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Degenerate evolution problems and Beta-type operators

by

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Abstract. The present paper is concerned with the study of the differential operator $Au(x) := \alpha(x)u''(x) + \beta(x)u'(x)$ in the space $C([0,1])$ and of its adjoint $Bv(x) := ((\alpha v)'(x) - \beta(x)v(x))'$ in the space $L^1(0,1)$, where $\alpha(x) := x(1-x)/2$ $(0 \le x \le 1)$. A careful analysis of their main properties is carried out in view of some generation results available in [6, 12, 20] and [25]. In addition, we introduce and study two different kinds of Beta-type operators as a generalization of similar operators defined in [18]. Among the corresponding approximation results, we show how they can be used in order to represent explicitly the solutions of the Cauchy problems associated with the operators A and $\widetilde{A}$, where $\widetilde{A}$ is equal to B up to a suitable bounded additive perturbation.

1. Introduction and notations. The present paper falls within a wide program of investigations whose main object is the interplay between constructive approximation processes and degenerate evolution problems by means of standard semigroup theory. More specifically, we are interested in representing explicitly the semigroups generated by some degenerate differential operators in terms of powers of suitable positive linear operators: as a direct consequence, the solutions of the initial value problems canonically associated with such differential operators may be represented in the same way, as well. This kind of approach, basically based upon Voronovskaya-type formulas and Trotter's theorem [26], has its roots in a paper by Altomare [1], dealing with the convergence of the powers of the classical Bernstein operators; actually, it turns out to be quite satisfactory in practical situations, since some qualitative properties of the relevant semigroups, such as asymptotic behaviour, regularity, saturation and so on, may sometimes be easily derived from the corresponding properties of the approximating operators.

A rather exhaustive treatment of this subject together with a systematic analysis of some classical approximation processes may be found in

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[3, Chapter 6]; however, several recent contributions in this direction have enlarged the class of the differential operators and, at the same time, of the approximating operators considered, in the framework of spaces of continuous functions on both bounded and unbounded intervals.

Without assumption of completeness, we mention, for instance, [2, 4, 5, 7, 8, 9] and [11] in this respect.

Our aim is to take the described approach for the differential operator

$$Au(x) := \alpha(x)u''(x) + \beta(x)u'(x), \quad 0 < x < 1, \tag{1.1}$$

in the space $C([0,1])$ and for the operator $\widetilde{A}$ defined by

$$\widetilde{A}v(x) := Bv(x) - (\alpha''(x) - \beta'(x))v(x), \quad 0 < x < 1, \tag{1.2}$$

in the space $L^1(0,1)$, B being the operator adjoint of A defined by

$$Bv(x) := \frac{d}{dx}((\alpha v)'(x) - \beta(x)v(x)), \quad 0 < x < 1, \tag{1.3}$$

and $\alpha(x) := x(1-x)/2$ $(0 \le x \le 1)$.

In a more suggestive fashion, we may say that $\widetilde{A}$ is obtained by perturbing the operator B, adjoint of A, with the term $(\alpha''(x) - \beta'(x))v(x)$ which will be bounded in our treatment. As a consequence, when dealing with $\widetilde{A}$, all the classical perturbation results will be available.

Differently, observe that in (1.1) an additive unbounded perturbation appears with respect to the simpler operator

$$A_0u(x) := \frac{x(1-x)}{2}u''(x), \quad 0 < x < 1, \tag{1.4}$$

already studied in [3, Chapter 6].

We also point out that the above operators A and $\widetilde{A}$ occur when studying the general evolution problems describing diffusion models in population genetics (see [14, 16] and also [24]).

Moreover, we recall that, in the framework of spaces of continuous functions, even more general operators than A have been considered in [10] and [20], but merely from the viewpoint of semigroup theory: such analysis, therefore, though useful to our investigations, lacks any explicit mention of approximation of solutions of the associated evolution problems, being primarily concerned with analyticity and compactness properties of the relevant semigroups.

The paper is organized as follows. In Section 2 we outline some basic properties of the operator A defined in (1.1) when acting on its maximal domain

$$D_{\mathrm{M}}(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \mid Au \in C([0,1])\} \tag{1.5}$$

or, alternatively, on the domain

$$D_{\mathrm{V}}(A) := \{u \in D_{\mathrm{M}}(A) \mid \lim_{x\to 0,1} Au(x) = 0\}, \tag{1.6}$$

i.e., assuming Ventcel's conditions at the boundary points 0 and 1. Besides the generation of a $\mathcal{C}_0$-semigroup as an application of the classical results by Clément and Timmermans [12] and Timmermans [25], we show that the subspace of all twice continuously differentiable functions in the corresponding domains is a core for A.

A similar analysis is carried out in Section 3 for the operator $\widetilde{A}$. In this respect, we refer to some recent results stated in [6], which may be regarded as the analogues in $L^1(0,1)$ of the cited theorems by Clément and Timmermans [12] and Timmermans [25].

In the last Section 4 we introduce and study two sequences of Beta-type operators acting on $C([0,1])$ and $L^1(0,1)$, respectively, and generalizing two sequences of positive linear operators introduced by Lupaş [18]. After showing that they are positive approximation processes on the corresponding spaces, we state some estimates of the rate of convergence in terms of the classical modulus of continuity $\omega(f,\cdot)$ for continuous functions and, in the framework of the space $L^1(0,1)$, in terms of the averaged modulus of smoothness $\tau(f,\cdot)_1$ (see [23]).

The Voronovskaya-type results (Theorem 4.4) provide a link with semigroup theory, being, in fact, the key tool in proving the main theorems of the section. More specifically, we prove that there exist $\mathcal{C}_0$-semigroups of positive contractions on $C([0,1])$ and $L^1(0,1)$, having A and $\widetilde{A}$ as generators, respectively, and which may be expressed in terms of powers of the Beta-type operators previously defined.

The notations we use throughout the paper are quite standard: besides the spaces $C([0,1])$, $L^1(0,1)$ and $L^\infty(0,1)$ endowed with the usual norms, we sometimes deal with the space $L^1_{\mathrm{loc}}(0,1)$ of all Lebesgue measurable functions on $[0,1]$ which are integrable on compact subsets of $]0,1[$ as well as with the space $AC(0,1)$ of all absolutely continuous functions on $[0,1]$. Accordingly, $AC_{\mathrm{loc}}(0,1)$ denotes the space of all continuous functions on $[0,1]$ which are absolutely continuous on compact subsets of $]0,1[$. As usual, $C^m([0,1])$ denotes the vector space of all real-valued m-times continuously differentiable functions on $[0,1]$ $(m \ge 1)$.

Sometimes, in the above notations, the interval $[0,1]$ will be replaced by a more general interval I of the real line. Finally, for every $p \in \mathbb{N}_0$, we denote by e_p the continuous function on $[0,1]$ defined by $e_p(x) := x^p$ for every $x \in [0,1]$, whereas, for each $x \in [0,1]$, ψ_x is the function defined by $\psi_x(t) := t - x$ $(0 \le t \le 1)$. The Landau symbols will be denoted by $o(\cdot)$ and $O(\cdot)$, as usual.

2. The differential operator A. In this section we deal with the second order differential operator

$$Au(x) := \frac{x(1-x)}{2}\, u''(x) + \beta(x)u'(x), \quad 0 < x < 1, \tag{2.1}$$

acting on $C([0,1])$, where β is a continuous function on $[0,1]$ which is Hölder continuous at 0 and 1. It is convenient to rewrite (2.1) in the form

$$Au(x) = \frac{x(1-x)}{2}\, u''(x) + ((1-x)\lambda(x) - x\mu(x))u'(x), \quad 0 < x < 1, \tag{2.2}$$

with $\lambda := \beta$ and $\mu := -\beta$.

Observe that, whenever $\lambda(x) = \mu(x) = 1/2$ for any $x \in [0,1]$, then one has

$$Au(x) = \frac{d}{dx}\left(\frac{x(1-x)}{2}\, u'(x)\right),$$

i.e., A is self-adjoint.

For generation results, we basically refer to the papers by Clément and Timmermans [12] and Timmermans [25], which in turn are very closely related to the pioneer work by Feller [15]. Accordingly, following the notations of those papers, after choosing $x_0 = 1/2$, we set for every $x \in\,]0,1[$,

$$W(x) := \exp\left(-2\int_{1/2}^{x}\left(\frac{\lambda(t)}{t} - \frac{\mu(t)}{1-t}\right)dt\right),$$

$$Q(x) := \frac{2}{x(1-x)W(x)}\int_{1/2}^{x} W(t)\,dt, \tag{2.3}$$

$$R(x) := W(x)\int_{1/2}^{x} \frac{2}{t(1-t)W(t)}\,dt,$$

and an easy computation yields $W(x) \approx K/x^{2\lambda(0)}$ as $x \to 0^+$ as well as

$$Q(x) \approx \begin{cases} \dfrac{2}{1-2\lambda(0)}\left(1 - \dfrac{M}{x^{1-2\lambda(0)}}\right) & \text{if } \lambda(0) \ne 1/2,\\ 2(\log x + \log 2) & \text{if } \lambda(0) = 1/2, \end{cases} \tag{2.4}$$

and

$$R(x) \approx \begin{cases} \dfrac{1}{\lambda(0)} - \dfrac{N}{\lambda(0)x^{2\lambda(0)}} & \text{if } \lambda(0) \ne 0,\\ 2(\log x + \log 2) & \text{if } \lambda(0) = 0, \end{cases} \tag{2.5}$$

where K, M, N are suitable strictly positive constants depending only on $\lambda(0)$. By replacing x by $1-x$ and $\lambda(0)$ by $\mu(1)$ one gets similar asymptotic relations for W, Q and R near 1 and therefore the cited results by Clément and Timmermans [12] and Timmermans [25], according to the terminology of Feller [15] (see also [13, p. 366]), read as follows:

(1) If $\lambda(0), \mu(1) \ge 1/2$, i.e., if the endpoints 0 and 1 are both entrance boundary points, then A is a generator on the maximal domain $D_{\rm M}(A)$ (see (1.5)).

(2) If $\lambda(0), \mu(1) \le 0$ or $0 < \lambda(0), \mu(1) < 1/2$, i.e., if the endpoints 0 and 1 are both exit or both regular boundary points, respectively, then A is a generator on the domain $D_{\rm V}(A)$ (see (1.6)).

In addition, some intermediate situations are allowed: more precisely, if $\lambda(0) < 1/2$ and $\mu(1) \ge 1/2$ or, conversely, $\lambda(0) \ge 1/2$ and $\mu(1) < 1/2$, then A is a generator on the domain

$$D_{\rm VM}(A) := \{u \in D_{\rm M}(A) \mid \lim_{x\to 0^+} Au(x) = 0\}$$

or, respectively, on the domain

$$D_{\rm MV}(A) := \{u \in D_{\rm M}(A) \mid \lim_{x\to 1^-} Au(x) = 0\}$$

(see [25, Theorem 4]).

For a probabilistic interpretation of the above classification we refer the reader to [13, Problem 1, p. 382]; see also [13, p. 162] for connections with Markov jump processes.

REMARK 2.1. Note that our operator A formally coincides with the operator A_1 considered in [10] and [20] with $m(x) := 1/2$ and $b(x) := 2(1-x)\lambda(x) - 2x\mu(x)$. Therefore, if $\lambda(0), \mu(1) < 1/2$, on account of [20, Theorems 3.3 and 4.1], $(A, D_{\rm V}(A))$ is the generator of a bounded analytic semigroup of angle $\pi/2$, which is compact, positive and contractive (see also [10, Theorems 3.1, 3.3 and 3.6]).

For a more general discussion of one-dimensional degenerate diffusion processes see also [13, pp. 371–372] and [13, Theorem 2.8, p. 375] in the case $d = 1$.

A first simple result is indicated in the next proposition.

PROPOSITION 2.2. *If $u \in C([0,1]) \cap C^2(]0,1[)$ and Au is continuous at* 0 *(resp. at* 1*), then*

$$\lim_{x\to 0^+} xu'(x) = 0 \quad \left(\textit{resp. } \lim_{x\to 1^-} (1-x)u'(x) = 0\right). \tag{2.6}$$

Proof. We have $Au(x) \approx (x/2)u''(x) + \lambda(0)u'(x)$ near 0 and therefore a simple integration by parts gives

$$\int_{x}^{1/2} Au(t)\,dt \approx \frac{1}{4}u'\left(\frac{1}{2}\right) - \frac{x}{2}u'(x) + \left(\lambda(0) - \frac{1}{2}\right)\left(u\left(\frac{1}{2}\right) - u(x)\right), \quad x \to 0^+,$$

where the term on the left hand side is convergent as $x \to 0^+$ since Au is continuous at 0 by assumption. As a consequence, the limit $l := \lim_{x\to 0^+} xu'(x)$ exists and is finite. Indeed, it is 0, since otherwise $u'(x) \approx l/x$ as $x \to 0^+$ and hence, for a suitable non-zero constant M, $u(x) \approx M \log x$ as $x \to 0^+$, contradicting the fact that $u \in C([0,1])$. The proof for the limit at 1 is similar. ∎

In order to go deeper into the properties of the operator A, we need to write down its relevant domains in a way easier to handle in practice. Indeed, a rather complete result will be stated below in the case $\lambda(0), \mu(1) \notin\,]0,1/2[$, but first we introduce some notations. We set

$$(2.7)\qquad D(A) := \begin{cases} D_{\mathrm{V}}(A) & \text{if } \lambda(0), \mu(1) < 1/2, \\ D_{\mathrm{M}}(A) & \text{if } \lambda(0), \mu(1) \geq 1/2, \\ D_{\mathrm{VM}}(A) & \text{if } \lambda(0) < 1/2,\ \mu(1) \geq 1/2, \\ D_{\mathrm{MV}}(A) & \text{if } \lambda(0) \geq 1/2,\ \mu(1) < 1/2, \end{cases}$$

and for a fixed $u \in C([0,1]) \cap C^2(]0,1[)$ we define the boundary conditions (N_λ) and (N_μ) at 0 and 1, respectively, as follows:

$$(N_\lambda)\begin{cases} u \in C^1([0,1/2]),\ u'(0) = 0,\ \lim\limits_{x\to 0^+} xu''(x) = 0 & \text{if } \lambda(0) < 0, \\ \lim\limits_{x\to 0^+} xu''(x) = 0 & \text{if } \lambda(0) = 0, \\ u \in C^1([0,1/2]),\ \lim\limits_{x\to 0^+} xu''(x) = 0 & \text{if } \lambda(0) \geq 1/2, \end{cases}$$

$$(N_\mu)\begin{cases} u \in C^1([1/2,1]),\ u'(1) = 0,\ \lim\limits_{x\to 1^-} (1-x)u''(x) = 0 & \text{if } \mu(1) < 0, \\ \lim\limits_{x\to 1^-} (1-x)u''(x) = 0 & \text{if } \mu(1) = 0, \\ u \in C^1([1/2,1]),\ \lim\limits_{x\to 1^-} (1-x)u''(x) = 0 & \text{if } \mu(1) \geq 1/2. \end{cases}$$

The following theorem, which partly restates Proposition 3.2 of [20] in a particular case, will play a fundamental role in what follows; it also has an interest of its own.

THEOREM 2.3. *Assume $\lambda(0), \mu(1) \notin\,]0,1/2[$ and consider $u \in C([0,1]) \cap C^2(]0,1[)$. Then $u \in D(A)$ if and only if u satisfies the boundary conditions (N_λ) and (N_μ). Moreover $C^2([0,1]) \cap D(A)$ is a core for A provided we further suppose $\lambda(x)/x = O(1)$ as $x \to 0^+$ in the case $\lambda(0) = 0$, and $\mu(x)/(1-x) = O(1)$ as $x \to 1^-$ in the case $\mu(1) = 0$.*

Proof. We need to work out the proof of the first part of the theorem only in the case $\lambda(0), \mu(1) \geq 1/2$ (i.e., dealing with the maximal domain $D_{\mathrm{M}}(A)$), since all the other possible characterizations of $D(A)$ directly follow from [20, Proposition 3.2]. We argue only around 0 and therefore refer to the assumptions on $\lambda(0)$ and to condition (N_λ). The proof around 1 is similar.

Thus, assume $\lambda(0) \geq 1/2$, choose $u \in D(A)$ and set $f := Au$; a direct computation yields

$$(1)\qquad \frac{2f(x)}{x(1-x)W(x)} = \frac{d}{dx}\left(\frac{u'(x)}{W(x)}\right), \qquad 0 < x < 1.$$

Integrating both sides of (1) from ε to x, with $0 < \varepsilon < x$, we find

$$(2)\qquad \int_\varepsilon^x \frac{2f(t)}{t(1-t)W(t)}\,dt = \frac{u'(x)}{W(x)} - \frac{u'(\varepsilon)}{W(\varepsilon)}.$$

The above integral is convergent as $\varepsilon \to 0^+$: indeed, 0 is an entrance boundary point by assumption and therefore the function $(t(1-t)W(t)/2)^{-1}$ is summable near 0 (see [15, p. 516]). This, in turn, implies that the integrand in (2) is summable near 0 as well because, in addition, $f \in C([0,1])$.

Consequently, the limit $l := \lim_{\varepsilon\to 0^+} u'(\varepsilon)/W(\varepsilon)$ exists and is finite. More precisely, we show that necessarily $l = 0$. Otherwise, we should have $u'(\varepsilon) \approx lW(\varepsilon) \approx lK/\varepsilon^{2\lambda(0)}$ as $\varepsilon \to 0^+$ and therefore, for a suitable non-zero constant M,

$$u(\varepsilon) \approx \begin{cases} M\varepsilon^{1-2\lambda(0)} & \text{if } \lambda(0) > 1/2, \\ M\log\varepsilon & \text{if } \lambda(0) = 1/2, \end{cases} \qquad \varepsilon \to 0^+,$$

which is not possible, because $u \in C([0,1])$.

Thus, passing to the limit as $\varepsilon \to 0^+$ in (2) yields

$$u'(x) = W(x)\int_0^x \frac{2f(t)}{t(1-t)W(t)}\,dt \approx \frac{K}{x^{2\lambda(0)}}\int_0^x \frac{2f(t)}{t(1-t)W(t)}\,dt$$

for every $x \in\,]0,1[$, whence, by L'Hôpital's rule, $\lim_{x\to 0^+} u'(x) = f(0)/\lambda(0)$, i.e., u is differentiable at 0.

This implies that $\lim_{x\to 0^+} xu''(x)$ exists and is finite. In addition, it must be 0, because, if not, a non-zero constant l could be found such that $u''(x) \approx l/x$ as $x \to 0^+$ and hence $u'(x) \approx N\log x$ as $x \to 0^+$ for a suitable non-zero constant N, yielding a contradiction since u' is continuous at 0.

Therefore $u \in D(A)$ satisfies (N_λ) and the proof of the first part of the theorem is complete.

As for the second part, without loss of generality we may and do restrict ourselves to showing that $C^2([0,1/2]) \cap D(A)$ is a core for A (here we continue to denote by $D(A)$ the domains defined in (2.7) relative to functions in $C([0,1/2]) \cap C^2(]0,1/2])$ with the corresponding boundary condition only at 0); more precisely, we have to prove that for each $u \in D(A)$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^2([0,1/2]) \cap D(A)$ converging to u in the graph norm of A. In this way, only the assumptions on $\lambda(0)$ will be involved.

Therefore first assume $\lambda(0) \geq 1/2$, fix $u \in D(A)$ and for every $n \in \mathbb{N}$ set $u_n(x) := u(x)$ if $x \in [1/n, 1/2]$ whereas

$$u_n(x) := u\left(\frac{1}{n}\right) + u'\left(\frac{1}{n}\right)\left(x - \frac{1}{n}\right) + \frac{1}{2}u''\left(\frac{1}{n}\right)\left(x - \frac{1}{n}\right)^2 \tag{3}$$

whenever $0 \le x \le 1/n$.

Clearly each $u_n \in C^2([0,1/2])$, because $u_n'(x) = u'(1/n) + u''(1/n)(x - 1/n)$ and $u_n''(x) = u''(1/n)$ whenever $0 \le x \le 1/n$. In addition, for every $n \in \mathbb{N}$ we easily estimate

$$\|u_n - u\| = \sup_{0\le x\le 1/n} |u_n(x) - u(x)| \le \omega\left(u, \frac{1}{n}\right) + \frac{1}{n}\left|u'\left(\frac{1}{n}\right)\right| + \frac{1}{2n^2}\left|u''\left(\frac{1}{n}\right)\right|$$

where the sum on the right hand side tends to 0 as $n \to \infty$ because of condition (N_λ). Thus we have shown that $\lim_{n\to\infty} \|u_n - u\| = 0$.

It remains to prove that $\lim_{n\to\infty} \|Au_n - Au\| = 0$. To this end, let $M := \sup_{0\le x\le 1/2} |(1-x)\lambda(x) - x\mu(x)|$ and observe that, since $\lim_{x\to 0^+} xu''(x) = 0$ in view of (N_λ), for every $\varepsilon > 0$ a positive $\delta \le 1/2$ can be found such that $x|u''(x)| < \varepsilon/4$ for any $x \in \,]0,\delta]$. Moreover $\lim_{n\to\infty}(1/n)u''(1/n) = \lim_{n\to\infty} \omega(u', 1/n) = 0$ and therefore there exists an integer $n_0 \ge 1/\delta$ such that

$$\frac{1}{n}\left|u''\left(\frac{1}{n}\right)\right| < \frac{\varepsilon}{4(M+1)}, \quad \omega\left(u', \frac{1}{n}\right) < \frac{\varepsilon}{4(M+1)}$$

for any $n \ge n_0$. Consequently, for any $n \ge n_0$ and $x \in \,]0, 1/n]$, we have

$$\begin{aligned}
&|Au_n(x) - Au(x)| \\
&\le x\left|u''\left(\frac{1}{n}\right) - u''(x)\right| + \frac{M}{n}\left|u''\left(\frac{1}{n}\right)\right| + M\left|u'\left(\frac{1}{n}\right) - u'(x)\right| \\
&\le \frac{1}{n}\left|u''\left(\frac{1}{n}\right)\right| + x|u''(x)| + \frac{M}{n}\left|u''\left(\frac{1}{n}\right)\right| + M\omega\left(u', \frac{1}{n}\right) < \varepsilon,
\end{aligned}$$

which also hold at 0 and 1, because Au_n and Au are continuous. Since $\|Au_n - Au\| = \sup_{0\le x\le 1/n} |Au_n(x) - Au(x)|$ for every $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \|Au_n - Au\| = 0$, as required.

In order to show the assertion when $\lambda(0) < 0$, fix $u \in D(A)$ and for every $n \in \mathbb{N}$ set

$$\overline{u}_n(x) := \begin{cases} u(x) & \text{if } 1/n \le x \le 1/2, \\ g_n(x) + u_n(x) & \text{if } 0 \le x \le 1/n, \end{cases} \tag{4}$$

where

$$g_n(x) := \frac{1}{3}n\left(u''\left(\frac{1}{n}\right) - nu'\left(\frac{1}{n}\right)\right)\left(x - \frac{1}{n}\right)^3$$

and $(u_n)_{n\in\mathbb{N}}$ is defined as in (3), still converging to u in the graph norm of A on account of (2.7) because $D_V(A) \subset D_M(A)$.

One can easily check that $(\overline{u}_n)_{n\in\mathbb{N}}$ is a sequence in $C^2([0,1/2]) \cap D(A)$, i.e., each $\overline{u}_n$ belongs to $C^2([0,1/2])$ and $\overline{u}_n'(0) = 0$. In addition, in view of condition (N_λ), a straightforward computation shows that

$$\sup_{0\le x\le 1/n} |g_n(x)| \to 0, \quad \sup_{0\le x\le 1/n} |Ag_n(x)| \to 0, \quad n \to \infty,$$

where now A still denotes an operator acting formally as in (2.2).

On the other hand, for every $n \in \mathbb{N}$ and $x \in [0, 1/n]$ we have $|\overline{u}_n(x) - u(x)| \le |g_n(x)| + |u_n(x) - u(x)|$ as well as $|A\overline{u}_n(x) - Au(x)| \le |Ag_n(x)| + |Au_n(x) - Au(x)|$. Taking the supremum over $[0,1/2]$ or, which is the same, over $[0,1/n]$ in the above estimates and letting $n \to \infty$ yield $\|\overline{u}_n - u\| \to 0$ and $\|A\overline{u}_n - Au\| \to 0$ as $n \to \infty$. Therefore, also in this case the proof is complete.

Finally, assume $\lambda(0) = 0$ and $\lambda(x)/x = O(1)$ as $x \to 0^+$ and recall that for the differential operator A_0 defined by

$$A_0u(x) := \frac{x}{2}u''(x), \quad 0 < x < 1/2, \tag{5}$$

for every $u \in D_V(A_0) := \{u \in C([0,1/2]) \cap C^2(]0,1/2]) \mid \lim_{x\to 0^+} xu''(x) = 0\}$, $C^2([0,1/2])$ is a core: really, for a fixed $u \in D_V(A_0)$, the sequence $(u_n)_{n\in\mathbb{N}}$ defined in (3) satisfies

$$\|u_n - u\| \to 0, \quad \|A_0u_n - A_0u\| \to 0, \quad n \to \infty \tag{6}$$

(see also [3, Theorem 6.2.6, p. 436 and Theorem 6.3.5, p. 457]).

We show that $(u_n)_{n\in\mathbb{N}}$ also works for A. Indeed, choose $u \in D(A) = D_V(A_0)$ and observe that our assumption yields $|(1-x)\lambda(x) - x\mu(x)| \le Kx$ for any $x \in [0,1/2]$, $K > 0$ being a suitable constant. Thus, for every $x \in \,]0,1/2[$, we get

$$\begin{aligned}
|Au_n(x) - Au(x)| &\le |A_0u_n(x) - A_0u(x)| + Kx|u_n'(x) - u'(x)| \\
&\le \|A_0u_n - A_0u\| + 2Kx\Big(\sup_{x\le t\le 1/2} |u_n''(t) - u''(t)| + \sup_{x\le t\le 1/2} |u_n(t) - u(t)|\Big) \\
&\le \|A_0u_n - A_0u\| + 4K \sup_{x\le t\le 1/2} |(t/2)u_n''(t) - (t/2)u''(t)| + K\|u_n - u\| \\
&\le (4K+1)\|A_0u_n - A_0u\| + K\|u_n - u\|.
\end{aligned}$$

Since Au_n and Au are continuous, the above estimate still holds at 0, and therefore $\|Au_n - Au\| \to 0$ as $n \to \infty$ on account of (6). The proof of the theorem is now complete. ∎

Remark 2.4. When $0 < \lambda(0) < 1/2$ or $0 < \mu(1) < 1/2$, Theorem 2.3 fails to hold. Indeed, a characterization of $D(A)$ in terms of conditions (N_λ) and (N_μ), respectively, is not available, neither is $C^2([0,1]) \cap D(A)$ a core for A. In order to show this, we argue near 0 and, at this end, we may and do assume

$$Au(x) \approx \frac{x}{2}u''(x) + \lambda(0)u'(x)$$

for any $u \in D(A) := \{u \in C([0,\delta]) \cap C^2(]0,\delta]) \mid \lim_{x\to 0^+} Au(x) = 0\}$, $\delta > 0$ small enough.

If we set $\varphi(x) := x^{1-2\lambda(0)}$ $(0 \le x \le \delta)$, we have $A(\varphi) \equiv 0$ and therefore $\varphi \in D(A)$. Nevertheless $\varphi \notin C^1([0,\delta])$ and moreover $\lim_{x\to 0^+} x\varphi''(x) = -\infty$, i.e., (N_λ) is not satisfied. In addition φ cannot be approximated in the graph norm of A by functions in $C^2([0,\delta]) \cap D(A)$, i.e., $C^2([0,\delta]) \cap D(A)$ is not a core, as claimed. Indeed, suppose that for every $\varepsilon > 0$ a function $u \in C^2([0,\delta]) \cap D(A)$ may be found such that $\|u-\varphi\| < \varepsilon$ and $\|Au - A\varphi\| = \|Au\| < \varepsilon$. As a consequence we get

$$-2\varepsilon < t^{1-2\lambda(0)} \frac{d}{dt}(u'(t)\, t^{2\lambda(0)}) < 2\varepsilon, \qquad t \in]0,\delta].$$

Dividing all members in the above inequality by $t^{1-2\lambda(0)}$ and then integrating between 0 and s $(0 < s \le \delta)$ give

$$-\frac{\varepsilon\, s^{2\lambda(0)}}{\lambda(0)} < u'(s)\, s^{2\lambda(0)} < \frac{\varepsilon\, s^{2\lambda(0)}}{\lambda(0)}.$$

A further integration yields

$$u(0) - \frac{\varepsilon x}{\lambda(0)} < u(x) < u(0) + \frac{\varepsilon x}{\lambda(0)}, \qquad x \in]0,\delta],$$

whence, also in view of the assumption $\|u-\varphi\| < \varepsilon$,

$$u(\delta) < \varepsilon(1 + \delta/\lambda(0)), \qquad u(\delta) > \varphi(\delta) - \varepsilon = \delta^{1-2\lambda(0)} - \varepsilon,$$

which cannot both hold for any $\varepsilon > 0$.

As a final remark, we point out that, however, if $u \in C^1([0,\delta]) \cap C^2(]0,\delta])$ satisfies $u'(0) = 0$, then $u \in D(A)$. This can be easily seen as follows: consider the function $f(x) := xu'(x)$ $(0 \le x \le \delta)$ and observe that $f'(0) = u'(0) = 0$. Moreover, from $f'(x) = u'(x) + xu''(x)$ for any $x \in]0,\delta]$, we obtain $\lim_{x\to 0^+} xu''(x) = 0$, which, in turn, yields $\lim_{x\to 0^+} Au(x) = 0$.

3. The operator $\widetilde{A}$. This section is devoted to the study of the operator $\widetilde{A}$ defined in (1.2) in the space $L^1(0,1)$. Specifically, if we set

$$(3.1) \qquad \alpha(x) := \frac{x\,(1-x)}{2}, \qquad \beta(x) := (1-x)\lambda(x) - x\mu(x) \qquad (0 \le x \le 1),$$

with λ and μ continuous functions on $[0,1]$, we are interested in the differential operator

$$(3.2) \qquad \widetilde{A}v(x) := Bv(x) - (\alpha''(x) - \beta'(x))v(x), \qquad 0 < x < 1,$$

where B is the adjoint of the operator A defined in (2.2), i.e.,

$$(3.3) \qquad Bv(x) := \frac{d}{dx}((\alpha v)'(x) - \beta(x)v(x)) = \frac{d}{dx}\left(\frac{(\alpha W v)'(x)}{W(x)}\right),$$

W being defined as in (2.3).

Note that (3.2) or, equivalently, (3.3), makes sense for every $v \in L^1_{\mathrm{loc}}(0,1)$ such that the functions αv and $(\alpha v)' - \beta v$ belong to $AC_{\mathrm{loc}}(0,1)$.

Moreover, if v has a second derivative in $]0,1[$, then an easy computation shows that explicitly

$$(3.4) \quad \widetilde{A}v(x) := \frac{x(1-x)}{2} v''(x) + ((1-x)(1-\lambda(x)) - x(1-\mu(x)))v'(x).$$

Henceforth we assume that $\beta \in AC(0,1)$ with $\beta' \in L^\infty(0,1)$ so that the perturbing term $(\alpha''(x) - \beta'(x))v(x)$ in (3.2) is bounded and therefore any generation result about B still holds true for $\widetilde{A}$.

Following [15, p. 516], the classification of the boundary points 0 and 1 for the operator B is identical to the one already carried out for the operator A in Section 2, since the corresponding functions W, Q and R are the same in both cases. Consequently, on account of [6, Theorems 2.2 and 3.2], we have the following results:

(1) If $\lambda(0), \mu(1) \le 0$, i.e., if 0 and 1 are both exit boundary points, then B is a generator on the maximal domain

$$(3.5) \qquad D_{\mathrm{M}}(B) := \{v \in L^1(0,1) \mid Bv \in L^1(0,1)\}.$$

(2) If 0 and 1 are both regular boundary points (whenever $\lambda(0), \mu(1) \in]0,1/2]$) or both entrance boundary points (whenever $\lambda(0), \mu(1) \ge 1/2$), then B is a generator on the domain

$$(3.6) \qquad D_{\mathrm{N}}(B) := \{v \in D_{\mathrm{M}}(B) \mid \lim_{x\to 0,1}((\alpha v)'(x) - \beta(x)v(x)) = 0\}.$$

In addition, if $\lambda(0) \le 0$ and $\mu(1) > 0$ or, conversely, $\lambda(0) > 0$ and $\mu(1) \le 0$, then B is a generator on the domain

$$D_{\mathrm{MN}}(B) := \{v \in D_{\mathrm{M}}(B) \mid \lim_{x\to 1^-}((\alpha v)'(x) - \beta(x)v(x)) = 0\}$$

or, respectively, on the domain

$$D_{\mathrm{NM}}(B) := \{v \in D_{\mathrm{M}}(B) \mid \lim_{x\to 0^+}((\alpha v)'(x) - \beta(x)v(x)) = 0\}.$$

As already mentioned, all the above generation results still hold true for the operator $\widetilde{A}$: the corresponding domains are exactly the same, with B replaced by $\widetilde{A}$ everywhere.

A useful result is stated in the following proposition.

PROPOSITION 3.1. *If $\lambda(0), \mu(1) \le 0$, then $C^2([0,1])$ is a core for $\widetilde{A}$.*

Proof. First of all, we recall that, on account of the above discussion, $\widetilde{A}$ is a generator on its maximal domain $D_{\mathrm{M}}(\widetilde{A})$. Now consider the operator

$$\widetilde{A}_c u(x) := \frac{x\,(1-x)}{2} u''(x) + ((1-x)(1-\lambda(x)) - x(1-\mu(x))u'(x)$$

as acting on $C([0,1])$. Observe that $\widetilde{A}_c$ is closely related to the operator A defined in (2.2), with $1-\lambda$ and $1-\mu$ instead of λ and μ, respectively. As a consequence, the assumption $\lambda(0),\mu(1)\le 0$ together with the notations and the results of Section 2 (see, in particular, Theorem 2.3) shows that $(\widetilde{A}_c, D_M(\widetilde{A}_c))$ is a generator on $C([0,1])$ having $C^2([0,1])$ as a core. This implies, by definition, that

$$\overline{C^2([0,1])}^{c} = D_M(\widetilde{A}_c),$$

where $\overline{C^2([0,1])}^{c}$ denotes the closure of $C^2([0,1])$ with respect to the graph norm of $\widetilde{A}_c$. Therefore, on account of the Lumer–Phillips theorem (see, e.g., [22, Theorem 4.3, p. 14]), we get

$$(I-\widetilde{A}_c)(\overline{C^2([0,1])}^{c}) = (I-\widetilde{A}_c)(D_M(\widetilde{A}_c)) = C([0,1]),$$

which implies that $(I-\widetilde{A}_c)(\overline{C^2([0,1])}^{c})$ is dense in $L^1(0,1)$. Since, in view of (3.4), $\widetilde{A}=\widetilde{A}_c$ when both act on $\overline{C^2([0,1])}^{c} = D_M(\widetilde{A}_c)\subset D_M(\widetilde{A})$, it immediately follows that $(I-\widetilde{A})(\overline{C^2([0,1])}^{c})$ is dense in $L^1(0,1)$, i.e., $\overline{C^2([0,1])}^{c}$ is a core for $\widetilde{A}$. But then, *a fortiori*, the closure $\overline{C^2([0,1])}$ of $C^2([0,1])$ with respect to the graph norm of $\widetilde{A}$ (which is contained in $D_M(\widetilde{A})$ anyhow) is also a core for $\widetilde{A}$, which obviously yields that $C^2([0,1])$ is itself a core, as required. ∎

4. Sequences of Beta-type operators and convergence of their powers. The main aim of this section is to show how the semigroups generated by the differential operators A and $\widetilde{A}$ on $C([0,1])$ and $L^1(0,1)$, respectively, may be expressed, in some cases, in terms of powers of suitable positive linear operators of integral type, acting on the corresponding spaces.

Let us start with the following two lemmas which will be an essential tool in defining and studying the operators we are going to deal with.

LEMMA 4.1. *If $f\in L^\infty(0,1)$ is continuous at* 0, *then*

$$\lim_{\alpha\to 0^+}\alpha\int_0^1 t^{\alpha-1}f(t)\,dt = \lim_{\alpha\to\infty}(\alpha+1)\int_0^1 (1-t)^\alpha f(t)\,dt = f(0). \tag{4.1}$$

Similarly we get

$$\lim_{\alpha\to 0^+}\alpha\int_0^1 (1-t)^{\alpha-1}f(t)\,dt = \lim_{\alpha\to\infty}(\alpha+1)\int_0^1 t^\alpha f(t)\,dt = f(1), \tag{4.2}$$

provided f is continuous at 1.

Proof. We may restrict ourselves to proving (4.1) since (4.2) follows immediately via a change of variable.

Indeed, for fixed $\varepsilon>0$, choose $\delta\in\,]0,1[$ with $|f(t)-f(0)|<\varepsilon$ if $t\in[0,\delta[$. For $\alpha>0$ we readily estimate

$$\begin{aligned}\Big|\alpha\int_0^1 t^{\alpha-1}f(t)\,dt - f(0)\Big| &= \Big|\alpha\int_0^1 t^{\alpha-1}(f(t)-f(0))\,dt\Big|\\ &\le \alpha\int_0^\delta t^{\alpha-1}|f(t)-f(0)|\,dt + \alpha\int_\delta^1 t^{\alpha-1}|f(t)-f(0)|\,dt\\ &\le \varepsilon\delta^\alpha + M(1-\delta^\alpha)\le \varepsilon + M(1-\delta^\alpha),\end{aligned}$$

M being an upper bound of $|f-f(0)|$ in $[\delta,1]$. Following essentially the same lines, we also get

$$\Big|(\alpha+1)\int_0^1(1-t)^\alpha f(t)\,dt - f(0)\Big| \le \varepsilon + M(1-\delta)^{\alpha+1}.$$

Now (4.1) immediately follows from the above estimates, since $1-\delta^\alpha\to 0$ as $\alpha\to 0^+$ and the same happens to $(1-\delta)^{\alpha+1}$ as $\alpha\to\infty$. ∎

LEMMA 4.2. *Let $\alpha,\beta\in C([0,1])$. If $\alpha,\beta\ge 1$ and $f\in L^1(0,1)$, then the function*

$$x\mapsto \int_0^1 t^{\alpha(x)-1}(1-t)^{\beta(x)-1}f(t)\,dt\qquad (0\le x\le 1) \tag{4.3}$$

is continuous on $[0,1]$. *The same holds true if f is continuous on* $[0,1]$, *with* $\alpha,\beta>0$.

Proof. First observe that our assumptions ensure that the integrand in (4.3) is in $L^1(0,1)$, anyway. To check continuity when $\alpha,\beta\in C([0,1])$, $\alpha,\beta\ge 1$ and $f\in L^1(0,1)$, fix $x_0\in[0,1]$, consider a sequence $(a_n)_{n\in\mathbb{N}}$ in $[0,1]$ converging to x_0 and for each $n\in\mathbb{N}$ set

$$g_n(t) := t^{\alpha(a_n)-1}(1-t)^{\beta(a_n)-1}f(t),\qquad 0<t<1.$$

Then obviously

$$\lim_{n\to\infty} g_n(t) = t^{\alpha(x_0)-1}(1-t)^{\beta(x_0)-1}f(t)\qquad \text{a.e. in } [0,1]$$

and moreover $|g_n(t)|\le|f(t)|$ for every $n\in\mathbb{N}$ and $t\in\,]0,1[$. Since $f\in L^1(0,1)$ by assumption, Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^1 g_n(t)\,dt = \int_0^1 t^{\alpha(x_0)-1}(1-t)^{\beta(x_0)-1}f(t)\,dt,$$

i.e., (4.3) is continuous at x_0.

A similar argument applies to the case $\alpha,\beta,f \in C([0,1])$, $\alpha,\beta > 0$: simply define $\alpha_0 := \min_{0\le x\le 1}\alpha(x)$, $\beta_0 := \min_{0\le x\le 1}\beta(x)$ and observe that

$$|g_n(t)| \le \|f\|\, t^{\alpha_0-1}(1-t)^{\beta_0-1}, \quad n\ge 1,\ 0<t<1,$$

where the function on the right hand side is in $L^1(0,1)$ because $\alpha_0,\beta_0>0$. ■

Now choose $\lambda,\mu\in C([0,1])$ and for every $n\in\mathbb{N}$ and $x\in[0,1]$ define

$$\text{(4.4)}\qquad \begin{aligned} \lambda_n(x) &:= nx+\lambda(x), & \mu_n(x) &:= n(1-x)+\mu(x),\\ \widetilde{\lambda}_n(x) &:= nx-\lambda(x), & \widetilde{\mu}_n(x) &:= n(1-x)-\mu(x). \end{aligned}$$

Moreover set

$$\text{(4.5)}\qquad \begin{aligned} F_n &:= \{x\in[0,1] \mid \lambda_n(x)\le 0\}, & G_n &:= \{x\in[0,1] \mid \mu_n(x)\le 0\},\\ \widetilde{F}_n &:= \{x\in[0,1] \mid \widetilde{\lambda}_n(x)< 0\}, & \widetilde{G}_n &:= \{x\in[0,1] \mid \widetilde{\mu}_n(x)< 0\}. \end{aligned}$$

For n large enough the sets F_n and G_n lie very "close" to the endpoints 0 and 1, respectively; more precisely, it can be easily shown that for any $\delta\in\,]0,1/2[$, there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$ one has $F_n\subset[0,\delta[$ as well as $G_n\subset\,]1-\delta,1]$ and, consequently, $F_n\cap G_n=\emptyset$. Similarly, $\widetilde{F}_n\cap\widetilde{G}_n=\emptyset$ if n is large enough (without loss of generality, if $n\ge n_0$, as before).

After these preliminaries we are in a position to define two sequences of linear operators of integral type associated with λ and μ; namely, for every $n\ge n_0$ we consider $\mathcal{B}_{n,\lambda,\mu}: C([0,1])\to C([0,1])$ such that for every $f\in C([0,1])$ and $x\in[0,1]$ we have

$$\text{(4.6)}\qquad \mathcal{B}_{n,\lambda,\mu}f(x) := \begin{cases} \dfrac{\int_0^1 t^{\lambda_n(x)-1}(1-t)^{\mu_n(x)-1}f(t)\,dt}{B(\lambda_n(x),\mu_n(x))} & \text{if } \lambda_n(x),\mu_n(x)>0,\\ f(0) & \text{if } \lambda_n(x)\le 0<\mu_n(x),\\ f(1) & \text{if } \mu_n(x)\le 0<\lambda_n(x), \end{cases}$$

and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}: L^1(0,1)\to C([0,1])$ defined by

$$\text{(4.7)}\qquad \widetilde{\mathcal{B}}_{n,\lambda,\mu}f(x) := \begin{cases} \dfrac{\int_0^1 t^{\widetilde{\lambda}_n(x)}(1-t)^{\widetilde{\mu}_n(x)}f(t)\,dt}{B(\widetilde{\lambda}_n(x)+1,\widetilde{\mu}_n(x)+1)} & \text{if } \widetilde{\lambda}_n(x),\widetilde{\mu}_n(x)\ge 0,\\ (\widetilde{\mu}_n(x)+1)\displaystyle\int_0^1(1-t)^{\widetilde{\mu}_n(x)}f(t)\,dt & \text{if } \widetilde{\lambda}_n(x)<0\le\widetilde{\mu}_n(x),\\ (\widetilde{\lambda}_n(x)+1)\displaystyle\int_0^1 t^{\widetilde{\lambda}_n(x)}f(t)\,dt & \text{if } \widetilde{\mu}_n(x)<0\le\widetilde{\lambda}_n(x), \end{cases}$$

for any $f\in L^1(0,1)$ and $x\in[0,1]$, where $B(u,v):=\int_0^1 t^{u-1}(1-t)^{v-1}dt$ $(u,v>0)$ is the standard Beta function.

As an historical remark, we recall that Lupaş [18, p. 63 and p. 37] studied the operators $\mathcal{B}_{n,\lambda,\mu}$ and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$ in the particular case $\lambda=\mu=0$. In addition, as a first generalization, the operators $\mathcal{B}_{n,\lambda,\mu}$ when $\mu=\lambda$ and $\lambda([0,1])\subset\,]0,1]$ have been considered in [5] (see also the discussion below).

Note that our definitions of $\mathcal{B}_{n,\lambda,\mu}$ and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$ are meaningful: indeed, the corresponding kernel is in $L^1(0,1)$ in (4.6) and continuous in (4.7). Clearly, those operators are positive and linear with $\|\mathcal{B}_{n,\lambda,\mu}\|=\|\widetilde{\mathcal{B}}_{n,\lambda,\mu}\|=1$, as may be easily verified.

Moreover, a careful application of Lemmas 4.1 and 4.2 ensures that really each $\mathcal{B}_{n,\lambda,\mu}$ maps $C([0,1])$ into itself, whereas each $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$ maps $L^1(0,1)$ into $C([0,1])$, as well.

Also observe that, on account of the previous discussion about the properties of the sets defined in (4.5), for any $n\ge n_0$ the definitions (4.6) and (4.7) cover all the possible cases, since for such n the situations "$\lambda_n(x),\mu_n(x)\le 0$" or "$\widetilde{\lambda}_n(x),\widetilde{\mu}_n(x)<0$", respectively, never occur.

Finally, it seems worth while pointing out that if $\lambda(0)>0$ then the sets F_n are definitely empty and so are the sets G_n if $\mu(1)>0$. Therefore, whenever $\lambda(0)>0$ and $\mu(1)>0$, we get definitely

$$\text{(4.8)}\qquad \mathcal{B}_{n,\lambda,\mu}f(x)=\frac{\int_0^1 t^{\lambda_n(x)-1}(1-t)^{\mu_n(x)-1}f(t)\,dt}{B(\lambda_n(x),\mu_n(x))},\qquad f\in C([0,1]),$$

and, by a similar argument for $\widetilde{F}_n$ and $\widetilde{G}_n$,

$$\text{(4.9)}\qquad \widetilde{\mathcal{B}}_{n,\lambda,\mu}f(x)=\frac{\int_0^1 t^{\widetilde{\lambda}_n(x)}(1-t)^{\widetilde{\mu}_n(x)}f(t)\,dt}{B(\widetilde{\lambda}_n(x)+1,\widetilde{\mu}_n(x)+1)},\qquad f\in L^1(0,1),$$

provided $\lambda(0)<0$ and $\mu(1)<0$.

A first approximation result together with some quantitative estimates of the rate of convergence is indicated in the next theorem.

Theorem 4.3. *We have*

$$\text{(4.10)}\qquad \lim_{n\to\infty}\|\mathcal{B}_{n,\lambda,\mu}f-f\|=0,\qquad f\in C([0,1]),$$

$$\text{(4.11)}\qquad \lim_{n\to\infty}\|\widetilde{\mathcal{B}}_{n,\lambda,\mu}f-f\|_1=0,\qquad f\in L^1(0,1),$$

i.e., the sequences $(\mathcal{B}_{n,\lambda,\mu})_{n\ge n_0}$ and $(\widetilde{\mathcal{B}}_{n,\lambda,\mu})_{n\ge n_0}$ are positive approximation processes on $C([0,1])$ and $L^1(0,1)$, respectively. In addition, for n large enough,

$$\text{(4.12)}\qquad \|\mathcal{B}_{n,\lambda,\mu}f-f\|\le K_1(\omega(f,1/\sqrt{n})+(\|\lambda\|+\|\mu\|)\,\omega(f,1/n)),$$

$$\text{(4.13)}\qquad \|\widetilde{\mathcal{B}}_{n,\lambda,\mu}f - f\|_1 \le 748\,\tau\left(f, \frac{K_2}{2\sqrt{n}}\right)_1,$$

the positive constants K_i $(i = 0, 1)$ depending only on the functions λ and μ.

Proof. Indeed, by the elementary properties of the Beta function, it is easy to check that for every $n \ge n_0$, $p \in \mathbb{N}$ and $x \in [0,1]$ one has

$$\text{(1)}\qquad \mathcal{B}_{n,\lambda,\mu}e_p(x) = \begin{cases} \displaystyle\prod_{k=0}^{p-1} \frac{\lambda_n(x)+k}{\lambda_n(x)+\mu_n(x)+k} & \text{if } \lambda_n(x), \mu_n(x) > 0,\\ 0 & \text{if } \lambda_n(x) \le 0 < \mu_n(x),\\ 1 & \text{if } \mu_n(x) \le 0 < \lambda_n(x). \end{cases}$$

In particular, choose $i = 1, 2$ and observe that for fixed $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\left|\prod_{k=0}^{i-1} \frac{\lambda_n(x)+k}{\lambda_n(x)+\mu_n(x)+k} - e_i(x)\right| < \varepsilon, \qquad n \ge n_1,\ x \in [0,1].$$

Furthermore one can find $\delta \in\,]0, 1/2[$ such that $e_i(x) < \varepsilon$ if $x \in [0, \delta[$ and $1 - e_i(x) < \varepsilon$ if $x \in\,]1-\delta, 1]$. As a consequence, since $F_n \subset [0,\delta[$ and $G_n \subset\,]1-\delta, 1]$ for all $n \ge n_0$, after setting $\overline{n} = \max\{n_0, n_1\}$, in view of (1) we obtain

$$|\mathcal{B}_{n,\lambda,\mu}e_i(x) - e_i(x)| < \varepsilon, \qquad n \ge \overline{n},\ x \in [0,1].$$

Obviously $\mathcal{B}_{n,\lambda,\mu}e_0 = e_0$ for every $n \ge n_0$ and therefore we have proved that

$$\lim_{n\to\infty} \mathcal{B}_{n,\lambda,\mu}e_i = e_i, \qquad i = 0, 1, 2,$$

uniformly on $[0,1]$. This implies (4.10) on account of Korovkin's theorem (see, e.g., [3, Proposition 4.2.4, p. 214]).

In order to prove (4.11), as before we compute

$$\text{(2)}\qquad \widetilde{\mathcal{B}}_{n,\lambda,\mu}e_p(x) = \begin{cases} \displaystyle\prod_{k=0}^{p-1} \frac{\widetilde{\lambda}_n(x)+k+1}{\widetilde{\lambda}_n(x)+\widetilde{\mu}_n(x)+k+2} & \text{if } \widetilde{\lambda}_n(x), \widetilde{\mu}_n(x) \ge 0,\\ \displaystyle\prod_{k=0}^{p-1} \frac{p}{\widetilde{\mu}_n(x)+k+2} & \text{if } \widetilde{\lambda}_n(x) < 0 \le \widetilde{\mu}_n(x),\\ \dfrac{\widetilde{\lambda}_n(x)+1}{\widetilde{\lambda}_n(x)+p+1} & \text{if } \widetilde{\mu}_n(x) < 0 \le \widetilde{\lambda}_n(x), \end{cases}$$

for every $n \ge n_0$, $p \in \mathbb{N}$ and $x \in [0,1]$.

Moreover the sets $\widetilde{F}_n$ and $\widetilde{G}_n$ defined in (4.5) are both measurable with $m(\widetilde{F}_n), m(\widetilde{G}_n) \to 0$ as $n \to \infty$, m being the Lebesgue measure. In view of (2), for all $n \ge n_0$ and $i = 1, 2$, since $\widetilde{F}_n$ and $\widetilde{G}_n$ may be supposed to be disjoint, we readily obtain

$$\begin{aligned} \int_0^1 &|\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_i(x) - e_i(x)|\,dx \\ &= \int_{\widetilde{F}_n} |\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_i(x) - e_i(x)|\,dx + \int_{\widetilde{G}_n} |\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_i(x) - e_i(x)|\,dx \\ &\quad + \int_{[0,1]\setminus(\widetilde{F}_n\cup\widetilde{G}_n)} |\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_i(x) - e_i(x)|\,dx \\ &\le m(\widetilde{F}_n) + m(\widetilde{G}_n) \\ &\quad + \int_{[0,1]\setminus(\widetilde{F}_n\cup\widetilde{G}_n)} \left|\prod_{k=0}^{i-1} \frac{\widetilde{\lambda}_n(x)+k+1}{\widetilde{\lambda}_n(x)+\widetilde{\mu}_n(x)+k+2} - e_i(x)\right| dx. \end{aligned}$$

Since clearly

$$\prod_{k=0}^{i-1} \frac{\widetilde{\lambda}_n(x)+k+1}{\widetilde{\lambda}_n(x)+\widetilde{\mu}_n(x)+k+2} - e_i(x) \to 0$$

uniformly on $[0,1]$, the above estimate implies that

$$\lim_{n\to\infty} \int_0^1 |\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_i(x) - e_i(x)|\,dx = 0, \qquad i = 1, 2.$$

The same happens for $i = 0$, because $\widetilde{\mathcal{B}}_{n,\lambda,\mu}e_0 = e_0$ and therefore (4.11) is established as well, on account of Korovkin's theorem (see, e.g., [3, Proposition 4.2.5, p. 215]).

The estimate (4.12) may be easily established as in [5, Theorem 1.2, formula (1.10)], on account of the definition (4.6) and the elementary inequality $\omega(f, t\delta) \le (1+t)\omega(f,\delta)$ $(t, \delta \ge 0)$. To prove (4.13), set $d_n(x) := \widetilde{\mathcal{B}}_{n,\lambda,\mu}\psi_x^2(x)$ for $n \ge n_0$ and $x \in [0,1]$. Then, in view of (2), a direct computation yields

$$d_n(x) \approx \begin{cases} \frac{1}{n}x(1-x) & \text{if } \widetilde{\lambda}_n(x), \widetilde{\mu}_n(x) \ge 0,\\ x^2 & \text{if } \widetilde{\lambda}_n(x) < 0 \le \widetilde{\mu}_n(x),\\ (1-x)^2 & \text{if } \widetilde{\mu}_n(x) < 0 \le \widetilde{\lambda}_n(x), \end{cases} \qquad n \to \infty,$$

with $x^2 \le \|\lambda\|^2/n^2$ and $(1-x)^2 \le \|\mu\|^2/n^2$ (see (4.4)). Therefore, for a suitable constant $K_2 > 0$, we get $\|d_n\| \le K_2/(4n)$ for n large enough and this implies (4.13) by virtue of [23, Proposition 4.3, p. 77]. ■

The next theorem establishes a Voronovskaya-type result for the operators $\mathcal{B}_{n,\lambda,\mu}$ and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$.

THEOREM 4.4. *Consider the differential operators A and $\widetilde{A}$ defined in* (2.2) *and* (3.2), *respectively. Then, if $\lambda(0),\mu(1)\ge 0$, for every $u\in C^2([0,1])$,*

$$\lim_{n\to\infty} n(\mathcal{B}_{n,\lambda,\mu}u(x)-u(x))=Au(x) \tag{4.14}$$

uniformly with respect to $x\in[0,1]$. Similarly, for every $v\in C^2([0,1])$,

$$\lim_{n\to\infty} n\int_0^1 (\widetilde{\mathcal{B}}_{n,\lambda,\mu}v(x)-v(x))\,dx=\int_0^1 \widetilde{A}v(x)\,dx \tag{4.15}$$

provided $\lambda(0),\mu(1)\le 1$.

Proof. In order to prove (4.14) we shall apply a result by Mamedov [19]. First note that a direct computation shows that for any $n\ge n_0$ and $x\in[0,1]$ such that $\lambda_n(x),\mu_n(x)>0$ one has

$$\begin{aligned}
\mathcal{B}_{n,\lambda,\mu}\psi_x(x)&=\frac{(1-x)\lambda(x)-x\mu(x)}{\lambda_n(x)+\mu_n(x)},\\
\mathcal{B}_{n,\lambda,\mu}\psi_x^2(x)&=\frac{1}{(\lambda_n(x)+\mu_n(x))(\lambda_n(x)+\mu_n(x)+1)}\\
&\quad\times[nx(1-x)+(1-x)^2(\lambda^2(x)+\lambda(x))\\
&\quad+x^2(\mu^2(x)+\mu(x))+2x(1-x)\lambda(x)\mu(x)],\\
\mathcal{B}_{n,\lambda,\mu}\psi_x^4(x)&=\frac{P(n,x,\lambda(x),\mu(x))}{\prod_{k=0}^{3}(\lambda_n(x)+\mu_n(x)+k)},
\end{aligned} \tag{1}$$

$P(n,x,\lambda(x),\mu(x))$ being a polynomial in the variables n, x, $\lambda(x)$, $\mu(x)$ where the power of n is at most 2. For a given $\varepsilon>0$, since $\lambda(0)\ge 0$, $\mu(1)\ge 0$ by assumption, there exists $\delta\in\,]0,1/2[$ such that $-\lambda(x)+x(\lambda(x)+\mu(x))<\varepsilon$ and $-\varepsilon<x(\lambda(x)+\mu(x))<\varepsilon$ for $x\in[0,\delta[$ as well as $-\varepsilon<\mu(x)+(x-1)(\lambda(x)+\mu(x))$ and $-\varepsilon<(x-1)(\lambda(x)+\mu(x))<\varepsilon$ for $x\in\,]1-\delta,1]$. We may assume $F_n\subset[0,\delta[$ and $G_n\subset\,]1-\delta,1]$ for all $n\ge n_0$ and therefore, by using (1) and (4.6), it is not difficult to verify that there exists $\overline{n}\in\mathbb{N}$ such that

$$|n\mathcal{B}_{n,\lambda,\mu}\psi_x(x)-\lambda(x)(1-x)+x\mu(x)|<\varepsilon,\qquad n\ge\overline{n},\ x\in[0,1],$$

i.e.,

$$\lim_{n\to\infty} n\mathcal{B}_{n,\lambda,\mu}\psi_x(x)=\lambda(x)(1-x)-x\mu(x)$$

uniformly on $[0,1]$. Arguing similarly one can easily check that in addition $n\mathcal{B}_{n,\lambda,\mu}\psi_x^2(x)\to x(1-x)$ and $\mathcal{B}_{n,\lambda,\mu}\psi_x^4(x)\to 0$ uniformly on $[0,1]$ as $n\to\infty$. Furthermore for every $n\ge n_0$ we have $\sup_{0\le x\le 1} n\mathcal{B}_{n,\lambda,\mu}\psi_x^2(x)<\infty$, which completes the proof of (4.14) by the result of Mamedov [19].

To accomplish (4.15) when $\lambda(0),\mu(1)\le 1$, first observe that, when acting on $C^2([0,1])$, the operator $\widetilde{A}$ may be written as in (3.4). Now, for fixed $n\ge n_0$ and $v\in C^2([0,1])$, we may estimate

$$\begin{aligned}
\Big|\int_0^1 [n(\widetilde{\mathcal{B}}_{n,\lambda,\mu}v(x)-v(x))-\widetilde{A}v(x)]\,dx\Big|
&\le \int_{\widetilde{F}_n} n|\widetilde{\mathcal{B}}_{n,\lambda,\mu}v(x)-v(x)|\,dx+\int_{\widetilde{G}_n} n|\widetilde{\mathcal{B}}_{n,\lambda,\mu}v(x)-v(x)|\,dx\\
&\quad+\|\widetilde{A}v\|(m(\widetilde{F}_n)+m(\widetilde{G}_n))\\
&\quad+\int_{[0,1]\setminus(\widetilde{F}_n\cup\widetilde{G}_n)} |n(\mathcal{B}_{n,1-\lambda,1-\mu}v(x)-v(x))-\widetilde{A}v(x)|\,dx\\
&=: I_1+I_2+I_3+I_4.
\end{aligned} \tag{2}$$

This is quite easy to see in view of (4.4)–(4.7): simply observe that if $x\in[0,1]\setminus(\widetilde{F}_n\cup\widetilde{G}_n)$, i.e., if $\widetilde{\lambda}_n(x),\widetilde{\mu}_n(x)\ge 0$, then obviously $(1-\lambda)_n(x)>0$, $(1-\mu)_n(x)>0$ and, in addition, $\widetilde{\mathcal{B}}_{n,\lambda,\mu}v(x)=\mathcal{B}_{n,1-\lambda,1-\mu}v(x)$.

Now we show that each I_i tends to 0 as $n\to\infty$. This is rather obvious for I_3 and I_4: indeed, $m(\widetilde{F}_n),m(\widetilde{G}_n)\to 0$ and the same happens for the integrand in I_4, uniformly on $[0,1]$, on account of the assumption $\lambda(0),\mu(1)\le 1$ and (4.14). Regarding I_1, since $\widetilde{F}_n\subset[0,\|\lambda\|/n]$, from (4.7) and the classical mean value theorem for integrals, for any $n\ge n_0$ we have

$$\begin{aligned}
I_1&\le n\int_0^{\|\lambda\|/n}\Big|(\widetilde{\mu}_n(x)+1)\int_0^1(1-t)^{\widetilde{\mu}_n(x)}v(t)\,dt-v(x)\Big|\,dx\\
&=\|\lambda\|\Big|(\widetilde{\mu}_n(s_n)+1)\int_0^1(1-t)^{\widetilde{\mu}_n(s_n)}v(t)\,dt-v(s_n)\Big|
\end{aligned} \tag{3}$$

for a suitable s_n between 0 and $\|\lambda\|/n$. Since $s_n\to 0$ and $\widetilde{\mu}_n(s_n)\to\infty$ as $n\to\infty$, in view of Lemma 4.1 and the continuity of v at 0, we conclude that the last term in (3) tends to 0, whence $I_1\to 0$, too. In a similar fashion one can prove that $I_2\to 0$ and therefore (4.15) is fully established. ∎

REMARK 4.5. Observe that for any compact subset K of $]0,1[$ and for any $f\in C^2(K)$,

$$\lim_{n\to\infty}\mathcal{B}'_{n,\lambda,\mu}f=f'$$

uniformly on K, provided, in addition, $\lambda,\mu\in C^1(K)$; here we still denote by f a C^2-continuation of f to $[0,1]$.

Indeed, for n large enough and $x\in K$, $\mathcal{B}_{n,\lambda,\mu}f(x)$ is defined as in (4.8) (see the discussion after definition (4.5)); moreover, differentiating under the integral sign yields

$$\mathcal{B}'_{n,\lambda,\mu}f(x) = \lambda'_n(x)(\mathcal{B}_{n,\lambda,\mu}(f\log)(x) - \mathcal{B}_{n,\lambda,\mu}f(x)\mathcal{B}_{n,\lambda,\mu}\log(x)) + \mu'_n(x)(\mathcal{B}_{n,\lambda,\mu}(f\log(1-\cdot))(x) - \mathcal{B}_{n,\lambda,\mu}f(x)\mathcal{B}_{n,\lambda,\mu}\log(1-\cdot)(x))$$

where again all the functions at which we evaluate $\mathcal{B}_{n,\lambda,\mu}$ have to be understood as C^2-continuations to $[0,1]$. A repeated application of (4.10) and (4.14) gives the assertion.

Finally, note that if $f \in C^2(]0,1[)$ then

$$\lim_{n\to\infty} \mathcal{B}'_{n,\lambda,\mu}f = f'$$

pointwise on $]0,1[$ and uniformly on every compact subset of $]0,1[$.

Now we are in a position to state the main results of this section about the existence of C_0-semigroups on $C([0,1])$ and $L^1(0,1)$, having A and $\widetilde{A}$ as generators and which may be expressed as the limit of powers of the operators $\mathcal{B}_{n,\lambda,\mu}$ and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$, respectively. To this end we consider the linear operators $Z_{\lambda,\mu} : D(Z_{\lambda,\mu}) \to C([0,1])$ and $\widetilde{Z}_{\lambda,\mu} : D(Z_{\lambda,\mu}) \to L^1(0,1)$ defined by

$$Z_{\lambda,\mu}(f) := \lim_{n\to\infty} n(\mathcal{B}_{n,\lambda,\mu}f - f), \tag{4.16}$$

$$\widetilde{Z}_{\lambda,\mu}(f) := \lim_{n\to\infty} n\,(\widetilde{\mathcal{B}}_{n,\lambda,\mu}f - f) \tag{4.17}$$

on the maximal domains

$$D(Z_{\lambda,\mu}) = \{f \in C([0,1]) \mid \lim_{n\to\infty} n(\mathcal{B}_{n,\lambda,\mu}(f) - f) \in C([0,1]) \text{ exists}\},$$
$$D(\widetilde{Z}_{\lambda,\mu}) = \{f \in L^1(0,1) \mid \lim_{n\to\infty} n(\widetilde{\mathcal{B}}_{n,\lambda,\mu}f - f) \in L^1(0,1) \text{ exists}\},$$

which are dense in the corresponding spaces, because $C^2([0,1])$ is a subset of them both by Theorem 4.4.

The first theorem relates to the operator A (see (2.2)), where the domain $D(A)$ is defined according to (2.7).

THEOREM 4.6. *Let $\lambda,\mu \in C([0,1])$ be such that $\lambda(0),\mu(1) \geq 1/2$ or, alternatively, $\lambda(0)=\mu(1)=0$ with $\lambda(x)/x=O(1)$ as $x \to 0^+$ and $\mu(x)/(1-x) = O(1)$ as $x \to 1^-$. There exists a C_0-semigroup $(T_{\lambda,\mu}(t))_{t\geq 0}$ of positive contractions on $C([0,1])$ with generator $(A, D(A))$ such that for every $t \geq 0$ and for every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, we have*

$$T_{\lambda,\mu}(t) = \lim_{n\to\infty} \mathcal{B}^{k(n)}_{n,\lambda,\mu} \quad \textit{strongly on } C([0,1]). \tag{4.18}$$

In particular, for every $t \geq 0$,

$$T_{\lambda,\mu}(t) = \lim_{n\to\infty} \mathcal{B}^{[nt]}_{n,\lambda,\mu} \quad \textit{strongly on } C([0,1]), \tag{4.19}$$

$[nt]$ being the integer part of nt.

Proof. Under the above assumptions on $\lambda(0)$ and $\mu(1)$, $(A, D(A))$ is the generator of a C_0-semigroup on $C([0,1])$, as already pointed out in Section 2. On account of Theorem 2.3, $C^2([0,1])$ is a core for A and therefore $(I-A)(C^2([0,1]))$ is dense in $C([0,1])$. Applying Voronovskaya's formula (4.14), in view of (4.16), we get $Z_{\lambda,\mu} = A$ on $C^2([0,1])$ and consequently $(I - Z_{\lambda,\mu})(C^2([0,1])) = (I-A)(C^2([0,1]))$ is dense in $C([0,1])$, which obviously implies that the range of $I - Z_{\lambda,\mu}$ is dense as well. Since $\|\mathcal{B}_{n,\lambda,\mu}\| = 1$ for every $n \geq n_0$, on account of a result by Trotter [26] (see also [17, Chapter 9, Theorem 3.6] as well as [22, Theorem 6.7, p. 96] or [13, Theorem 6.5, p. 31]), we conclude that the operator $(Z_{\lambda,\mu}, D(Z_{\lambda,\mu}))$ is closable and its closure $(\overline{Z}_{\lambda,\mu}, D(\overline{Z}_{\lambda,\mu}))$ is the generator of a C_0-semigroup $(T_{\lambda,\mu}(t))_{t\geq 0}$ of contractions on $C([0,1])$ satisfying (4.18).

Obviously every $T_{\lambda,\mu}(t)$ is positive. Moreover, $\overline{Z}_{\lambda,\mu} = A$ on $C^2([0,1])$, proving that $C^2([0,1])$ is also a core for $\overline{Z}_{\lambda,\mu}$. Therefore $D(\overline{Z}_{\lambda,\mu}) = D(A)$ and $\overline{Z}_{\lambda,\mu} = A$, as required. ∎

In a very similar way, by using Proposition 3.1 together with formula (4.15), one can prove the following result concerning the operator $(\widetilde{A}, D_{\mathrm{M}}(\widetilde{A}))$ defined according to (3.2) and (3.5) (with B replaced by $\widetilde{A}$).

THEOREM 4.7. *Let $\lambda,\mu \in C([0,1])$ be such that $\lambda(0),\mu(1) \leq 0$. There exists a C_0-semigroup $(\widetilde{T}_{\lambda,\mu}(t))_{t\geq 0}$ of positive contractions on $L^1(0,1)$ with generator $(\widetilde{A}, D_{\mathrm{M}}(\widetilde{A}))$ such that for every $t \geq 0$ and for every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, we have*

$$\widetilde{T}_{\lambda,\mu}(t) = \lim_{n\to\infty} \widetilde{\mathcal{B}}^{k(n)}_{n,\lambda,\mu} \quad \textit{strongly on } L^1(0,1). \tag{4.20}$$

In particular, for every $t \geq 0$,

$$\widetilde{T}_{\lambda,\mu}(t) = \lim_{n\to\infty} \widetilde{\mathcal{B}}^{[nt]}_{n,\lambda,\mu} \quad \textit{strongly on } L^1(0,1). \tag{4.21}$$

As a direct consequence of Theorems 4.6 and 4.7, by classical semigroup arguments (see, e.g., [21, Chapter A–II] or [22, Chapter 4]), the following Cauchy problems:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), & t \geq 0, \\ u(0) = u_0, & u_0 \in D(A), \end{cases} \qquad \begin{cases} \dfrac{dv(t)}{dt} = \widetilde{A}v(t), & t \geq 0, \\ v(0) = v_0, & v_0 \in D_{\mathrm{M}}(\widetilde{A}), \end{cases}$$

have unique classical solutions $u(\cdot)$ and $v(\cdot)$ such that

$$(4.22)\qquad \begin{aligned} u(t) &= T_{\lambda,\mu}(t)u_0 = \lim_{n\to\infty} \mathcal{B}_{n,\lambda,\mu}{}^{[nt]}u_0,\\ v(t) &= \widetilde{T}_{\lambda,\mu}(t)v_0 = \lim_{n\to\infty} \widetilde{\mathcal{B}}^{[nt]}_{n,\lambda,\mu}v_0, \end{aligned} \qquad t \ge 0.$$

REMARK 4.8. From (4.22) it follows that a better acquaintance with the properties of the Beta-type operators $\mathcal{B}_{n,\lambda,\mu}$ and $\widetilde{\mathcal{B}}_{n,\lambda,\mu}$ would allow one to derive some qualitative information about the solutions $u(\cdot)$ and $v(\cdot)$ of the above Cauchy problems. In our case, the functions λ and μ appearing in the definitions of our operators are very general; however, we point out that for particular λ and μ a finer and rather exhaustive analysis is already well known: we refer the reader to [5, 18] and to the references quoted therein in this respect.

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