k}}(x),\qquad j\in\mathbb{N}_0,\ x\in\mathbb{R}^n.$$

Then the quasi-norms we have in mind are given by

$$(3.4)\quad \begin{aligned}\|\mathcal{P}\|_{P_m(\Omega,B^s_{pq})}&=\|\{2^{js}A_j(\mathcal{P})\}_{j=0}^\infty\|_{\ell_q(L_p)},\\ \|\mathcal{P}\|_{P_m(\Omega,F^s_{pq})}&=\|\{2^{js}A_j(\mathcal{P})\}_{j=0}^\infty\|_{L_p(\ell_q)}.\end{aligned}$$

Let $P_m(\Omega,B^s_{pq})$ and $P_m(\Omega,F^s_{pq})$ be the linear subspaces of $P_m(\Omega)$ consisting of all $\mathcal{P}$ for which the introduced quasi-norms are finite. These subspaces are quasi-Banach spaces with those quasi-norms.

We now construct what we call an extension operator for these local polynomial spaces. Let

$$\widetilde{\Gamma}_j=\{\mathbf{k}\in\mathbb{Z}^n: 3Q_{j,\mathbf{k}}\cap\Omega\ne\emptyset\}\subset\Gamma_j,\qquad j\in\mathbb{N}_0.$$

Take an $\eta\in C^\infty(Q_{0,0})$ with integral 1 and define a smooth partition of unity $\eta_{j,\mathbf{k}}=\eta_j*\chi_{j,\mathbf{k}}$, so that

$$\operatorname{supp}\eta_{j,\mathbf{k}}\subset 3Q_{j,\mathbf{k}},\qquad \sum_{\mathbf{k}\in\mathbb{Z}^n}\eta_{j,\mathbf{k}}\equiv 1\quad\text{on }\mathbb{R}^n.$$

Introduce a sequence of cut-off functions

$$\omega^j=\sum_{\mathbf{k}\in\widetilde{\Gamma}_{j+1}}\eta_{j+1,\mathbf{k}},\qquad j\in\mathbb{N}_0.$$

Note that

$$(3.5)\qquad \omega^j\equiv 1\quad\text{on }\Omega,\qquad \operatorname{supp}\omega^j\subset\bigcup_{\mathbf{k}\in\widetilde{\Gamma}_j}Q_{j,\mathbf{k}}$$

(to see the latter property, notice that if $\mathbf{u}\in\widetilde{\Gamma}_{j+1}$ and $3Q_{j+1,\mathbf{u}}\cap Q_{j,\mathbf{k}}\ne\emptyset$, then $3Q_{j+1,\mathbf{u}}\subset 3Q_{j,\mathbf{k}}$ and, consequently, $\mathbf{k}\in\widetilde{\Gamma}_j$).

Let $\mathcal{P}$ be of the form (3.1). For $x \in \mathbb{R}^n$ put

$$
\begin{aligned}
&E^j(\mathcal{P})(x) = \sum_{\mathbf{k}\in\Gamma_j} P_{j,\mathbf{k}}(x)\eta_{j,\mathbf{k}}(x), \quad j \in \mathbb{N}_0,\\
(3.6)\qquad &F^0(\mathcal{P})(x) = E^0(\mathcal{P})(x),\\
&F^j(\mathcal{P})(x) = \omega^j(x)(E^j(\mathcal{P})(x) - E^{j-1}(\mathcal{P})(x)), \quad j \in \mathbb{N}.
\end{aligned}
$$

Further, let

$$
(3.7)\qquad E(\mathcal{P}) = \sum_{j=0}^{\infty} F^j(\mathcal{P}) \in \mathcal{S}',
$$

provided that the series converges in $\mathcal{S}'$. It is clear that E is a linear operator on the set $D(E)$ of all $\mathcal{P}$ for which the convergence in (3.7) occurs.

THEOREM 3.1. (a) *Let $0 < p, q \le \infty$ and $s > (n/p - n)_+$. Then $P_m(\Omega, B^s_{pq}) \subset D(E)$ and*

$$
(3.8)\qquad \|E(\mathcal{P})\|_{B^s_{pq}} \le C(s,p,q,n,m,\eta)\|\mathcal{P}\|_{P_m(\Omega,B^s_{pq})}
$$

for any $\mathcal{P} \in \mathcal{P}_m(\Omega, B^s_{pq})$.

(b) *Let $0 < p < \infty$, $0 < q \le \infty$ and $s > (n/\min(p,q) - n)_+$. Then $P_m(\Omega, F^s_{pq}) \subset D(E)$ and the estimate (3.8) is true with F^s_{pq} instead of B^s_{pq}.*

REMARK 3.2. As we mentioned above, in applications $P_{j,\mathbf{k}}$ will approximate a function f given in Ω. This function will belong to a restriction space $B^s_{pq}[\Omega]$ or $F^s_{pq}[\Omega]$. Assume that we have managed to construct the $P_{j,\mathbf{k}}$ in such a way that the family $\mathcal{P}$ is in the corresponding space $P_m(\Omega, B^s_{pq})$ or $P_m(\Omega, F^s_{pq})$. Then by the theorem $E(\mathcal{P})$ belongs to B^s_{pq} or F^s_{pq}. However, because of (3.5) we have

$$
\sum_{j=0}^{l} F^j(\mathcal{P})|_\Omega = E^l(\mathcal{P})|_\Omega,
$$

and it is easily seen from the definition of $E^l(\mathcal{P})$ that, if the $P_{j,\mathbf{k}}$ approximate f sufficiently well, there is a good chance that $E^l(\mathcal{P})$ will converge to f in, for instance, $L_1(\Omega, \mathrm{loc})$. After having verified this convergence, we may assert that $E(\mathcal{P})$ is an extension of f. This explains the importance of the theorem. While proving the main results of the paper in Section 5, we will essentially follow the plan we have just described.

Proof (of Theorem 3.1). For $\mathcal{P} \in P_m(\Omega)$ and $F^j = F^j(\mathcal{P})$ put

$$
d_{j,\mathbf{k}} = \max_{|\alpha|\le S} 2^{-j|\alpha|}\|D^\alpha F^j\|_{3Q_{j,\mathbf{k}}}, \quad S = [s] + 1.
$$

We claim that under the conditions of the theorem one has

$$
(3.9)\qquad
\begin{aligned}
&\Big\|\Big\{2^{js}\sum_{\mathbf{k}\in\Gamma_j} d_{j,\mathbf{k}}\chi_{j,\mathbf{k}}\Big\}_{j=0}^{\infty}\Big\|_{\ell_q(L_p)} \le C\|\mathcal{P}\|_{P_m(\Omega,B^s_{pq})},\\
&\Big\|\Big\{2^{js}\sum_{\mathbf{k}\in\Gamma_j} d_{j,\mathbf{k}}\chi_{j,\mathbf{k}}\Big\}_{j=0}^{\infty}\Big\|_{L_p(\ell_q)} \le C\|\mathcal{P}\|_{P_m(\Omega,F^s_{pq})}.
\end{aligned}
$$

As soon as we prove this, the theorem will follow from Lemma 2.3.

Recall the n-dimensional *Markov inequality*: for every fixed $m \in \mathbb{N}_0$ there is a constant C such that

$$
(3.10)\qquad \|D^\alpha P\|_Q \le C\ell(Q)^{-|\alpha|}\|P\|_Q, \quad |\alpha| \le m,
$$

for every polynomial $P \in \Pi_m$ and every cube $Q \subset \mathbb{R}^n$ ($\ell(Q)$ is the side length of Q).

Put $a_{j,\mathbf{k}} = 0$ for $\mathbf{k} \notin \Gamma_j$. If $j = 0$, then it follows from (3.10) and Leibniz's rule that

$$
(3.11)\qquad d_{0,\mathbf{k}} \le C \sum_{\varrho(\mathbf{u},\mathbf{k})\le 2} a_{0,\mathbf{u}}.
$$

Let now $j \in \mathbb{N}$. It follows from (3.5) that

$$
(3.12)\qquad \operatorname{supp} F^j \subset \bigcup_{\mathbf{k}\in\widetilde{\Gamma}_j} Q_{j,\mathbf{k}}.
$$

Let $\mathbf{k} \in \widetilde{\Gamma}_j$ and $x \in Q_{j,\mathbf{k}}$. By Leibniz's rule

$$
(3.13)\qquad \max_{|\alpha|\le S} 2^{-j|\alpha|}|D^\alpha F^j(x)| \le C \max_{|\alpha|\le S} 2^{-j|\alpha|}|D^\alpha(E^j(x) - E^{j-1}(x))|
$$

(we have used the obvious estimate $\|D^\alpha\omega^j\|_{L_\infty} \le C2^{j|\alpha|}$ for $|\alpha| \le S$, and set $E^j = E^j(\mathcal{P})$). Further ($\mathbf{k}'$ was defined after (3.2))

$$
\begin{aligned}
(3.14)\qquad &|D^\alpha(E^j(x) - E^{j-1}(x))|\\
&= \Big|D^\alpha\Big(\sum_{\varrho(\mathbf{u},\mathbf{k})\le 1} P_{j,\mathbf{u}}(x)\eta_{j,\mathbf{u}}(x) - \sum_{\varrho(\mathbf{v},\mathbf{k}')\le 1} P_{j-1,\mathbf{v}}(x)\eta_{j-1,\mathbf{v}}(x)\Big)\Big|\\
&\le \Big|D^\alpha \sum_{\varrho(\mathbf{u},\mathbf{k})=1} (P_{j,\mathbf{u}}(x) - P_{j,\mathbf{k}}(x))\eta_{j,\mathbf{u}}(x)\Big|\\
&\quad + |D^\alpha(P_{j,\mathbf{k}}(x) - P_{j-1,\mathbf{k}'}(x))|\\
&\quad + \Big|D^\alpha \sum_{\varrho(\mathbf{v},\mathbf{k}')=1} (P_{j-1,\mathbf{k}'}(x) - P_{j-1,\mathbf{v}}(x))\eta_{j-1,\mathbf{v}}(x)\Big|,
\end{aligned}
$$

where we have used the obvious fact that if $\mathbf{k} \in \widetilde{\Gamma}_j$, then $\mathbf{k}' \in \widetilde{\Gamma}_{j-1}$, and therefore $\varrho(\mathbf{u},\mathbf{k}) = 1$, $\varrho(\mathbf{v},\mathbf{k}') = 1$ imply $\mathbf{u} \in \Gamma_j$, $\mathbf{v} \in \Gamma_{j-1}$.

By applying (3.10) and Leibniz's rule to each term on the right-hand side of (3.14), we easily obtain

$$\max_{|\alpha|\le S} 2^{-j|\alpha|}|D^\alpha(E^j(x)-E^{j-1}(x))| \le C(a_{j,\mathbf{k}}+a_{j-1,\mathbf{k}'}).$$

Hence, in view of (3.12) and (3.13),

$$d_{j,\mathbf{k}} \le C \sum_{\varrho(\mathbf{u},\mathbf{k})\le 1} (a_{j,\mathbf{u}}+a_{j-1,\mathbf{u}'}). \tag{3.15}$$

By a standard argument we now deduce (3.9) from (3.11) and (3.15). ∎

4. Polynomial projections. In this section we continue the realization of the program outlined in the introduction and in Remark 3.2. Assume that Ω is a d-thick open set, and let Q be a cube with side length 1 whose center x_Q is in Ω. We are limiting ourselves to cubes of side length 1 because the d-thickness condition is homogeneous. In Section 5 the construction will be easily transferred by dilation to all scales. We have

$$H_d(Q\cap\Omega)\ge\varepsilon>0. \tag{4.1}$$

It is well known that for any Borel E one has $H_d(E)=\sup H_d(K)$ over all compact $K\subset E$ (see Rogers [17], Chapter 2:7). We take a compact $K\subset Q\cap\Omega$ such that $H_d(K)\ge\varepsilon/2$. By a well-known theorem of Frostman [10] (see also Adams and Hedberg [2], Theorem 5.1.12) there exists a positive Borel measure μ with $\mu(K)=1$ such that

$$\mu(B(x,r))\le\frac{c(n)}{H_d(K)}\,r^d \quad \text{for all } B(x,r). \tag{4.2}$$

The Q and μ will be fixed in the rest of the section. We will construct, by integration with respect to μ, polynomials approximating a given function f defined in $Q\cap\Omega$.

Case $d>n-1$. For any continuous function g on $\mathbb{R}^n$ denote by $Z(g)$ the set of its zeros. We start by proving two lemmas about such sets corresponding to polynomials. The first one says essentially that $Z(P)$ has Hausdorff dimension $\le n-1$ for any nontrivial P.

Lemma 4.1. *Let $P\in\Pi_m$ be a nontrivial polynomial in n variables, and let $Z=Z(P)\cap Q$. Cover Q with a mesh of closed cubes with disjoint interiors and side length δ, $0<\delta\le 1$. Then the number of cubes of this mesh intersecting Z is $\le C(m,n)\delta^{1-n}$.*

Proof. This is an intuitively clear fact, so we are brief. The assertion is trivial for all n if $m=0$, and also for all m if $n=1$. Acting by induction, it suffices to prove the assertion for given m and n under the assumption that it is already proved for degree m in dimension $n-1$, and for degree $m-1$ in dimension n.

Let Q_δ be a cube of the mesh intersecting Z. If ∇P does not vanish in Q_δ, then Z must intersect a face of Q_δ (if Z does not meet a face and $n\ge 2$, then the sign of P must be constant on the boundary of Q, for instance positive; then the point of Q at which P attains its minimum, which is non-positive, is a critical point, and that contradicts the hypothesis). Consider P as a polynomial of $n-1$ variables on the hyperplane containing this face. Since the total number of such hyperplanes is of order δ^{-1}, by the induction hypothesis we conclude that the number of such cubes Q_δ does not exceed $C\delta^{-1}\delta^{1-(n-1)}=C\delta^{1-n}$.

Further, since $D_{x_i}P$ are polynomials of degree $m-1$, by the other induction hypothesis the number of cubes Q_δ on which ∇P vanishes is also not greater than $C\delta^{1-n}$. ∎

The next lemma gives an upper estimate of the Hausdorff content of the set on which a given polynomial is small.

Lemma 4.2. *Let $P\in\Pi_m$ be such that $\|P\|_Q=1$. Then for any δ the set $\{x\in Q:|P(x)|\le\delta^{2^{m-1}}\}$ can be covered by at most $C(m,n)\delta^{1-n}$ balls of radius δ.*

Proof. The assertion follows by induction from Lemma 4.1 and from the inclusion

$$\{x\in Q:|P(x)|\le\delta^2\}\subset U_\delta(Z(P))\cup\{x\in Q:|\nabla P(x)|\le C\delta\},$$

where $C=C(m,n)$ is sufficiently large and U_δ denotes the δ-neighborhood. To see this inclusion, note that $|\nabla^2P|$ is uniformly bounded in Q by a constant by Markov's inequality (3.10). Therefore, if for some $x\in Q$ we have $|P(x)|\le\delta^2$ and $|\nabla P(x)|\ge C\delta$ with C large enough, then by Taylor's formula there is a zero of P in the δ-neighborhood of x. ∎

Let now P be as in Lemma 4.2. It follows from Lemma 4.2 and (4.2) that

$$\mu(\{x\in Q:|P(x)|\le\delta^{2^{m-1}}\})\le C(\varepsilon,m,n)\delta^{d+1-n}\to 0 \quad (\delta\to 0).$$

Hence

$$\int|P|^2\,d\mu\ge\int_{\{|P|>\delta^{2^{m-1}}\}}|P|^2\,d\mu\ge\delta^{2^m}(1-C(\varepsilon,m,n)\delta^{d+1-n}) \ge C'(\varepsilon,m,n,d) \tag{4.3}$$

if we choose δ so that $C(\varepsilon,m,n)\delta^{d+1-n}=1/2$. This inequality implies that in the space Π_m there exists an $L_2(\mu)$-orthonormal basis $\{P_\alpha\}_{|\alpha|\le m}$ such that

$$\|P_\alpha\|_Q\le C(\varepsilon,m,n,d) \quad \text{for all } \alpha. \tag{4.4}$$

To see this, we take any basis of Π_m of sup-norm 1 and apply the Gram–Schmidt orthonormalization procedure. Then (4.3) shows that the "normalization" part of the procedure does not increase the supremum norm by more than a bounded factor, and the "orthogonalization" does not increase it at all, since it involves only a bounded number of terms.

We now define the operator $L : C(Q\cap\Omega)\to\Pi_m$ by the formula

$$Lf(x)=\sum_{|\alpha|\le m}P_\alpha(x)\int_K P_\alpha(y)f(y)\,d\mu(y). \tag{4.5}$$

It is easily seen that L is a projection, i.e. $L^2=L$. Before we study the properties of L in detail, we introduce its twin $\widetilde{L}$ working for $d\le n-1$.

CASE $d\le n-1$. In this case we will use the projection from $C^m(Q\cap\Omega)$ to Π_m given by the formula

$$\widetilde{L}f(x)=\int_K T_y^m f(x)\,d\mu(y)\equiv\sum_{|\alpha|\le m}\frac{1}{\alpha!}\int_K(x-y)^\alpha D^\alpha f(y)\,d\mu(y). \tag{4.6}$$

($T_y^m f$ denotes the mth order Taylor polynomial of f at y.) In general, it would not be possible to construct such a projection by using fewer than m derivations. (An example: all derivatives of the polynomial x_1^m up to order $m-1$ vanish on the $(n-1)$-dimensional hyperplane $\{x_1=0\}$.) In principle, we could use $\widetilde{L}$ for $d>n-1$ as well, but the results obtained would be worse than those with the use of L.

PROPERTIES. We now establish a series of estimates for L and $\widetilde{L}$ which will be extensively used in Section 5.

LEMMA 4.3. *For any $f\in C^{m+1}$ and $a\ge1$,*

$$\|f-Lf\|_{aQ}\le C(\varepsilon,n,m,d,a)\|\nabla^{m+1}f\|_{aQ}\quad(d>n-1); \tag{4.7a}$$

$$\|f-\widetilde{L}f\|_{aQ}\le C(n,m,a)\|\nabla^{m+1}f\|_{aQ}. \tag{4.7b}$$

Proof. By Taylor's formula,

$$|f(x)-T_y^m f(x)|\le C(n,m)\|\nabla^{m+1}f\|_{aQ} \tag{4.8}$$

for all $x,y\in aQ$. From this we get (4.7b) by integrating over $d\mu(y)$. To get (4.7a), fix y and write

$$\begin{aligned}\|f-Lf\|_{aQ}&\le\|f-T_y^m f\|_{aQ}+\|Lf-T_y^m f\|_{aQ}\\&=\|f-T_y^m f\|_{aQ}+\|L(f-T_y^m f)\|_{aQ}\\&\le(1+C(\varepsilon,n,m,d,a))\|f-T_y^m f\|_{aQ},\end{aligned}$$

where we have used (4.4). It remains to invoke (4.8) once more. ∎

LEMMA 4.4. *Let Ψ be a rapidly decreasing function, that is, $|\Psi(x)|\le C_N(1+|x|)^{-N}$ for all N. Then for $1\le r\le\infty$ (with r' being the index conjugate to r),*

$$\|\Psi_l*\mu\|_{L_{r'}}\le C(\varepsilon,n,\Psi)\,2^{l(n-d)/r},\qquad l\in\mathbb{N}. \tag{4.9}$$

Proof. It suffices to prove the estimate when Ψ is the characteristic function of the ball $B(0,1)$. In this case $\Psi_l*\mu(x)=2^{ln}\mu(B(x,2^{-l}))$, whence (4.9) for $r=1$ follows. For $r=\infty$, (4.9) is obvious by Fubini's theorem. The general case follows by trivial interpolation. ∎

REMARK 4.5. An equivalent form of writing (4.9) is

$$\mu\in B^{-(n-d)/r}_{r',\infty}.$$

Since the latter space is dual to $B^{(n-d)/r}_{r,1}$ (see Triebel [23], 2.11.2), it follows that we can extend L by continuity to this space, and $\widetilde{L}$ to the space $B^{m+(n-d)/r}_{r,1}$.

LEMMA 4.6. *Let $0<r<\infty$ and $\varkappa=n/r-d/\max(1,r)$. Then for any $f\in\mathcal{S}$ such that $\operatorname{supp}\widehat{f}\subset\{|\xi|\le2^l\}$, $l\in\mathbb{N}_0$, and any $N>0$, $a\ge1$,*

$$\|Lf\|_{aQ}\le C\,2^{l\varkappa}\Big(\int_{\mathbb{R}^n}\frac{|f(x)|^r}{(1+|x-x_Q|)^N}\,dx\Big)^{1/r}\qquad(d>n-1),$$

$$\|\widetilde{L}f\|_{aQ}\le C\,2^{l\varkappa}\sum_{|\alpha|\le m}\Big(\int_{\mathbb{R}^n}\frac{|D^\alpha f(x)|^r}{(1+|x-x_Q|)^N}\,dx\Big)^{1/r},$$

where C depends only on ε, n, m, d, N, r, a.

Proof. It suffices to show that under the assumptions of the lemma for any continuous function g,

$$\Big|\int_K gf\,d\mu\Big|\le C(\varepsilon,n,N,r)2^{l\varkappa}\Big(\int_{\mathbb{R}^n}\frac{|f(x)|^r}{(1+|x-x_Q|)^N}\,dx\Big)^{1/r}\|g\|_Q. \tag{4.10}$$

Without loss of generality, we may assume that $x_Q=0$. Let $\Phi\in\mathcal{S}$ be a function invariant under reflection in the origin and such that $\operatorname{supp}\widehat{\Phi}\subset\{|\xi|\le2\}$ and $\widehat{\Phi}\equiv1$ for $|\xi|\le1$. Then $f\equiv f*\Phi_l$, whence

$$\begin{aligned}\Big|\int_K gf\,d\mu\Big|&=\big|f*\Phi_l*((g\mu)(-\cdot))(0)\big|\\&=\Big|\int_{\mathbb{R}^n}f(x)(\Phi_l*(g\mu))(x)\,dx\Big|.\end{aligned} \tag{4.11}$$

For $r\ge1$ we estimate the integrand by absolute value to get

$$\le\|g\|_Q\int|f(x)|\cdot|\Phi_l|*\mu(x)\,dx.$$

We further note that since $\operatorname{supp}\mu \subset Q$, the convolution $|\Phi_l| * \mu$ is rapidly decreasing outside $3Q$, and therefore the estimate can be continued as

$$\le \|g\|_Q\Big(C_N \int_{\mathbb{R}^n\setminus 3Q} \frac{|f(x)|}{(1+|x|)^N}\,dx + \int_{3Q} |f(x)|\cdot|\Phi_l| * \mu(x)\,dx\Big).$$

Now by applying Hölder's inequality and Lemma 4.4, we obtain (4.10) (with a different, but still arbitrarily large value of N).

For $r<1$ we first apply a Nikol′skiĭ-type inequality (see Triebel [23], 1.3.2, (1)), which says that whenever $u\in\mathcal{S}$ has Fourier transform supported in $\{|\xi|\le 2^l\}$, one has

$$\|u\|_{L_1} \le C(n,r)2^{ln(1/r-1)}\|u\|_{L_r}, \qquad r<1. \tag{4.12}$$

In our case $u(x) = f(x)(\Phi_l * (g\mu))(x)$, and $\operatorname{supp}\widehat{u} \subset \{|\xi|\le 3\cdot 2^l\}$. Hence, we may apply (4.12) to conclude that the right-hand side of (4.11) is

$$\le C2^{ln(1/r-1)}\Big(\int |f(x)|^r|\Phi_l * (g\mu)(x)|^r\,dx\Big)^{1/r}.$$

Estimating by absolute value as before, we find that this does not exceed

$$C2^{ln(1/r-1)}\|g\|_Q\Big(C_N \int_{\mathbb{R}^n\setminus 3Q} \frac{|f(x)|^r}{(1+|x|)^N}\,dx + \int_{3Q} |f(x)|^r\,dx\cdot \||\Phi_l| * \mu\|_{3Q}^r\Big)^{1/r},$$

whence (4.10) follows by the case $r=1$ of Lemma 4.4. ∎

Remark 4.7. It should be noted that the projections (4.5) and (4.6) have earlier been used by Jonsson [12] in his study of restrictions of function spaces on closed sets preserving Markov's inequality. Also, the projections given by (4.6) have been applied by Hedberg while proving Poincaré type inequalities depending on capacities (see Adams and Hedberg [2], Section 8.2, Equation (8.2.6)).

5. Main result. Let A^s_{pq} be one of the spaces B^s_{pq} or F^s_{pq} with parameters satisfying (2.6). In particular, by Remark 2.4, $A^s_{pq}\subset L_1^{\mathrm{loc}}$. Let Ω be an open subset of $\mathbb{R}^n$. We define the notions of the restriction space $A^s_{pq}[\Omega]$ and of the linear extension operator for $A^s_{pq}[\Omega]$ analogously to how it was done in the introduction for $H^s_p[\Omega]$. We are now interested in whether or not

(5.1) there exists a linear extension operator for $A^s_{pq}[\Omega]$.

We will prove the following theorem, which is the main result of this paper. In particular, it will imply Theorem 1.1 by specializing $H^s_p = F^s_{p,2}$, $1<p<\infty$. Put $u=p$ for $A^s_{pq}=B^s_{pq}$, and $u=\min(p,q)$ for $A^s_{pq}=F^s_{pq}$.

Theorem 5.1. *Let Ω be d-thick. Then (5.1) is true if*

(a) $d>n-1$ and $s>n/u-d/\max(1,u)$.
(b) $d\le n-1$ and $s-[s]>n/u-d/\max(1,u)$.

It should be mentioned that in the case $d=n$, $p\ge 1$, $A^s_{pq}=B^s_{pq}$ the existence of a linear extension operator was established earlier by Shvartsman [20].

Proof. We follow the plan outlined in Remark 3.2. We will define a linear operator

$$\Lambda : A^s_{pq}[\Omega] \to P_m(\Omega, A^s_{pq}) \tag{5.2}$$

and will prove that it is bounded. Then we will put

$$\mathrm{ext} = E\circ\Lambda. \tag{5.3}$$

By Theorem 3.1 we will conclude that $\mathrm{ext} : A^s_{pq}[\Omega]\to A^s_{pq}$ is a bounded linear map, and it will only remain to prove that $\mathrm{re}_\Omega\circ\mathrm{ext} = \mathrm{id}$.

Step 1. We consider the case $d\le n-1$; minor changes in the argument necessary for $d>n-1$ will be indicated at the end of the proof. We retain all the notation from Section 3. To define the operator Λ, we should for each $(j,\mathbf{k})\in\Gamma$ define a linear map

$$f\mapsto P_{j,\mathbf{k}}(f).$$

We have $5Q_{j,\mathbf{k}}\cap\Omega\ne\emptyset$. Consider a cube $Q^*_{j,\mathbf{k}}$ of side length 2^{-j} centered somewhere in this nonempty intersection. By d-thickness,

$$H_d(Q^*_{j,\mathbf{k}}\cap\Omega)\ge\varepsilon 2^{-jd},$$

hence, in view of the obvious homogeneity of the Hausdorff content,

$$H_d(2^jQ^*_{j,\mathbf{k}}\cap 2^j\Omega)\ge\varepsilon>0.$$

We now apply the theory of Section 4 to $Q=2^jQ^*_{j,\mathbf{k}}$. We consider the projection

$$\widetilde{L}\equiv\widetilde{L}_{j,\mathbf{k}} : C^m(2^jQ^*_{j,\mathbf{k}}\cap 2^j\Omega)\to\Pi_m.$$

This projection depends on $j,\mathbf{k}$, but it satisfies the estimates of Lemma 4.3 and 4.6 with constants independent of $j,\mathbf{k}$. We take $\boxed{m=[s]}$.

Let $f\in C^m(\Omega)$. Then $f(2^{-j}\cdot)\in C^m(2^jQ^*_{j,\mathbf{k}}\cap 2^j\Omega)$. We put

$$P_{j,\mathbf{k}}(f) = (\widetilde{L}_{j,\mathbf{k}}(f(2^{-j}\cdot)))(2^j\cdot). \tag{5.4}$$

In other words, we first dilate, then apply the projection, and finally shrink back.

The formula (5.4) defines the operator Λ originally on $C^m(\Omega)$. However, we claim that in fact Λ can be extended (by continuity, or by understanding the integration against the measure μ in (4.6) in the sense of the duality pairing) to every space A^s_{pq} with parameters satisfying the assumptions of Theorem 5.1(b). Indeed, by Remark 4.5 this will be proved if we verify the embedding

$$A^s_{pq} \subset B^{m+(n-d)/\theta}_{\theta,1}, \quad \theta = \max(1,p). \tag{5.5}$$

We use the following elementary fact (see Triebel [23], 2.3.2, Proposition 2):

$$A^s_{pq} \subset B^{\overline{s}}_{p,1} \quad \text{for } s > \overline{s}. \tag{5.6}$$

For $p \ge 1$, (5.5) follows from (5.6), because we have $s > m + (n-d)/p$. If $p < 1$, we first apply a Sobolev-type embedding (see [23], 2.7.1, (1) and (2))

$$A^s_{pq} \subset A^{s+n-n/p}_{1,q} \quad (p < 1). \tag{5.7}$$

Since we have $s + n - n/p > m + n - d$ for $p < 1$, we get (5.5) by applying (5.6) to the space on the right-hand side of (5.7). For later reference we note that the argument just given shows a little bit more. Namely, we can change $m + (n-d)/\theta$ on the right-hand side of (5.5) to a slightly greater number so that the embedding will remain valid.

Step 2. In the previous step we constructed a linear operator $\Lambda : A^s_{pq}[\Omega] \to P_m(\Omega)$, and we are now going to prove that it acts boundedly into $P_m(\Omega, A^s_{pq})$, i.e.,

$$\|\Lambda g\|_{P_m(\Omega, A^s_{pq})} \le C\|g\|_{A^s_{pq}[\Omega]} \tag{5.8}$$

with C independent of g. It suffices to prove for all $f \in A^s_{pq}$ the estimate

$$\|\Lambda f\|_{P_m(\Omega, A^s_{pq})} \le C\|f\|_{A^s_{pq}}, \tag{5.9}$$

since (5.8) follows from (5.9) by taking the infimum over all f satisfying $f|_\Omega = g$.

Let Φ, φ be as in Definition 2.1. We use the representation

$$f = \Phi_j * f + \sum_{l=j+1}^{\infty} \varphi_l * f. \tag{5.10}$$

By (5.5), this series converges in $B^{m+(n-d)/\theta}_{\theta,1}$. Therefore for all $(j,\mathbf{k}) \in \Gamma$,

$$P_{j,\mathbf{k}}(f) = P_{j,\mathbf{k}}(\Phi_j * f) + \sum_{l=j+1}^{\infty} P_{j,\mathbf{k}}(\varphi_l * f),$$

and, consequently (see (3.2)),

$$a_{j,\mathbf{k}}(\Lambda f) \le a_{j,\mathbf{k}}(\Lambda(\Phi_j * f)) + \sum_{l=j+1}^{\infty} a_{j,\mathbf{k}}(\Lambda(\varphi_l * f)). \tag{5.11}$$

Let $j \in \mathbb{N}$ (for $j = 0$ the argument would differ slightly because of the different definition of $a_{0,\mathbf{k}}$ in (3.2)). In view of (5.4), by Lemma 4.3 we have

for any fixed $a \ge 1$ the estimate

$$\begin{aligned} \|P_{j,\mathbf{k}}(\Phi_j * f) - \Phi_j * f\|_{aQ^*_{j,\mathbf{k}}} &\le C\|\nabla^{m+1}((\Phi_j * f)(2^{-j}\cdot))\|_{a2^jQ^*_{j,\mathbf{k}}} \\ &= C2^{-j(m+1)}\|\nabla^{m+1}(\Phi_j * f)\|_{aQ^*_{j,\mathbf{k}}}. \end{aligned} \tag{5.12}$$

Analogous estimates are valid for $P_{j,\mathbf{k}}$ and $Q^*_{j,\mathbf{k}}$ replaced by $P_{j-1,\mathbf{k}'}$ and $Q^*_{j-1,\mathbf{k}'}$, or by $P_{j,\mathbf{u}}$ and $Q^*_{j,\mathbf{u}}$, where $u \in \Gamma_j$, $\varrho(\mathbf{u},\mathbf{k}) = 1$. Now take a so large that all the cubes $aQ^*_{j,\mathbf{k}}$, $aQ^*_{j-1,\mathbf{k}'}$, $aQ^*_{j,\mathbf{u}}$ cover $Q_{j,\mathbf{k}}$, say, $a = 100$. Then it follows from (3.2) by the triangle inequality and by (5.12) (together with its analogues just mentioned) that

$$a_{j,\mathbf{k}}(\Lambda(\Phi_j * f)) \le C2^{-j(m+1)}\|\nabla^{m+1}(\Phi_j * f)\|_{1000Q_{j,\mathbf{k}}}. \tag{5.13}$$

We now deal with the remaining terms in (5.11). The functions $(\varphi_l * f)(2^{-j}\cdot)$ have their Fourier transforms supported in $\{|\xi| \le 2^{l+1-j}\}$, hence Lemma 4.6 shows that ($x_{j,\mathbf{k}} \equiv x_{Q_{j,\mathbf{k}}}$)

$$\begin{aligned} &\|P_{j,\mathbf{k}}(\varphi_l * f)\|_{Q_{j,\mathbf{k}}} \\ &\quad \le C2^{(l-j)\varkappa} \sum_{|\alpha| \le m} \bigg(\int_{\mathbb{R}^n} \frac{|D^\alpha((\varphi_l * f)(2^{-j}\cdot))(x)|^r}{(1 + |x - 2^j x_{j,\mathbf{k}}|)^N}\,dx\bigg)^{1/r}, \end{aligned} \tag{5.14}$$

where $0 < r < \infty$, $\varkappa = \varkappa(r) = n/r - d/\max(1,r)$. Analogous estimates are again true for $P_{j,\mathbf{u}}$ and $P_{j-1,\mathbf{k}'}$. By summing them all together and changing variables $2^{-j}x \mapsto x$, we arrive at

$$\begin{aligned} &a_{j,\mathbf{k}}(\Lambda(\varphi_l * f)) \\ &\quad \le C2^{(l-j)\varkappa} \sum_{|\alpha| \le m} \bigg(2^{jn}2^{(l-j)|\alpha|} \int_{\mathbb{R}^n} \frac{|((D^\alpha\varphi)_l * f)(x)|^r}{(1 + 2^j|x - x_{j,\mathbf{k}}|)^N}\,dx\bigg)^{1/r} \end{aligned} \tag{5.15}$$

To rewrite the estimates obtained in a more compact form, we introduce suitable Peetre's maximal functions (see (2.1)) by

$$f^*_j(x) = \sum_{|\alpha| = m+1} (D^\alpha\Phi)^*_{j,\lambda} f(x) + \sum_{|\alpha| \le m} (D^\alpha\varphi)^*_{j,\lambda} f(x),$$

where $\lambda > n/\min(p,q)$ is a fixed number. Then it follows from (5.11), (5.13), and (5.15) (see also (2.4)) that

$$a_{j,\mathbf{k}}(\Lambda f) \le C \sum_{l=j}^{\infty} 2^{(l-j)(m+\varkappa)} \bigg(2^{jn} \int_{\mathbb{R}^n} \frac{|f^*_l(x)|^r}{(1 + 2^j|x - x_{j,\mathbf{k}}|)^N}\,dx\bigg)^{1/r} \tag{5.16}$$

(N has become λ smaller). This is the crucial estimate. The reader will easily check that the analogous estimate for $j = 0$ is also true if we define f^*_0 appropriately.

Let M be the usual Hardy–Littlewood maximal operator, and M_r, $0 < r < \infty$, be defined by $M_r f = (M(|f|^r))^{1/r}$. If we choose $N > n$, then by the well-known majorant property of M (e.g. Stein and Weiss [21], Chapter 2, (3.9)) (5.16) implies (see (3.3))

$$A_j(\Lambda f) \le C \sum_{l=j}^{\infty} 2^{(l-j)(m+\varkappa)} M_r(f_l^*). \tag{5.17}$$

So far $0 < r < \infty$ have not been fixed, and all constants in the estimates depend on r. We now pick (as we may) an r satisfying

$$\text{(i) } r < u \quad \text{and} \quad \text{(ii) } s - m > \varkappa = \frac{n}{r} - \frac{d}{\max(1,r)}. \tag{5.18}$$

Consider the case of $A_{pq}^s = F_{pq}^s$. In view of (5.18)(ii), $m + \varkappa - s < 0$, and if we rewrite (5.17) as

$$2^{js} A_j(\Lambda f) \le C \sum_{l=j}^{\infty} 2^{(l-j)(m+\varkappa-s)} 2^{ls} M_r(f_l^*),$$

the following estimate clearly follows (see (3.4)):

$$\|\Lambda f\|_{P_m(\Omega, A_{pq}^s)} \le C \|\{2^{js} M_r(f_j^*)\}_{j=0}^{\infty}\|_{L_p(\ell_q)}.$$

By the Fefferman–Stein vector-valued maximal theorem ([7]), which is applicable in view of (5.18)(i) (recall $u = \min(p,q)$), we conclude that

$$\|\{2^{js} M_r(f_j^*)\}\|_{L_p(\ell_q)} \le C \|\{2^{js} f_j^*\}\|_{L_p(\ell_q)}.$$

Finally, Lemma 2.2 can be applied with $D^\alpha \Phi$ ($|\alpha| = m+1$) or $D^\alpha \varphi$ instead of ψ (the condition (2.2) is satisfied), and this gives us the estimate

$$\|\{2^{js} f_j^*\}\|_{L_p(\ell_q)} \le C \|f\|_{F_{pq}^s}.$$

The last three estimates prove (5.9) for $A_{pq}^s = F_{pq}^s$. If $A_{pq}^s = B_{pq}^s$, then the same estimates are true with $L_p(\ell_q)$ replaced with $\ell_q(L_p)$. In this case the simple scalar-valued maximal theorem suffices, and we can take $u = p$. Thus, the boundedness of Λ is proved.

STEP 3. We now define the operator ext by (5.3). Then (i) both E and Λ are linear; (ii) in the previous step we proved (5.2); (iii) E is a bounded linear operator $P_m(\Omega, A_{pq}^s) \to A_{pq}^s$ by Theorem 3.1. Hence, ext is a bounded linear operator from $A_{pq}^s[\Omega]$ to A_{pq}^s. We claim that ext also has the extension property, i.e., $\mathrm{re}_\Omega \circ \mathrm{ext} = \mathrm{id}$. As we already noticed in Remark 3.2, this property will be proved if we show that $E^j(\Lambda f) \to f$ in $L_1(\Omega, \mathrm{loc})$ as $j \to \infty$.

We will show that $E^j(\Lambda f) - \Phi_j * f \to 0$ in $L_1(\Omega, \mathrm{loc})$. Since $\Phi_j * f \to f$ in L_1^{loc} for any locally integrable f, this will give the desired result.

We take another look at the argument of Step 2. We are about to estimate the distance between $P_{j,\mathbf{k}}(f)$ and $\Phi_j * f$. The estimate (5.12) informs us about the distance between $P_{j,\mathbf{k}}(\Phi_j * f)$ and $\Phi_j * f$, and (5.14) says that the $P_{j,\mathbf{k}}(\varphi_l * f)$ are not too large. Actually, if we express the right-hand sides of (5.12) and (5.14) via f_j^*, then we arrive analogously to (5.17) at

$$\|E^j(\Lambda f) - \Phi_j * f\|_{L_1(Q)} \le C \Big\| \sum_{l=j}^{\infty} 2^{(l-j)(m+\varkappa)} M_r(f_l^*) \Big\|_{L_1(Q)}, \tag{5.19}$$

where Q is any cube contained in Ω. Here we are not bound to choose r as in (5.18); in fact, we now take $r = \max(1,p) - \delta$, $\delta > 0$ small. Now $L_{\max(1,p)}(Q) \subset L_1(Q)$, and M_r is bounded on $L_{\max(1,p)}$. It follows that the right-hand side of (5.19) does not exceed

$$2^{-j(m+\varkappa)} \sum_{l=j}^{\infty} 2^{l(m+\varkappa)} \|f_l^*\|_{L_{\max(1,p)}(Q)} \le 2^{-j(m+\varkappa)} \|f\|_{B^{m+\varkappa}_{\max(1,p),1}},$$

where the latter estimate follows by Lemma 2.2. Thus, if we prove that $f \in B^{m+\varkappa}_{\max(1,p),1}$, we are done. Recall that in Step 1 we proved the embedding (5.5) with $\theta = \max(1,p)$, and noticed that the same embedding with a slightly greater smoothness parameter of the target space is true. Since $\varkappa(\max(1,p)) = (n-d)/\max(1,p) = (n-d)/\theta$, it follows that if we take δ sufficiently small, then A_{pq}^s is embedded into $B^{m+\varkappa}_{\max(1,p),1}$, as desired.

STEP 4. So far we proved part (b) of the theorem. In the case of (a) one needs to make the following changes in the above argument. First, in the definition of the operator Λ (Step 1) we must change $\widetilde{L}$-type projections to L-type ones (see Section 4). The proof of the boundedness of Λ in $A_{pq}^s[\Omega]$ then follows the same scheme. However, while applying Lemma 4.6 we get an analogue of (5.14) in which instead of the sum over $|\alpha| \le m$ on the right-hand side there is only the term corresponding to $\alpha = 0$. By continuing the estimation, we arrive at the counterpart of (5.17) of the form

$$A_j(\Lambda f) \le C \sum_{l=j}^{\infty} 2^{(l-j)\varkappa} M_r(f_l^*).$$

Hence the boundedness of Λ in the desired range of s follows. Now to prove that ext has the extension property we repeat the argument of Step 3 putting $m = 0$ throughout. This concludes the proof of the theorem. ■

REMARK 5.2. It follows from the above proof that under the conditions of the theorem the following equivalence is true:

$$C^{-1} \|f\|_{A_{pq}^s[\Omega]} \le \|\Lambda f\|_{P_m(\Omega, A_{pq}^s)} \le C \|f\|_{A_{pq}^s[\Omega]} \quad \text{for all } f \in A_{pq}^s[\Omega].$$

Moreover, it can be seen from the proof that the equivalence is preserved for all $f \in A_{pq}^s(\Omega, \mathrm{loc})$, i.e., for f such that $\varphi f \in A_{pq}^s$ for every $\varphi \in C_0^\infty(\Omega)$. In other words, we get the following intrinsic characterization: the space $A^s_{pq}[\Omega]$ consists of exactly those $f \in A^s_{pq}(\Omega, \mathrm{loc})$ for which $\Lambda f \in P_m(\Omega, A^s_{pq})$.

REMARK 5.3. We finally note that the methods of this paper can be applied to study restrictions to (or traces on) sets Ω satisfying the condition (1.2) which are not assumed to be open. Of course, the definition of the restriction space should be modified in this case. E.g., for any $f \in L_1^{\mathrm{loc}}$ let $\widetilde{f}$ denote the function given by

$$\widetilde{f}(x) = \lim_{\varrho \to 0} \frac{1}{|B(x,\varrho)|} \int_{B(x,\varrho)} f(y)\, dy \tag{5.20}$$

for all $x \in \mathbb{R}^n$ for which the limit exists, and $\widetilde{f}(x) = 0$ otherwise. It is well known that if f belongs to H^s_p, then the set where the limit does not exist has H^s_p capacity ($\equiv (p,s)$-capacity) zero, and $\widetilde{f}$ is quasi-continuous with respect to the H^s_p capacity (see, e.g., Adams and Hedberg [2]). Analogous results are valid for Besov and Triebel–Lizorkin spaces (see Dorronsoro [6], Netrusov [15]). We set $\mathrm{Re}_\Omega f = \widetilde{f}|_\Omega$. By using the known relations between the capacities and Hausdorff contents (see [2], Chapter 5, and Adams [1]), one can show that for $s > n/u - d/\max(1,u)$ the restriction space $A^s_{pq}[\Omega]$ makes sense as the linear space of equivalence classes of functions on Ω with equivalence relation $f \sim g \Leftrightarrow H_d(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. The restriction space can be normed by

$$\|g\|_{A^s_{pq}[\Omega]} = \inf\{\|f\|_{A^s_{pq}} : \mathrm{Re}_\Omega f \sim g\}.$$

Analogously for $m \in \mathbb{N}$ and $s > m + n/u - d/\max(1,u)$ the *jet restriction space* $A^s_{pq}[\Omega, m]$ is defined by means of the *jet restriction operator* $\mathrm{Re}^m_\Omega f = \{\widetilde{D^\alpha f}|_\Omega\}_{|\alpha| \le m}$.

By a suitable adaptation of the methods developed above, one can prove

THEOREM 5.4. *Let Ω be a (not necessarily open) Borel subset of $\mathbb{R}^n$ satisfying the d-thickness condition* (1.2).

(a) *If $d > n-1$ and $s > n/u - d/\max(1,u)$, then there exists a bounded linear operator* $\mathrm{Ext} : A^s_{pq}[\Omega] \to A^s_{pq}$ *such that* $\mathrm{Re}_\Omega \circ \mathrm{Ext} = \mathrm{id}$.

(b) *If $d \le n-1$ and $s - [s] > n/u - d/\max(1,u)$, then there exists a bounded linear operator* $\mathrm{Ext} : A^s_{pq}[\Omega, m] \to A^s_{pq}$ *such that* $\mathrm{Re}^m_\Omega \circ \mathrm{Ext} = \mathrm{id}$, $m = [s]$.

It should be noted that for *closed* d-thick sets Ω (in the case when $A^s_{pq} = F^s_{pq}$ one must also impose the condition that Ω have empty interior), this theorem can be derived from Jonsson [12], Theorems 5 and 7, by using results of Bylund [5].

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Received February 8, 1999
Revised version January 24, 2000 (4257)

STUDIA MATHEMATICA 140 (2) (2000)

Restriction of an operator to the range of its powers

by

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Abstract. Let T be a bounded linear operator acting on a Banach space X. For each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In [1] and [2] we have characterized operators T such that for a given integer n, the operator T_n is a Fredholm or a semi-Fredholm operator. We continue those investigations and we study the cases where T_n belongs to a given regularity in the sense defined by Kordula and Müller in [10]. We also consider the regularity of operators with topological uniform descent.

1. Introduction. Let $L(X)$ be the Banach algebra of bounded linear operators acting on a Banach space X and let $T \in L(X)$. We denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) *operator*. A *semi-Fredholm operator* is an upper or a lower semi-Fredholm operator. We let $\Phi_+(X)$ (resp. $\Phi_-(X)$) denote the set of upper (resp. lower) semi-Fredholm operators. If both $\alpha(T)$ and $\beta(T)$ are finite then T is called a *Fredholm operator* and the index of T is defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$.

For each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm (resp. semi-Fredholm) operator, then T is called a *B-Fredholm operator* (resp. a *semi-B-Fredholm*) operator. In [1] the author has studied this class of operators and proved [1, Theorem 2.7] that $T \in L(X)$ is a B-Fredholm operator if and only if $T = Q \oplus F$, where Q is a nilpotent operator and F is a Fredholm operator. In [2] we have proved the same result for semi-B-Fredholm operators acting on Hilbert spaces.

Recall that an operator $T \in L(X)$ has a *generalized inverse* if there is an $S \in L(X)$ such that $TST = T$. In this case T is also called a *relatively*

2000 *Mathematics Subject Classification*: 47A53, 47A55.