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On having a countable cover by sets of small local diameter

by

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Abstract. A characterization of topological spaces admitting a countable cover by sets of small local diameter close in spirit to known characterizations of fragmentability is obtained. It is proved that if X and Y are Hausdorff compacta such that $C(X)$ has an equivalent p-Kadec norm and $C_{\mathrm{p}}(Y)$ has a countable cover by sets of small local norm diameter, then $C_{\mathrm{p}}(X\times Y)$ has a countable cover by sets of small local norm diameter as well.

1. Introduction. The topological properties of the space (E,w) where E is a Banach space and w is its weak topology (as well as of the spaces $C_{\mathrm{p}}(X)$ of continuous functions on X with the pointwise topology) have proved to be important for studying the Banach space E (or $C(X)$) itself. These properties are closely related to the properties of mappings into the Banach space in question (e.g. existence of Baire class 1 selectors, co-Namioka property, single-valuedness almost everywhere of uscos, cf. [9], [7] and others), the descriptive properties of its subsets (e.g. coincidence of the Borel σ-algebras generated by the weak and the norm topologies, cf. [2], [16] and others). Most striking perhaps is the close connection with the renorming properties of the space. We recall some definitions:

DEFINITION 1.1. Let E be a linear space and τ be a linear topology on it. A norm $\|\cdot\|$ on E is said to be *τ-Kadec* (or just *Kadec* if τ is the weak topology) if τ and the norm topology coincide on the unit sphere. A norm is said to be *locally uniformly rotund* (LUR) if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in the unit sphere and x is a point in the unit sphere such that $\lim_{n\to\infty}\|x_n+x\|=2$ then $\{x_n\}_{n=1}^{\infty}$ converges to x in norm.

The LUR norms which are τ-lower semicontinuous are τ-Kadec.

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In [6] Jayne, Namioka and Rogers introduced and deeply studied the notion of σ-fragmentability. In particular they proved that every Banach space admitting an equivalent Kadec norm is σ-fragmentable (using differences of weakly closed sets). Moreover, they showed that such a Banach space satisfies a stronger condition they called "having a countable cover by sets of small local diameter":

DEFINITION 1.2 (see [6]). Let (X,τ) be a topological space and ϱ be a metric on it. We say that X *has a countable cover by sets of small local ϱ-diameter* (or ϱ-SLD for short) if for every $\varepsilon > 0$ one can split the space into countably many parts,

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon},$$

in such a way that for every positive integer n and every point $x \in X_n^{\varepsilon}$ there exists a τ-open set U containing x and satisfying

$$\varrho\text{-diam}(U \cap X_n^{\varepsilon}) < \varepsilon.$$

Moltó, Orihuela and Troyanski [11] characterized the Banach spaces E admitting an equivalent LUR norm as those spaces E for which (E,w) has a special kind of $\|\cdot\|$-SLD, namely the weakly open sets U appearing in the definition of the SLD property should be open halfspaces.

The topological properties of spaces having ϱ-SLD were deeply studied in [12], [5]. M. Raja [16] proved that in the context of a Banach space with weak topology the $\|\cdot\|$-SLD property is almost equivalent to the existence of a Kadec renorming:

THEOREM 1.3 (cf. [15], [16]). *Let E be a Banach space, τ be a vector topology coarser than the norm topology and $\overline{B}_E^{\tau}$ be bounded. Then the following are equivalent*:

(i) (E,τ) *has* $\|\cdot\|$-*SLD*;

(ii) *For every constant* $c > 1$ *there exists a nonnegative symmetric positive homogeneous τ-lower semicontinuous function F on E with*

$$\|\cdot\| \le F \le c\|\cdot\|$$

and such that the norm topology and τ coincide on the set

$$S = \{x \in E : F(x) = 1\}.$$

This paper is devoted to the property of having a countable cover by sets of small local diameter. In the second section we show that, roughly speaking, for a topological space (X,τ) the existence of a metric ϱ on it such that X has ϱ-SLD is equivalent to (X,τ) being a Gruenhage space (see Definition 2.1 below). The last notion appeared more than ten years ago in [4] under the name of a "topological space admitting a σ-distributively point-finite T_0-separating cover". In [18] a characterization was given for such spaces close in spirit to the characterization of fragmentable spaces obtained in [17].

We hope that the investigation we present here may illuminate the common features as well as the substantial differences between spaces admitting ϱ-SLD and fragmentable (σ-fragmentable) spaces. It turns out that there exists an analogy between the relation σ-fragmentability$\leftrightarrow$fragmentability, established by Kenderov and Moors in [10], and the relation SLD$\leftrightarrow$Gruenhage spaces, established in the second section. Moreover, in the Banach space context, the analogue of Theorem 1.4 of [10] (the existence of a fragmenting metric which majorizes the weak topology is equivalent to the existence of a fragmenting metric which majorizes the norm topology) holds true (cf. Corollary 2.6). In the third section we use these topological results to prove that if $C(X)$ admits an equivalent p-Kadec norm and $C_{\rm p}(Y)$ has SLD with respect to the norm for two Hausdorff compact spaces X and Y, then $C_{\rm p}(X \times Y)$ has $\|\cdot\|$-SLD. In fact, we do not need the full power of the assumption on $C(X)$. We can use the function F built in Raja's theorem (Theorem 1.3) provided that in addition it is *norm continuous*.

It was shown recently (cf. [13], [19], [14]) that if X and Y are Hausdorff compacta and $C_{\rm p}(X)$, $C_{\rm p}(Y)$ are σ-fragmentable, then so is $C_{\rm p}(X \times Y)$. It would be interesting to know whether $C_{\rm p}(X \times Y)$ has $\|\cdot\|$-SLD provided $C_{\rm p}(X)$ and $C_{\rm p}(Y)$ have this property. Another open question concerns the stability of the existence of an equivalent p-Kadec renorming under this operation. A natural question is also: if X and Y are Hausdorff compacta such that $C(X)$ and $C(Y)$ admit a (pointwise lower semicontinuous) LUR renorming, does $C(X \times Y)$ admit a (pointwise lower semicontinuous) LUR norm as well? This is answered affirmatively in a forthcoming work of V. D. Babev and the author. The proof strongly relies on the construction implemented in the proof of Theorem 3.1 below.

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2. Topological characterization of SLD. The following definition can be found in [18].

DEFINITION 2.1. A well ordered family $\mathcal{U} = \{U_\xi : 1 \le \xi \le \overline{\xi}\}$ of subsets of a topological space X is said to be a *G-relatively open partition* of X if

(i) U_1 is open in X;

(ii) $\{U_\xi : 2 \le \xi < \overline{\xi}\}$ is a disjoint family of relatively open subsets of $X \setminus U_1$;

(iii) $U_{\overline{\xi}} = X \setminus (\bigcup_{\xi<\overline{\xi}} U_\xi)$.

The family $\{U_\xi : 2 \le \xi < \bar{\xi}\}$ will be called the *middle level* of $\mathcal{U}$.

The family $\mathcal{U}$ of subsets of X will be called a *σ-G-relatively open partition* if it is the union of countably many G-relatively open partitions $\mathcal{U}^n$, $n = 1, 2, \ldots$, of X. We say that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}^n$ *separates the points* of X if whenever x and y are two distinct points of X there exists a positive integer n such that x and y belong to different elements of $\mathcal{U}^n$. A topological space X is said to be a *Gruenhage space* if it admits a separating σ-G-relatively open partition.

Having a σ-G-relatively open partition $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}^n$, we denote by $\varrho(\mathcal{U})$ the following metric on X:

$$\varrho(\mathcal{U})(x, y) = \begin{cases} (\min\{n : \mathcal{U}^n \text{ separates } x \text{ and } y\})^{-1} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

PROPOSITION 2.2. *Let (X, τ) be a regular topological space and ϱ be a metric on it, satisfying at least one of the following two conditions:*

(a) *ϱ is lower semicontinuous with respect to τ;*
(b) *τ_ϱ (the topology generated by ϱ) is stronger than τ.*

If (X, τ) has a countable cover by sets of small local ϱ-diameter, then there exists a separating σ-G-relatively open partition $\mathcal{U}$ of X. Moreover, $\tau_{\varrho(\mathcal{U})} \ge \tau_\varrho$ in case (a) *and $\tau_{\varrho(\mathcal{U})} \ge \tau$ in case* (b).

Proof. In this proof, unless otherwise specified, all topological notions refer to the original topology τ.

Let $l \in \mathbb{N}$. We consider the cover of X consisting of all ϱ-open balls with radius $1/l$. This cover has a σ-discrete (with respect to τ_ϱ) refinement consisting of ϱ-open sets. Denote it by $\mathcal{B}^l = \bigcup_{n=1}^{\infty} \mathcal{B}_n^l$, where $\mathcal{B}_n^l$ are ϱ-discrete families of ϱ-open sets of ϱ-diameter less than or equal to $2/l$. On the other hand, by assumption, for every $m \in \mathbb{N}$ the space can be expressed as $X = \bigcup_{k=1}^{\infty} X_k^m$ in such a way that for all $k \in \mathbb{N}$ and $x \in X_k^m$ there exists an open set U containing x with $\varrho\text{-diam}(U \cap X_k^m) \le 1/m$. Now for each quadruple (l, n, m, k) we define a G-relatively open partition $\mathcal{U}^{lmnk}$ of (X, τ) in the following way:

$$U_1^{lnmk} = X \setminus \overline{X_k^m},$$
$$U_B^{lnmk} = \{x \in \overline{X_k^m} : \text{there exists } U \text{ open}, x \in U,\ \emptyset \ne U \cap X_k^m \subset B\}$$

for every $B \in \mathcal{B}_n^l$. Some of these sets may be empty, but all of them are obviously relatively open subsets of $X \setminus U_1^{lnmk} = \overline{X_k^m}$. Note that

$$U_{B_1}^{lnmk} \cap U_{B_2}^{lnmk} = \emptyset \quad \text{if } B_1 \ne B_2.$$

Indeed, if $V = U_{B_1}^{lnmk} \cap U_{B_2}^{lnmk}$ is not empty, then $V \cap X_k^m \ne \emptyset$ because V is relatively open and X_k^m is dense in $\overline{X_k^m}$. On the other hand, $V \cap X_k^m \subset U_{B_i}^{lnmk} \cap X_k^m \subset B_i$ for $i = 1, 2$. Hence $V \cap X_k^m \subset B_1 \cap B_2 = \emptyset$ (as $\mathcal{B}_n^l \supset \{B_1, B_2\}$ is discrete), a contradiction. So $\{U_B^{lnmk} : B \in \mathcal{B}_n^l\}$ is a disjoint family of relatively open subsets of $\overline{X_k^m}$. The last element of this G-relatively open partition $\mathcal{U}^{lnmk}$ will consist of the remaining points.

Let us see that $\{\mathcal{U}^{lnmk} : l, m, n, k = 1, 2, \ldots\}$ separates the points of X. We choose $x \ne y$ in X. In case (a) we fix $l \in \mathbb{N}$ such that $2/l < \varrho(x, y)$. In case (b) we fix an open set V separating x and y, that is, $x \in V$, $y \notin \overline{V}$ (X is Hausdorff). Then $\tau_\varrho \ge \tau$ implies that there exists $l \in \mathbb{N}$ such that the ball $B(x, 2/l)$ is contained in V. Now for both cases, using the fact that $\mathcal{B}^l$ is a cover of X, we find n and $B \in \mathcal{B}_n^l$ with $x \in B$. The set B is ϱ-open and so there exists $m \in \mathbb{N}$ such that $\overline{B}_\varrho(x, 1/m) \subset B$. As $X = \bigcup_{k=1}^{\infty} X_k^m$ for this m, we finally fix k with $x \in X_k^m$. Let $U \ni x$ be an open set satisfying $\varrho\text{-diam}(X_k^m \cap U) \le 1/m$. Then $\emptyset \ne X_k^m \cap U \subset B$. Therefore $x \in U_B^{lnmk}$. Note that $\varrho\text{-diam}(B) \le 2/l$ yields $\varrho\text{-diam}(U_B^{lnmk} \cap X_k^m) \le 2/l$.

Is it possible to have $y \in U_B^{lnmk}$ as well? In case (a) the lower semicontinuity of ϱ yields

$$\varrho\text{-diam}(\overline{X_k^m} \cap U) = \varrho\text{-diam}(X_k^m \cap U)$$

where U is an open set with $U_B^{lnmk} = U \cap \overline{X_k^m}$. Indeed, assume the contrary, i.e.

$$\varrho\text{-diam}(\overline{X_k^m} \cap U) > \varrho\text{-diam}(X_k^m \cap U) = d.$$

Then there exist z_1, z_2 in $\overline{X_k^m} \cap U$ with $\varrho(z_1, z_2) > d$. Now by the lower semicontinuity of ϱ we find open sets V_i with $z_i \in V_i \subset U$, $i = 1, 2$, such that $\varrho(y_1, y_2) > d$ whenever $y_1 \in V_1$, $y_2 \in V_2$. But $z_i \in \overline{X_k^m}$ shows that we can select two points $y_i \in X_k^m \cap V_i$, $i = 1, 2$. Now $\varrho(y_1, y_2) > d$ contradicts $y_1, y_2 \in X_k^m \cap U$ and $\varrho\text{-diam}(X_k^m \cap U) \le d$.

Using the above equality, we can estimate

$$\varrho\text{-diam}(U_B^{lnmk}) = \varrho\text{-diam}(U \cap \overline{X_k^m}) = \varrho\text{-diam}(U_B^{lnmk} \cap X_k^m) \le 2/l,$$

which shows that $\varrho(x, y) > 2/l$ yields $y \notin U_B^{lnmk}$. Moreover, if we choose l arbitrarily small, the above reasoning shows that there exist n, m, k and a set B such that the element U_B^{lnmk} of the G-relatively open partition $\mathcal{U}^{lnmk}$ contains x and is contained in $B_\varrho(x, 2/l)$. This means that $\tau_{\varrho(\mathcal{U})} \ge \tau_\varrho$.

In case (b) the estimate $\varrho\text{-diam}(U_B^{lnmk} \cap X_k^m) \le 2/l$ together with the choice of l gives $U_B^{lnmk} \cap X_k^m \subset V$. Therefore

$$U_B^{lnmk} = U_B^{lnmk} \cap \overline{X_k^m} \subset \overline{U_B^{lnmk} \cap X_k^m} \subset \overline{V}.$$

As $y \notin \overline{V}$, this finishes the proof that $\mathcal{U}^{lnmk}$ separates x and y. Again, letting V be an arbitrary open neighbourhood of x, we derive the existence of l, n, m, k and $U_B^{lnmk} \in \mathcal{U}_{lnmk}$ with $x \in U_B^{lnmk} \subset \overline{V}$. The last inclusion and the regularity of X yield that $\tau_{\varrho(\mathcal{U})} \ge \tau$.

REMARK 2.3. Note that in the proposition above for every point x and every neighbourhood (ϱ-neighbourhood in case (a)) of x we found a

G-relatively open partition and its element *in the middle level* containing x and contained in the neighbourhood in question. Thus we proved that the union of the middle levels of the partitions constructed above is a σ-isolated (in τ) network for τ (τ_ϱ in case (a)). For the definition of σ-isolated network and related results see [12].

PROPOSITION 2.4. *Let (X,τ) be a topological space and $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}^n$ be a σ-G-relatively open partition which separates the points of X. Then X has a countable cover by sets of small local $\varrho(\mathcal{U})$-diameter. Moreover, if ϱ is a metric on X with $\tau_{\varrho(\mathcal{U})}\ge\tau_\varrho$, then X has a countable cover by sets of small local ϱ-diameter.*

Proof. Let $\mathcal{U}^n=\{U_\xi^n:1\le\xi<\xi_n+1\}$. We define

$$E_1^n=U_1^n,\quad E_2^n=\bigcup_{1<\xi<\xi_n}U_\xi^n,\quad E_3^n=U_{\xi_n}^n.$$

Let $\varepsilon>0$ and n be such that $1/n<\varepsilon$. We put

$$X_{i_1\ldots i_n}^{\varepsilon}=E_{i_1}^1\cap\ldots\cap E_{i_n}^n$$

where $i_j\in\{1,2,3\}$ for every $j\in\{1,\ldots,n\}$. It is clear that

$$X=\bigcup\{X_{i_1\ldots i_n}^{\varepsilon}:\ i_j\in\{1,2,3\},\ j\in\{1,\ldots,n\}\}.$$

Note that these sets are only finitely many. Fix x in $X_{i_1\ldots i_n}^{\varepsilon}$. For a fixed $k\in\{1,\ldots,n\}$ we define V_k to be U_1^k if $x\in U_1^k$, $U_1^k\cup U_\xi^k$ if $x\in U_\xi^k$ for some $\xi\in(1,\xi_k)$, and all of X if $x\in U_{\xi_k}^k$. The sets V_k are open in X and contain x. Their intersection $V=\bigcap_{k=1}^{n}V_k$ is open, too, and contains x.

We now prove that $V\cap X_{i_1\ldots i_n}^{\varepsilon}$ has small $\varrho(\mathcal{U})$-diameter. Indeed, let $k\in\{1,\ldots,n\}$. We know that $x\in V\subset V_k$. If $i_k=1$, then $X_{i_1\ldots i_n}^{\varepsilon}\subset E_{i_k}^k=U_1^k$, so $V\cap X_{i_1\ldots i_n}^{\varepsilon}\subset U_1^k\in\mathcal{U}^k$. If $i_k=2$, then $X_{i_1\ldots i_n}^{\varepsilon}\subset\bigcup_{1<\xi<\xi_k}U_\xi^k$ and so $V\subset V_k=U_1^k\cup U_\xi^k$, where $x\in U_\xi^k$, $\xi\in(1,\xi_k)$. Thus $V\cap X_{i_1\ldots i_n}^{\varepsilon}\subset U_\xi^k$. If $i_k=3$, then $X_{i_1\ldots i_n}^{\varepsilon}\subset U_{\xi_k}^k\in\mathcal{U}^k$. Therefore $\mathcal{U}^k$ does not separate any two points of $V\cap X_{i_1\ldots i_n}^{\varepsilon}$, meaning that

$$\varrho(\mathcal{U})\text{-diam}(V\cap X_{i_1\ldots i_n}^{\varepsilon})\le\frac{1}{n+1}<\frac{1}{n}.$$

Let ϱ be a metric on X with $\tau_\varrho\le\tau_{\varrho(\mathcal{U})}$. We fix $\varepsilon>0$ and put

$$X_n^\varepsilon=\{x\in X:\varrho(x,y)<\varepsilon \text{ whenever } y\in X \text{ satisfies } \varrho(\mathcal{U})(x,y)<1/n\}.$$

It is clear that $X=\bigcup_{n=1}^{\infty}X_n^\varepsilon$ because of the relation between the topologies generated by ϱ and $\varrho(\mathcal{U})$. Now by the first part of the proof we can split each X_n^ε into finitely many parts so that the points in every part have relative neighbourhoods of $\varrho(\mathcal{U})$-diameter less than $1/n$. Then the same relative neighbourhoods will have ϱ-diameter less than 2ε.

REMARK 2.5. It is worth noting that in Proposition 2.4 the space X with respect to $\varrho(\mathcal{U})$ has a stronger property than just having a countable cover of sets of small local diameter. Namely for each $\varepsilon>0$ one can split the space in finitely many (not countably many as usual) subsets. There is an analogy with simplifying the property of "σ-fragmentability" to "fragmentability" when passing to a metric generating a stronger topology.

COROLLARY 2.6. *Let E be a Banach space, w be its weak topology and A be a nonempty subset of E. Then (A,w) has $\|\cdot\|_E$-SLD if and only if (A,w) has ϱ-SLD for some metric ϱ with $\tau_\varrho\ge w$.*

Indeed, the "only if" part is trivial, and to prove the "if" part we note that by Remark 2.3, A has a σ-isolated network for the weak topology. We apply (d)⇒(a) of Corollary 2 in [12] to finish the proof.

The following proposition requires no proof.

PROPOSITION 2.7. *Let $\mathcal{U}=\{U_\xi:1\le\xi\le\overline{\xi}\}$ be a G-relatively open partition of a topological space X.*

(a) *If X_0 is a subspace of X, then the family $\mathcal{U}\cap X_0:=\{U_\xi\cap X_0:1\le\xi\le\overline{\xi}\}$ is a G-relatively open partition of X_0.*

(b) *If $\mathcal{V}=\{V_\eta:1\le\eta\le\overline{\eta}\}$ is a G-relatively open partition of X, then*

$$\mathcal{U}\cap\mathcal{V}:=\{U_1\cup V_1\}\cup\{U_\xi\cap V_\eta:2\le\xi<\overline{\xi},\ 2\le\eta<\overline{\eta}\}\cup\{U_{\overline{\xi}}\cup V_{\overline{\eta}}\}$$

is a G-relatively open partition of X.

(c) *If $\mathcal{V}^\xi=\{V_\eta^\xi:1\le\eta\le\overline{\eta^\xi}\}$ is a G-relatively open partition of U_ξ for every $\xi\in[2,\overline{\xi})$, then the family*

$$\mathcal{U}^{\mathcal{V}}:=\Big\{U_1\cup\bigcup_{\xi\in[2,\overline{\xi})}V_1^\xi\Big\}\cup\{V_\eta^\xi:2\le\eta<\overline{\eta^\xi},\ 2\le\xi<\overline{\xi}\}\cup\Big\{U_{\overline{\xi}}\cup\bigcup_{\xi\in[2,\overline{\xi})}V_{\overline{\eta^\xi}}^\xi\Big\}$$

is a G-relatively open partition of X.

3. A stability property for SLD

THEOREM 3.1. *Let X and Y be Hausdorff compact spaces such that $C_{\rm p}(X)$ admits an equivalent p-Kadec norm and $C_{\rm p}(Y)$ has $\|\cdot\|$-SLD. Then $C_{\rm p}(X\times Y)$ has $\|\cdot\|$-SLD.*

Proof. We will develop a structure on $C_{\rm p}(X)$ for our purposes, and on $C_{\rm p}(Y)$ just the definition of $\|\cdot\|$-SLD will be sufficient. We will work in a (maybe) more general situation than having an equivalent Kadec norm, namely we will assume the following:

There is a nonnegative symmetric, homogeneous, norm continuous p-lower semicontinuous function F on $C_{\rm p}(X)$ with $\|h\|\le F(h)\le2\|h\|$ whenever $h\in C(X)$ and such that the norm topology and the pointwise topology coincide on the set $S=\{h\in C(X):F(h)=1\}$.

In the whole proof of the theorem $\varepsilon > 0$ will be fixed. For the fixed ε we will construct a σ-G-relatively open partition of $C_p(X \times Y)$ satisfying: for every $f \in C(X \times Y)$ there exists a middle level set in one of the countably many G-relatively open partitions which contains f and has norm diameter not greater than ε. This will be sufficient to prove the theorem because of Proposition 2.4 (and the possibility to give ε the values $1/n$, $n = 1, 2, \ldots$).

The following lemma is an adaptation of Lemma 2.4 of [6], with the norm replaced by the function F.

LEMMA 3.2. *Let h be in S and U be a pointwise neighbourhood of h with*

$$\operatorname{diam}(S \cap U) < \varepsilon.$$

Then there exists an elementary pointwise neighbourhood $W \subset U$ of h with

$$\operatorname{diam}(B \cap U) < \varepsilon$$

where $B = \{g \in C(X) : F(g) \le 1\}$.

Proof. Choose δ with $0 < \delta < 1$ so that

$$2\delta + \operatorname{diam}(S \cap U) < \varepsilon.$$

Then choose an elementary pointwise neighbourhood V of zero with $h + V \subset U$. Since V is a norm neighbourhood of zero, we can choose η with $0 < \eta < \delta < 1$ and $\{h \in C(X) : \|h\| \le \eta\} \subset \frac{1}{2}V$. But

$$\eta B = \{h \in C(X) : F(h) \le \eta\} \subset \{h \in C(X) : \|h\| \le \eta\} \subset \tfrac{1}{2}V.$$

As $F(h) = 1$, we have $h \notin (1-\eta)B$, a pointwise closed set. Let V' be an elementary neighbourhood of zero such that $(h + V') \cap (1-\eta)B = \emptyset$. Put

$$W := \left(h + \tfrac{1}{2}V\right) \cap (h + V').$$

This set is an elementary neighbourhood of h contained in U. We estimate the diameter of $B \cap W$. Suppose $h_1 \in B \cap W$. Write $h_1' = h_1/F(h_1)$. Then

$$F(h_1' - h_1) = F\left(\frac{h_1}{F(h_1)} - h_1\right) = |1 - F(h_1)| < \eta$$

because $h_1 \in B \cap W$ implies $F(h_1) > 1 - \eta$ ($h_1 \notin (1-\eta)B$) and $1 \ge F(h_1)$ ($h_1 \in B$). Thus $h_1' - h_1 \in \eta B \subset \frac{1}{2}V$ and $h_1 \in h + \frac{1}{2}V$, so that

$$h_1' = h_1 + (h_1' - h_1) \in h + \tfrac{1}{2}V + \tfrac{1}{2}V \subset h + V \subset U.$$

Moreover, $F(h_1') = 1$ and so h_1' is in S. Having $h_1' \in S \cap U$, we can similarly put $h_2' = h_2/F(h_2)$ for a second point $h_2 \in B \cap W$. Again $h_2' \in S \cap U$. Thus

$$\begin{aligned}\|h_1 - h_2\| &\le \|h_1 - h_1'\| + \|h_1' - h_2'\| + \|h_2' - h_2\| \\ &\le F(h_1' - h_1) + \operatorname{diam}(S \cap U) + F(h_2' - h_2) \\ &< 2\eta + \operatorname{diam}(S \cap U) < 2\delta + \operatorname{diam}(S \cap U) < \varepsilon.\end{aligned}$$

This shows that $\operatorname{diam}(B \cap W) < \varepsilon$ as required.

We will call a subset $W \subset C(X)$ a *finitely determined pointwise open set* if

$$W = \bigcap_{i=1}^{l} \Big(\bigcup_{j=1}^{m_i} \{h \in C(X) : h(x'_{ij}) > \alpha_{ij}\} \cup \bigcup_{k=1}^{n_i} \{h \in C(X) : h(x''_{ik}) < \beta_{ik}\} \Big)$$

where x'_{ij} and x''_{ik} are points in X and α_{ij}, β_{ik} are reals. Some of the unions may be empty. Then we set

$$W^{(-\delta)} = \bigcap_{i=1}^{l} \Big(\bigcup_{j=1}^{m_i} \{h : h(x'_{ij}) > \alpha_{ij} + \delta\} \cup \bigcup_{k=1}^{n_i} \{h : h(x''_{ik}) < \beta_{ik} - \delta\} \Big),$$

$$W^{(+\delta)} = \bigcap_{i=1}^{l} \Big(\bigcup_{j=1}^{m_i} \{h : h(x'_{ij}) > \alpha_{ij} - \delta\} \cup \bigcup_{k=1}^{n_i} \{h : h(x''_{ik}) < \beta_{ik} + \delta\} \Big).$$

LEMMA 3.3. *There exists a pointwise open set U containing S and countably many G-relatively open partitions $\mathcal{U}^n$ of $C_p(X)$ with first element $C(X) \setminus B$ and middle levels $\{U^n_\xi : 2 \le \xi < \overline{\xi^n}\}$, $n = 1, 2, \ldots$, having the following properties*:

(i) $\bigcup_{n=1}^{\infty} \bigcup_{\xi \in [2, \overline{\xi^n})} U^n_\xi = B \cap U$;

(ii) $\{U^n_\xi : 2 \le \xi < \overline{\xi^n},\ n \in \mathbb{N}\}$ *is a locally finite family in the norm topology*;

(iii) *for every n there exists $\delta_n > 0$ such that for each k-tuple $\widetilde{\xi} = (\xi_1 \ldots \xi_k)$, $2 \le \xi_1 < \ldots < \xi_k < \overline{\xi^n}$, there exist k disjoint pointwise open sets $W_1, \ldots, W_k$ which are finitely determined, $W_i^{(-\delta_n)} \supset U^n_{\xi_i}$ and*

$$\operatorname{diam}(W_i^{(+\delta_n)} \cap B) < \varepsilon \quad \textit{for every } i = 1, \ldots, k.$$

Proof. We apply the construction in the proof of Stone's theorem (see for example Theorem 4.4.1 of [3]) to the family of elementary pointwise open sets given by the previous lemma. Let $h \in S$. Let W_h be an elementary pointwise neighbourhood of h satisfying $\operatorname{diam}(W_h \cap B) < \varepsilon$ (it exists by the previous lemma). Similarly, for every n let W^n_h be an elementary pointwise open set containing h, contained in W_h and satisfying $\operatorname{diam}(W^n_h \cap B) < 1/2^n$. Well order $S = [0, \overline{\xi})$ and consider the family $\{W_\xi\}_{\xi \in [0,\overline{\xi})}$. It covers S. We define inductively

$$U^n_\xi := \Big(\bigcup_{h \in A^n_\xi} W^n_h \Big) \cap B$$

where A^n_ξ consists of all $h \in S$ satisfying:

(1) $\{g \in C(X) : \|h - g\| \le 3/2^n\} \subset W_\xi$;

(2) $h \notin W_\eta$ whenever $\eta < \xi$;

(3) $h \notin U^l_\eta$ for every $l < n$ and every $\eta \in [0, \overline{\xi})$.

The sets U^n_ξ are relatively pointwise open in B. We put

$$U := \bigcup_{n=1}^{\infty} \bigcup_{\xi \in [0,\overline{\xi})} \bigcup_{h \in A^n_\xi} W^n_h.$$

Obviously

$$U \cap B = \bigcup_{n=1}^{\infty} \bigcup_{\xi \in [0,\overline{\xi})} U^n_\xi.$$

This set contains S. Indeed, let $h \in S$ and let ξ be the first ordinal with $h \in W_\xi$. Choose n so large that $B_{\|\cdot\|}(h, 3/2^n) \subset W_\xi$. So, either $h \in U^l_\eta$ for some $l < n$ and $\eta \in [0, \overline{\xi})$, or $h \in U^n_\xi$.

We prove that the families $\{U^n_\xi : \xi \in [0, \overline{\xi})\}$ are discrete in the norm topology for every n. It is sufficient to show that if $h_1 \in U^n_{\xi_1}$ and $h_2 \in U^n_{\xi_2}$, $\xi_1 \neq \xi_2$, then $\|h_1 - h_2\| > 1/2^n$. Indeed,

$$\|h_1 - h_2\| \geq \|\overline{h}_1 - \overline{h}_2\| - \|\overline{h}_1 - h_1\| - \|\overline{h}_2 - h_2\| > 1/2^n$$

where $h_i \in W^n_{\overline{h}_i}$, $\overline{h}_i \in A^n_{\xi_i}$ for $i = 1, 2$.

To finish the proof of (ii), we show that every h in $U \cap B$ has a norm neighbourhood which does not intersect U^m_ξ for m sufficiently large. As $h \in U \cap B$, there exist n and $\xi \in [0, \overline{\xi})$ with $h \in U^n_\xi$. Hence there is $l \in \mathbb{N}$ such that

$$\{g \in B : \|g - h\| \leq 1/2^l\} \subset U^n_\xi.$$

We show that

$$\{g \in B : \|g - h\| \leq 1/2^{l+n}\} \cap U^m_\eta = \emptyset$$

whenever $m \geq l + n$ and $\eta \in [0, \overline{\xi})$. Indeed, $A^m_\eta \cap U^n_\xi = \emptyset$ for $m > n$ by condition (3) in our definition. Therefore $\|h - \overline{h}\| > 1/2^l$ for every $\overline{h} \in A^m_\eta$. Now $l + n \geq l + 1$ and $m \geq l + 1$ imply

$$\{g \in C(X) : \|h - g\| \leq 1/2^{l+n}\} \cap \{g \in C(X) : \|\overline{h} - g\| \leq 1/2^m\} = \emptyset$$

whenever $\overline{h} \in A^m_\eta$. Remembering that $\operatorname{diam}(B \cap W^m_{\overline{h}}) < 1/2^m$, we have

$$U^m_\eta \cap \{g \in B : \|g - h\| \leq 1/2^{l+n}\} = \emptyset,$$

thus finishing the proof of (ii).

To prove (iii), we set $\delta_n = 1/2^{n+1}$ and fix $\widetilde{\xi} = (\xi_1 \ldots \xi_k)$, $0 \leq \xi_1 < \ldots < \xi_k < \overline{\xi}$. Then we define

$$W_1 := W^{(-3\delta_n)}_{\xi_1}, \quad W_2 := W^{(-3\delta_n)}_{\xi_2} \setminus \overline{W^{(-3\delta_n)}_{\xi_1}}, \ \ldots$$

$$W_k := W^{(-3\delta_n)}_{\xi_k} \setminus \overline{\bigcup_{i=1}^{k-1} W^{(-3\delta_n)}_{\xi_i}}$$

These sets are finitely determined because the W_ξ are elementary. Moreover,

$$W^{(-\delta_n)}_i = \Big(W^{(-3\delta_n)}_{\xi_i} \setminus \overline{\bigcup_{j=1}^{i-1} W^{(-3\delta_n)}_{\xi_j}}\Big)^{(-\delta_n)} = W^{(-4\delta_n)}_{\xi_i} \setminus \overline{\bigcup_{j=1}^{i-1} W^{(-2\delta_n)}_{\xi_j}}.$$

Thus if $h \in U^n_{\xi_i}$, then there exists $\overline{h} \in A^n_{\xi_i}$ with $\|h - \overline{h}\| < 1/2^n = 2\delta_n$ and

$$\{g \in C(X) : \|g - \overline{h}\| \leq 6\delta_n\} \subset W_{\xi_i},$$

hence

$$h \in \{g \in C(X) : \|g - \overline{h}\| \leq 2\delta_n\} \subset W^{(-4\delta_n)}_{\xi_i}.$$

On the other hand, for $j < i$ we have $\overline{h} \notin W_{\xi_j}$ by condition (2) and so $h \notin \overline{W^{(-2\delta_n)}_{\xi_j}}$. Therefore $W^{(-\delta_n)}_i \supset U^n_{\xi_i}$. The second part of the assertion in (iii) follows from $W^{(+\delta_n)}_i \subset W_{\xi_i}$.

To finish the proof of the lemma it remains to renumber the families $\{U^n_\xi : \xi \in [0, \overline{\xi})\}$ removing the empty sets and moving the beginning of the ordinal interval to 2.

LEMMA 3.4. *Every norm compact subset of $C(X)$ can be covered by $B_{\|\cdot\|}(0, \varepsilon/2)$ and by finitely many members of the family $\{qU^s_\xi : 2 \leq \xi < \overline{\xi^s},\ s \in \mathbb{N},\ q \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rational numbers. The families $\{U^s_\xi : 2 \leq \xi < \overline{\xi^s}\}$, $s = 1, 2, \ldots$, are built in the previous lemma to be the middle levels of the G-relatively open partitions $\mathcal{U}^s$, $s = 1, 2, \ldots$, of $C_{\mathrm{p}}(X)$.*

Proof. This is the key point where we need the continuity of the function F which is replacing an equivalent pointwise Kadec norm. Let K be a norm compact subset of $C(X)$. We consider the set

$$K' = K \setminus \{h \in C(X) : \|h\| < \varepsilon/2\}$$

and its "projection" on S:

$$K^S = \{h/F(h) : h \in K'\}.$$

Since F is continuous and it is not less than $\varepsilon/2$ on K', K^S is norm compact as a continuous image of a compact space. We denote by K^H the sector of B generated by K':

$$K^H := \{\alpha h : h \in K^S,\ \alpha \in [\varepsilon/2, 1]\}.$$

It is compact as well. We assert that there exists $\gamma > 0$ such that $(1-\gamma)B \supset K^H \setminus U$ where U is the pointwise open set from the previous lemma. Indeed, assume the contrary, i.e. there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset K^H \setminus U$ with $F(h_n) \to 1$. Since K^H is compact and U is open, there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ tending to a point h_0 in $K^H \setminus U$. Now the continuity of F implies that $F(h_0) = 1$, that is, $h_0 \in S$. This contradicts $U \supset S$. A consequence of the above fact and of the continuity of the function F is that

$$U \cap B \supset K^H \setminus \operatorname{int}(1-\gamma_0)B \quad \text{whenever } \gamma_0 < \gamma.$$

Let $M = \max\{F(h) : h \in K\}$. We find rational numbers γ_0 and M_0 such that $0 < \gamma_0 < \gamma$ and

$$(1-\gamma)M \le (1-\gamma_0)M_0 < M \le M_0.$$

As $\{(1-\gamma_0)^n M_0\}_{n=0}^{\infty}$ tends to zero, we can choose n_0 with $(1-\gamma_0)^{n_0} M_0 < \varepsilon/2$. Then

$$K \subset \Big(\bigcup_{n=0}^{n_0} K^n\Big) \cup \{h \in C(X) : \|h\| \le \varepsilon/2\}$$

where

$$K^n := \{\alpha h : h \in K^S,\ \alpha \in [(1-\gamma_0)^{n+1} M_0, (1-\gamma_0)^n M_0]\}.$$

By the above, the norm compact set K^n is contained in $q_n(U \cap B)$ where $q_n = (1-\gamma_0)^n M_0$. Now we use the fact that the family $\{U_\xi^s : 2 \le \xi < \overline{\xi^s},\ s \in \mathbb{N}\}$ is locally finite and covers $B \cap U$ (condition (i)) to prove that K^n can be covered by finitely many members of the family $\{q_n U_\xi^s : 2 \le \xi < \overline{\xi^s},\ s \in \mathbb{N}\}$.

Lemma 3.5. *For every $f \in C(X \times Y)$ the operator $P_f : Y \to (C(X), \|\cdot\|)$ defined by $[P_f(y)](x) = f(x,y)$, i.e. $P_f(y) \equiv f(\cdot, y)$, is continuous. In particular, the set $P_f(Y_0)$ is norm compact in $C(X)$ for every closed subset Y_0 of Y.*

Proof. Let $\{y_\alpha\}_{\alpha \in A}$ be a net in Y converging to y_0. Assume that $\{P_f(y_\alpha)\}_{\alpha \in A}$ does not converge to $P_f(y_0)$. Passing to a subnet if necessary we can assume that

$$\|P_f(y_\alpha) - P_f(y_0)\| \ge \varepsilon_0$$

for some fixed $\varepsilon_0 > 0$. Then for every $\alpha \in A$ there exists a point x_α in the compact space X with

$$|f(x_\alpha, y_\alpha) - f(x_\alpha, y_0)| > \varepsilon_0/2.$$

We can assume that $\{x_\alpha\}_{\alpha \in A}$ converges to a point x_0 of X. But this contradicts the continuity of f.

Lemma 3.6. *Let Y_0 be a closed subset of Y. Then the set*

$$\{f \in C(X \times Y) : P_f(Y_0) \cap U \neq \emptyset\}$$

is pointwise open whenever U is a pointwise open subset of $C(X)$.

Proof. Let f be in the set in question, i.e. there exists a point y_f in Y_0 such that $f(\cdot, y_f) \in U$. Hence there exists an elementary neighbourhood of $f(\cdot, y_f)$ in $C_p(X)$ which is contained in U:

$$\{h \in C(X) : |h(x_i) - f(x_i, y_f)| < \alpha_i,\ i = 1, \ldots, s\} \subset U.$$

Then

$$\{g \in C(X \times Y) : |g(x_i, y_f) - f(x_i, y_f)| < \alpha_i,\ i = 1, \ldots, s\}$$

is an elementary neighbourhood of f in $C_p(X \times Y)$ and $g(\cdot, y_f) \in U$, that is, $P_g(Y_0) \cap U \neq \emptyset$ for every g in this neighbourhood.

Construction: Step 1. Given a G-relatively open partition $\mathcal{U}$ of $C_p(X)$ and a closed subset Y_0 of Y we construct countably many G-relatively open partitions $\widetilde{\mathcal{U}}^k$, $k = 1, 2, \ldots$, of $C_p(X \times Y)$ sorting the functions with respect to the behaviour of the sets $P_f(Y_0)$.

Let $\mathcal{U} = \{U_1\} \cup \{U_\xi : 2 \le \xi < \overline{\xi}\} \cup \{U_{\overline{\xi}}\}$. Put

$$\deg_{\mathcal{U}}(f, Y_0) := |\{\xi \in [2, \overline{\xi}) : P_f(Y_0) \cap U_\xi \neq \emptyset\}|$$

for every $f \in C(X \times Y)$. Then we define

$$\widetilde{U}_1^k := \{f \in C(X \times Y) : P_f(Y_0) \cap U_1 \neq \emptyset\} \cup \{f \in C(X \times Y) : \deg_{\mathcal{U}}(f, Y_0) > k\}.$$

As $U_1 \cup U_\xi$ is open in $C_p(X)$ for every $\xi \in [2, \overline{\xi})$, the above set is open in $C_p(X \times Y)$ by Lemma 3.6. Let $\xi_1 < \ldots < \xi_k$ be in $[2, \overline{\xi})$. We put

$$\widetilde{U}_{\xi_1 \ldots \xi_k}^k := \{f \in C(X \times Y) : P_f(Y_0) \cap U_{\xi_i} \neq \emptyset,\ i = 1, \ldots, k\} \setminus \widetilde{U}_1^k,$$

$$\widetilde{U}_{\overline{\xi}}^k := C(X \times Y) \setminus \Big(\widetilde{U}_1^k \cup \bigcup_{2 \le \xi_1 < \ldots < \xi_k < \overline{\xi}} \widetilde{U}_{\xi_1 \ldots \xi_k}^k\Big).$$

Thus we have defined

$$\widetilde{\mathcal{U}}^k := \{\widetilde{U}_1^k\} \cup \{\widetilde{U}_{\xi_1 \ldots \xi_k}^k : 2 \le \xi_1 < \ldots < \xi_k < \overline{\xi}\} \cup \{\widetilde{U}_{\overline{\xi}}^k\},$$

which is a G-relatively open partition of $C_p(X \times Y)$. Indeed, the set $\{f \in C(X \times Y) : P_f(Y_0) \cap (U_1 \cup U_{\xi_i}) \neq \emptyset$ for every $i = 1, \ldots, k\}$ is open in $C_p(X \times Y)$ and if $P_f(Y_0) \cap U_1 \neq \emptyset$, then f is in $\widetilde{U}_1^k$. So $\widetilde{U}_{\xi_1 \ldots \xi_k}^k$ is relatively open in $C_p(X \times Y) \setminus \widetilde{U}_1^k$. Moreover, if $(\xi_1 \ldots \xi_k) \neq (\eta_1 \ldots \eta_k)$ (in the sense that $\xi_i \neq \eta_i$ for at least one i), then the corresponding elements of $\widetilde{\mathcal{U}}^k$ have empty intersection, because $\deg_{\mathcal{U}}(f, Y_0) > k$ implies that f is in $\widetilde{U}_1^k$. Note that the middle level is empty in the case $k = 0$.

Construction: Step 2. Let $\mathcal{U}$ be a G-relatively open partition of $C_p(X)$ having the additional property from Lemma 3.3, namely: There exists $\delta > 0$ such that for every k-tuple $\widetilde{\xi} = (\xi_1 \dots \xi_k)$, where $2 \le \xi_1 < \dots < \xi_k < \overline{\xi}$, there exist k disjoint finitely determined pointwise open sets $W_{\widetilde{\xi},i}$, $i = 1, \dots, k$, in $C(X)$ satisfying

$$W_{\widetilde{\xi},i}^{(-\delta)} \supset U_{\xi_i}, \quad \|\cdot\|\text{-diam}(W_{\widetilde{\xi},i}^{(+\delta)} \setminus U_1) < \varepsilon, \quad i = 1, \dots, k.$$

For every k we will construct countably many G-relatively open partitions $\widetilde{\mathcal{W}}^{kl}$ of $C_p(X \times Y)$, which are refinements of the partition $\widetilde{\mathcal{U}}^k$ from Step 1, having the properties:

(a) the union of the middle levels of $\widetilde{\mathcal{W}}^{kl}$, $l = 1, 2, \dots$, is the middle level of $\widetilde{\mathcal{U}}^k$;

(b) for every set V in the middle level of some $\widetilde{\mathcal{W}}^{kl}$ there exists a closed subset Y_0^V of Y_0 such that $Y_0 \setminus Y_0^V$ is nonempty and for every $y \in Y_0 \setminus Y_0^V$ the norm diameter of $\{P_f(y) : f \in V\}$ is less than ε. Moreover, $P_f(Y_0^V) \subset U_{\overline{\xi}}$ whenever $f \in V$.

For the above δ by Proposition 2.2 and Remark 2.3 there exist countably many G-relatively open partitions $\mathcal{V}^m$ of $C_p(Y)$ such that the union of their middle levels covers $C(Y)$ and every middle level set in them has diameter (in the uniform norm of $C(Y)$) less than δ. Note that this is the only place where we use the fact that $C_p(Y)$ has a countable cover by sets of small local diameter. We lift the partitions $\mathcal{V}^m$ to G-relatively open partitions $\widetilde{\mathcal{V}}^{m,x}$ of $C_p(X \times Y)$: if $\mathcal{V}^m = \{V_\eta : 1 \le \eta < \overline{\eta}\}$, we put $\widetilde{\mathcal{V}}^{m,x} = \{\widetilde{V}_\eta : 1 \le \eta < \overline{\eta}\}$ where

$$\widetilde{V}_\eta = \{f \in C(X \times Y) : f(x, \cdot) \in V_\eta\}.$$

Now fix k and $\widetilde{\xi} = (\xi_1 \dots \xi_k)$, $2 \le \xi_1 < \dots < \xi_k < \overline{\xi}$. We first construct countably many G-relatively open partitions $\widetilde{\mathcal{W}}^{\widetilde{\xi},l}$ of the middle level set $\widetilde{U}^k_{\widetilde{\xi}}$. Let $C_{\widetilde{\xi}}$ be the finite subset of X involved in the definition of the finitely determined pointwise open sets $W_{\widetilde{\xi},i}$, $i = 1, \dots, k$. If $s = |C_{\widetilde{\xi}}|$, our countable index $l = \{l(x)\}_{x \in C_{\widetilde{\xi}}}$ consists of s-tuples of positive integers. We define

$$\widetilde{\mathcal{W}}^{\widetilde{\xi},l} := \Big(\bigcap_{x \in C_{\widetilde{\xi}}} \widetilde{\mathcal{V}}^{l(x),x}\Big) \cap \widetilde{U}^k_{\widetilde{\xi}}$$

where the intersection of G-relatively open partitions is taken in the sense of Proposition 2.7(b) and then every element of it is intersected with $\widetilde{U}^k_{\widetilde{\xi}}$ (see Proposition 2.7(a)). The partitions $\widetilde{\mathcal{W}}^{k,l}$ are obtained by collecting $\widetilde{\mathcal{W}}^{\widetilde{\xi},l}$ for the different $\widetilde{\xi} = (\xi_1 \dots \xi_k)$ as in Proposition 2.7(c) (the positive integers k and l are fixed). Note that the nature of the countable index l is different for different $\widetilde{\xi}$, but we think about it as about a positive integer.

Let us check condition (a). Indeed, if f is in the middle level set $\widetilde{U}^k_{\widetilde{\xi}}$ and $x \in C_{\widetilde{\xi}}$, then there exists $l_f(x) \in \mathbb{N}$ such that $f(x, \cdot)$ is in a middle level set of $\mathcal{V}^{l_f(x)}$. We put $l_f := \{l_f(x)\}_{x \in C_{\widetilde{\xi}}}$. Then f is in a middle level set of $\widetilde{\mathcal{W}}^{\widetilde{\xi},l_f}$.

For (b), we fix a middle level set V of $\widetilde{\mathcal{W}}^{kl}$ contained in $\widetilde{U}^k_{\widetilde{\xi}}$ and define

$$Y_0^V := \Big\{y \in Y_0 : P_f(y) \notin \bigcup_{i=1}^k U_{\xi_i} \text{ for every } f \in V\Big\}.$$

Note that $f \in V \subset \widetilde{U}^k_{\widetilde{\xi}}$ implies that $P_f(Y_0) \subset C(X) \setminus U_1$. But $\bigcup_{i=1}^k U_{\xi_i}$ is a relatively open subset of $C(X) \setminus U_1$, so the set

$$\Big\{y \in Y_0 : P_f(y) \notin \bigcup_{i=1}^k U_{\xi_i}\Big\}$$

is closed in Y_0 because of the continuity of P_f. Therefore Y_0^V is closed as an intersection of closed sets. By the definitions of Y_0^V and $\widetilde{U}^k_{\widetilde{\xi}}$ we have

$$P_f(Y_0^V) \subset C(X) \setminus \Big(U_1 \cup \bigcup_{i=1}^k U_{\xi_i}\Big) \subset U_{\overline{\xi}}$$

for every $f \in V$ (because $P_f(Y_0) \cap U_\xi = \emptyset$ whenever $\xi \in [2, \overline{\xi})$ is not in $\widetilde{\xi}$ and f is in $\widetilde{U}^k_{\widetilde{\xi}}$).

Fix y in the set $Y_0 \setminus Y_0^V$ (obviously nonempty) and let $f \in V$. Since $y \notin Y_0^V$, there exists $\overline{f} \in V$ with $P_{\overline{f}}(y) \in \bigcup_{i=1}^k U_{\xi_i}$. If x is in the finite set $C_{\widetilde{\xi}}$, then f and $\overline{f}$ being both in V yields that $f(x, \cdot)$ and $\overline{f}(x, \cdot)$ are in the same middle level set of $\mathcal{V}^s$ for some positive integer s. Therefore

$$\|f(x, \cdot) - \overline{f}(x, \cdot)\|_{C(Y)} < \delta.$$

In particular, $|f(x, y) - \overline{f}(x, y)| < \delta$ for every x in $C_{\widetilde{\xi}}$. Let $i \in \{1, \dots, k\}$ be such that $P_{\overline{f}}(y) \equiv \overline{f}(\cdot, y) \in U_{\xi_i}$. But then $U_{\xi_i} \subset W_{\widetilde{\xi},i}^{(-\delta)}$ and the inequalities above imply that $P_f(y) \equiv f(\cdot, y) \in W_{\widetilde{\xi},i}$. On the other hand, $V \subset \widetilde{U}^k_{\widetilde{\xi}}$ yields that $f(\cdot, y) \notin U_1$. Hence

$$\{P_f(y) : f \in V\} \subset W_{\widetilde{\xi},i} \setminus U_1.$$

But the norm diameter of $W_{\widetilde{\xi},i} \setminus U_1$ is less than ε, which finishes the proof of (b).

Let $\widetilde{\mathcal{W}}^t$ be a G-relatively open partition of $C_p(X \times Y)$ labelled by some finite sequence of positive integers t and such that to every middle level set V of $\widetilde{\mathcal{W}}^t$ a closed subset Y_0^V of Y is assigned. Suppose we are also given a G-relatively open partition $q\mathcal{U}^n$ of $C_p(X)$ among the ones built in

Lemma 3.3. Then by a *passage* through our two-step construction we will mean the following procedure:

We fix V in the middle level of $\widetilde{\mathcal{W}}^t$ and start the two-step construction with the initial G-relatively open partition $\mathcal{U}$ of $C_p(X)$ to be $q\mathcal{U}^n$ and Y_0 to be Y_0^V. As a result we get countably many G-relatively open partitions of $C_p(X\times Y)$ called $\widetilde{\mathcal{W}}^{kl}_{Y_0^V}$. We intersect them with V to obtain countably many G-relatively open partitions of the topological space (V,p):

$$\widetilde{\mathcal{W}}^{kl}_V = \widetilde{\mathcal{W}}^{kl}_{Y_0^V}\cap V.$$

Now for (k,l) fixed we have a G-relatively open partition of every middle level set V of $\widetilde{\mathcal{W}}^t$. Then there exists a G-relatively open partition $\widetilde{\mathcal{W}}^{(t,q,n,l,k)}$ of the whole $C_p(X\times Y)$ whose middle level is the union of the middle levels of $\widetilde{\mathcal{W}}^{kl}_V$ for all V (see Proposition 2.7(c)). The new label is a finite sequence of positive integers beginning with t and continuing with the quadruple (q,n,k,l) whose first two elements identify the G-relatively open partition of $C_p(X)$ we start with and whose next two elements identify one of the resulting countably many partitions obtained after the passage.

We are ready to define the desired σ-G-relatively open partition of $C_p(X\times Y)$ by repeating the above described passages finitely many times. The labels of the partitions are finite sequences of positive integers of the form

$$(q_1n_1k_1l_1, q_2n_2k_2l_2,\ldots,q_sn_sk_sl_s).$$

They are divided into quadruples each of which corresponds to a passage. Thus we start the first passage with $Y_0=Y$ and some G-relatively open partition of $C(X)$ of the form $q_1\mathcal{U}^{n_1}$. After the two-step construction we obtain countably many G-relatively open partitions of $C_p(X\times Y)$ called $\widetilde{\mathcal{W}}^{(q_1n_1k_1l_1)}$ (we still have no G-relatively open partition of $C_p(X\times Y)$ to refine). To start the second passage we fix one of the partitions just obtained in one passage, choose a partition $q_2\mathcal{U}^{n_2}$ of $C(X)$ and apply the construction to obtain the partitions $\widetilde{\mathcal{W}}^{(q_1n_1k_1l_1,q_2n_2k_2l_2)}$. Note that suitable closed sets Y_0^V are already assigned to the middle level sets V of $\widetilde{\mathcal{W}}^{(q_1n_1k_1l_1)}$ by the second step of the construction. We can continue in this way s times to obtain a G-relatively open partition of $C_p(X\times Y)$ labelled with the above sequence.

It remains to prove that for every $f\in C(X\times Y)$, (among the constructed partitions) there exists a partition $\widetilde{\mathcal{W}}$ such that the element $V\in\widetilde{\mathcal{W}}$ containing f has norm diameter not greater than ε. Indeed, let $K=P_f(Y)$. It is a norm compact subset of $C(X)$ by Lemma 3.5. Now Lemma 3.4 yields the existence of finitely many rational numbers q_m, $m=0,1,\ldots,m_0$, and finitely many positive integers $\{n_i^m : i=1,\ldots,\overline{n^m};\ m=0,1,\ldots,m_0\}$ such that K^m is contained in the union of the middle levels of the partitions $q_m\mathcal{U}^{n_i^m}$, $i=1,\ldots,\overline{n^m}$, for every fixed $m=0,1,\ldots,m_0$ (for the definition of K^m see the proof of Lemma 3.4). The G-relatively open partition $\widetilde{\mathcal{W}}$ we are seeking for will be obtained in $s=\sum_{m=0}^{m_0}\overline{n^m}$ passages. Its label begins with the couple $(q_0n_1^0)$. Remembering the definition of q_0 we see that $K\equiv P_f(Y)$ does not intersect the first element of $q_0\mathcal{U}^{n_1^0}$ (equal to $C(X)\setminus q_0B$). As the middle level of $q_0\mathcal{U}^{n_1^0}$ is discrete, K intersects at most finitely many of its elements (and at least one by the choice of the partition), hence there exist $k_1^0\in\mathbb{N}$ and a k_1^0-tuple $\tilde{\xi}$ of ordinals such that f belongs to the corresponding middle level set of the partition constructed in the first step. Proceeding with the second step, we find $l_1^0\in\mathbb{N}$ such that f is in a middle level set of the corresponding partition (by (a)). Thus after the first passage we have built a G-relatively open partition $\widetilde{\mathcal{W}}^{(q_0n_1^0k_1^0l_1^0)}$ such that f is in a set of its middle level. We proceed in the same way doing $\overline{n^0}$ passages with initial $C(X)$-partitions $q_0\mathcal{U}^{n_i^0}$, $i=1,\ldots,\overline{n^0}$. The function f belongs to a middle level set V^0 of the resulting partition. Moreover, condition (b) in the second steps gives that

$$\operatorname{diam}(\{P_g(y): g\in V^0\})<\varepsilon$$

whenever $y\in Y\setminus Y_0^{V^0}$. Also, $P_f(Y_0^{V^0})$ does not intersect the union of the middle levels of $q_0\mathcal{U}^{n_i^0}$, $i=1,\ldots,\overline{n^0}$, hence it does not intersect K^0. Therefore $P_f(Y_0^{V_0})$ does not intersect the first element $C(X)\setminus q_1B$ of $q_1\mathcal{U}^{n_i^1}$, $i=1,\ldots,\overline{n^1}$. Then we can make $\overline{n^1}$ passages keeping f in some middle level set V^1 of one of the resulting G-relatively open partitions and so on. We repeat this procedure m_0 times, obtaining a label t. Then the partition $\widetilde{W}$, labelled by $(t,\varepsilon/2,1)$, contains f in some middle level set V. It satisfies

$$\operatorname{diam}(\{P_g(y): g\in V\})<\varepsilon$$

for every $y\in Y\setminus Y_0^V$ and

$$P_g(Y_0^V)\subset\{h\in C(X):\|h\|<\varepsilon/2\},\qquad g\in V.$$

The last statement implies that the diameter of the set $\{P_g(y): g\in V\}$ is less than ε for every y in the compact space Y. Therefore the uniform norm diameter of V is not greater than ε and the theorem is proved.

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Degenerate evolution problems and Beta-type operators

by

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Abstract. The present paper is concerned with the study of the differential operator $Au(x) := \alpha(x)u''(x) + \beta(x)u'(x)$ in the space $C([0,1])$ and of its adjoint $Bv(x) := ((\alpha v)'(x) - \beta(x)v(x))'$ in the space $L^1(0,1)$, where $\alpha(x) := x(1-x)/2$ $(0 \le x \le 1)$. A careful analysis of their main properties is carried out in view of some generation results available in [6, 12, 20] and [25]. In addition, we introduce and study two different kinds of Beta-type operators as a generalization of similar operators defined in [18]. Among the corresponding approximation results, we show how they can be used in order to represent explicitly the solutions of the Cauchy problems associated with the operators A and $\widetilde{A}$, where $\widetilde{A}$ is equal to B up to a suitable bounded additive perturbation.

1. Introduction and notations. The present paper falls within a wide program of investigations whose main object is the interplay between constructive approximation processes and degenerate evolution problems by means of standard semigroup theory. More specifically, we are interested in representing explicitly the semigroups generated by some degenerate differential operators in terms of powers of suitable positive linear operators: as a direct consequence, the solutions of the initial value problems canonically associated with such differential operators may be represented in the same way, as well. This kind of approach, basically based upon Voronovskaya-type formulas and Trotter's theorem [26], has its roots in a paper by Altomare [1], dealing with the convergence of the powers of the classical Bernstein operators; actually, it turns out to be quite satisfactory in practical situations, since some qualitative properties of the relevant semigroups, such as asymptotic behaviour, regularity, saturation and so on, may sometimes be easily derived from the corresponding properties of the approximating operators.

A rather exhaustive treatment of this subject together with a systematic analysis of some classical approximation processes may be found in

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