STUDIA MATHEMATICA 140 (1) (2000)

Variational integrals for elliptic complexes

by

FLAVIA GIANNETTI and ANNA VERDE (Napoli)

Abstract. We discuss variational integrals which are defined on differential forms associated with a given first order elliptic complex. This general framework provides us with better understanding of the concepts of convexity, even in the classical setting $\mathcal{D}'(\mathbb{R}^n,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^n) \xrightarrow{\text{curl}} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^{n\times n})$.

1. Introduction. This work can be regarded as a sequel to the paper [IS3] where the theory of quasiharmonic fields is developed using singular integrals, in particular the n-dimensional Hilbert transform. The estimates in that paper play an important role in the theory of elliptic partial differential equations, largely pertaining to the higher integrability of the gradient for nonuniformly elliptic PDEs. The intent of this paper is to continue this theme from a more general and unifying perspective. We believe that the more general setting presented in our paper provides a better understanding of several unanswered questions in [IS3], especially those concerning the L^p-norm of the Hilbert transform and sharp estimates for elliptic PDEs. Although we do not pursue these questions here they lead naturally to a study of certain variational integrals. A principal feature of our setting is that we look at an elliptic complex of first order differential operators

$$\mathcal{D}'(\mathbb{R}^n,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n,\mathbf{W})$$

where $\mathbf{U},\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional inner product spaces.

Such complexes are viewed here, in many ways, as generalizations of the classical exact sequence of the gradient and rotation operator

$$\mathcal{D}'(\mathbb{R}^n,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^n) \xrightarrow{\text{curl}} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^{n\times n}). \tag{1.1}$$

There has been some recent and related work concerning differential forms and the exterior derivative operators [I]. The reader should also consult [FM] for even more general setting as we consider here only the variational integrals which we believe have a good chance of being quasiconvex.

2000 *Mathematics Subject Classification*: Primary 58G05; Secondary 49J40, 58A14.

Among the desirable results, we obtain, in analogy with the div-curl decomposition of a vector field, the Hodge type decomposition of $F \in L^p(\mathbb{R}^n, \mathbf{V})$ associated with a given elliptic complex, that is,

$$F = \mathcal{P}u + \mathcal{Q}^* w$$

where $u \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $w \in W^{1,p}(\mathbb{R}^n, \mathbf{W})$.

Guided by [IS3], we define the Hilbert transform $S : L^p(\mathbb{R}^n, \mathbf{V}) \to L^p(\mathbb{R}^n, \mathbf{V})$ by the rule

$$SF = \mathcal{P}u - \mathcal{Q}^* w.$$

Of course, the value of this operator goes beyond the mere analogy with the familiar Hilbert transform on the real line. Indeed, as shown in [IS3] for the complex (1.1), this transform plays an essential role in the study of sharp L^p-bounds for elliptic systems of PDEs.

A central question concerning the p-norms of S leads us to the Burkholder functional

$$\mathcal{E}[F] = \int_{\mathbb{R}^n} [(p-1)|SF| - |F|][|SF| + |F|]^{p-1}, \qquad p \ge 2.$$

It is important to realize that this functional is convex in the so-called singular directions. Continuing the analogy with the vectorial Calculus of Variations we arrive at a challenging conjecture that convexity in singular directions might imply quasiconvexity. If true, this would have far reaching implications for the regularity theory of PDEs, quasiconformal mappings, and much more. But we have reserved these matters for our subsequent studies.

2. Some first order partial differential operators. This section is dedicated to an exposition of basic differential operators to be discussed later on. We begin with a quite general setting.

Let $\mathbf{U}$ and $\mathbf{V}$ be finite-dimensional vector spaces over the field of real numbers. We assume that both spaces are equipped with inner products, denoted by $\langle \ , \ \rangle_{\mathbf{U}}$ and $\langle \ , \ \rangle_{\mathbf{V}}$, respectively. The space of infinitely differentiable functions on $\mathbb{R}^n$ with values in $\mathbf{U}$ will be denoted by $C^\infty(\mathbb{R}^n, \mathbf{U})$. Other spaces, such as $L^2(\mathbb{R}^n, \mathbf{U})$ and Sobolev classes $W^{1,p}(\mathbb{R}^n, \mathbf{U})$ will be discussed as well.

Let $\mathcal{L} : C^\infty(\mathbb{R}^n, \mathbf{U}) \to C^\infty(\mathbb{R}^n, \mathbf{V})$ be a differential operator of first order with constant coefficients. More explicitly,

$$\mathcal{L} = \sum_{k=1}^n A_k \frac{\partial}{\partial x_k}$$

where A_k, $k = 1, \ldots, n$, are given linear transformations from $\mathbf{U}$ into $\mathbf{V}$.

The formal adjoint $\mathcal{L}^* : C^\infty(\mathbb{R}^n, \mathbf{V}) \to C^\infty(\mathbb{R}^n, \mathbf{U})$ is defined by the rule

$$\int_{\mathbb{R}^n} \langle \mathcal{L}^* v, u \rangle_{\mathbf{U}} = \int_{\mathbb{R}^n} \langle v, \mathcal{L}u \rangle_{\mathbf{V}}$$

for $u \in C^\infty(\mathbb{R}^n, \mathbf{U})$ and $v \in C^\infty(\mathbb{R}^n, \mathbf{V})$. Thus

$$\mathcal{L}^* = -\sum_{k=1}^n A_k^t \frac{\partial}{\partial x_k}$$

where $A_k^t : \mathbf{V} \to \mathbf{U}$ is the transpose of A_k.

Our basic example, which actually originated this work, is that of the *gradient operator*

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) : C^\infty(\mathbb{R}^n, \mathbb{R}) \to C^\infty(\mathbb{R}^n, \mathbb{R}^n)$$

and its adjoint

$$-\operatorname{div} : C^\infty(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R})$$

defined by

$$\operatorname{div} f = \frac{\partial f^1}{\partial x_1} + \ldots + \frac{\partial f^n}{\partial x_n} \quad \text{for } f = (f^1, \ldots, f^n).$$

More generally, the differential

$$D : C^\infty(\mathbb{R}^n, \mathbb{R}^m) \to C^\infty(\mathbb{R}^n, \mathbb{R}^{m \times n})$$

assigns to a mapping $h = (h^1, \ldots, h^m) : \mathbb{R}^n \to \mathbb{R}^m$ its Jacobian matrix

$$Dh(x) = \left[\frac{\partial h^j(x)}{\partial x_i} \right] \in \mathbb{R}^{m \times n}.$$

Its formal adjoint $D^* : C^\infty(\mathbb{R}^n, \mathbb{R}^{m \times n}) \to C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is the *divergence operator* on matrix fields, that is,

$$-D^* F = (\operatorname{div} F^1, \ldots, \operatorname{div} F^m)$$

where $F^1, \ldots, F^m$ are the row vectors of the matrix F.

We shall also consider the so-called *rotation operator*

$$\operatorname{curl} : C^\infty(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$$

defined by

$$\operatorname{curl} f = \left[\frac{\partial f^i}{\partial x_j} - \frac{\partial f^j}{\partial x_i} \right], \qquad i, j = 1, \ldots, n,$$

for $f = (f^1, \ldots, f^n)$.

An interesting class of PDOs arises in the study of differential forms. We refer to [ISS] for notation and definitions used here. The book by H. Cartan [C] is an excellent reference for the material used here.

For an integer $l = 0, 1, \ldots, n$, we consider the space $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ of l-covectors in $\mathbb{R}^n$. Differential l-forms on $\mathbb{R}^n$ are simply functions defined on $\mathbb{R}^n$ with values in $\Lambda^l(\mathbb{R}^n)$. Now we take $\mathbf{U} = \Lambda^{l-1}(\mathbb{R}^n)$ and $\mathbf{V} = \Lambda^l(\mathbb{R}^n)$, $l = 1, \ldots, n$. The basic PDO on forms is the *exterior differentiation*

$$d : C^\infty(\mathbb{R}^n, \Lambda^{l-1}) \to C^\infty(\mathbb{R}^n, \Lambda^l)$$

and its formal adjoint, called the *Hodge codifferential*:

$$d^* : C^\infty(\mathbb{R}^n, \Lambda^l) \to C^\infty(\mathbb{R}^n, \Lambda^{l-1}),$$

It is also worth extending these operators to the exterior algebra $\Lambda = \Lambda(\mathbb{R}^n) = \bigoplus_{l=0}^n \Lambda^l(\mathbb{R}^n)$:

$$d, d^* : C^\infty(\mathbb{R}^n, \Lambda) \to C^\infty(\mathbb{R}^n, \Lambda).$$

Then we have the *Dirac operators*, defined by

$$\begin{cases} \partial^+ = d + d^* : C^\infty(\mathbb{R}^n, \Lambda) \to C^\infty(\mathbb{R}^n, \Lambda), \\ \partial^- = d - d^* : C^\infty(\mathbb{R}^n, \Lambda) \to C^\infty(\mathbb{R}^n, \Lambda). \end{cases}$$

There are many more examples of operators in applied PDEs which fit well the theme of this paper, but we shall not discuss them here.

3. Short elliptic complexes. Let $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ be finite-dimensional inner product spaces. We consider a sequence of differential operators of first order in n independent variables with constant coefficients

$$\mathcal{D}'(\mathbb{R}^n, \mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, \mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, \mathbf{W}). \tag{3.1}$$

Note that spaces of Schwartz distributions are being used here.

More precisely, if $u \in \mathcal{D}'(\mathbb{R}^n, \mathbf{U})$ and $v \in \mathcal{D}'(\mathbb{R}^n, \mathbf{V})$, then

$$\mathcal{P}u = \sum_{k=1}^n A_k \frac{\partial u}{\partial x_k}, \quad \mathcal{Q}v = \sum_{k=1}^n B_k \frac{\partial v}{\partial x_k}$$

where $A_k \in L(\mathbf{U}, \mathbf{V})$, and $B_k \in L(\mathbf{V}, \mathbf{W})$ for $k = 1, \ldots, n$. The *symbols* $\mathcal{P} = \mathcal{P}(\xi)$ and $\mathcal{Q} = \mathcal{Q}(\xi)$ are linear functions of $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ with values in $L(\mathbf{U}, \mathbf{V})$ and in $L(\mathbf{V}, \mathbf{W})$, respectively. They are explicitly given by

$$\mathcal{P}(\xi) = \sum_{k=1}^n \xi_k A_k, \quad \mathcal{Q}(\xi) = \sum_{k=1}^n \xi_k B_k.$$

The complex (3.1) is said to be *elliptic* if the sequence of symbols

$$\mathbf{U} \xrightarrow{\mathcal{P}(\xi)} \mathbf{V} \xrightarrow{\mathcal{Q}(\xi)} \mathbf{W} \tag{3.2}$$

is *exact*, i.e. $\operatorname{Im} \mathcal{P}(\xi) = \ker \mathcal{Q}(\xi)$ for all $\xi \neq 0$. The dual sequence consists of the formal adjoint operators

$$\mathcal{D}'(\mathbb{R}^n, \mathbf{U}) \xleftarrow{\mathcal{P}^*} \mathcal{D}'(\mathbb{R}^n, \mathbf{V}) \xleftarrow{\mathcal{Q}^*} \mathcal{D}'(\mathbb{R}^n, \mathbf{W}),$$

$$\mathcal{P}^* v = -\sum_{k=1}^n A_k^* \frac{\partial v}{\partial x_k}, \quad \mathcal{Q}^* w = -\sum_{k=1}^n B_k^* \frac{\partial w}{\partial x_k}.$$

Since $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ have inner products, the dual spaces $\mathbf{U}^*, \mathbf{V}^*$ and $\mathbf{W}^*$ are identified with $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$, respectively. The dual complex is elliptic if the original complex is.

Given an elliptic complex we have the associated *Laplace–Beltrami operator*

$$-\Delta = \mathcal{P}\mathcal{P}^* + \mathcal{Q}^*\mathcal{Q} : \mathcal{D}'(\mathbb{R}^n, \mathbf{V}) \to \mathcal{D}'(\mathbb{R}^n, \mathbf{V}).$$

Its symbol is a quadratic form with values in $L(\mathbf{V}, \mathbf{V})$,

$$\begin{aligned} \Delta(\xi) &= \Big(\sum_{j=1}^n \xi_j A_j\Big) \circ \Big(\sum_{k=1}^n \xi_k A_k^*\Big) + \Big(\sum_{j=1}^n \xi_j B_j^*\Big) \circ \Big(\sum_{k=1}^n \xi_k B_k\Big) \\ &= \sum_{j,k=1}^n \xi_j \xi_k (A_j A_k^* + B_j^* B_k) \\ &= \sum_{j,k=1}^n \xi_j \xi_k (A_j A_k^* + A_k A_j^* + B_j^* B_k + B_k^* B_j). \end{aligned}$$

To abbreviate, we denote by $C_{jk} : \mathbf{V} \to \mathbf{V}$ the operators in the latter expression, so that

$$\Delta(\xi) = \sum_{j,k=1}^n \xi_j \xi_k C_{jk}.$$

Now consider an arbitrary vector field $F = (f^1, \ldots, f^n) \in L^2(\mathbb{R}^n, \mathbf{V})$. We can solve the *Poisson equation*

$$\Delta \varphi = F$$

for φ whose second derivatives are L^2-integrable on $\mathbb{R}^n$. As a matter of fact, these derivatives can be expressed in terms of F by using singular integrals. To enable explicit calculation of those integrals we assume, without loss of generality, that both φ and F belong to $C_0^\infty(\mathbb{R}^n, \mathbf{V})$. The calculation goes as follows.

Let $\widehat{\varphi}$ and $\widehat{F}$ denote the Fourier transforms of φ and F. Then

$$\Big(\sum_{j,k=1}^n \xi_j \xi_k C_{jk}\Big) \widehat{\varphi} = \widehat{F}$$

and hence $\widehat{\varphi}(\xi) = \Delta^{-1}(\xi) \widehat{F}(\xi)$. It follows from the identities $(\partial^2 \varphi / \partial x_i \partial x_j)^\wedge = \xi_i \xi_j \widehat{\varphi}$ that

$$\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right)^\wedge = \xi_i\xi_j\Delta^{-1}(\xi)\widehat{F}. \tag{3.3}$$

In order to go further we need the following uniform bound:

$$|\xi_i\xi_j\Delta^{-1}(\xi)| \le c \quad \text{or equivalently} \quad |\Delta^{-1}(\xi)| \le c|\xi|^{-2}. \tag{3.4}$$

The proof is based on looking at Cramer's formula for the inverse matrix:

$$|\Delta^{-1}(\xi)| = \left|\frac{\operatorname{adj}\Delta(\xi)}{\det\Delta(\xi)}\right| \le \frac{c|\xi|^{2(n-1)}}{\det\Delta(\xi)} \le \frac{c}{|\xi|^2}.$$

Now, having (3.4), we can take the Fourier inverse of (3.3) to get

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = (\xi_i\xi_j\Delta^{-1}(\xi)\widehat{F})^\vee.$$

In this way, we arrive at the convolution operator for the second order derivatives of φ:

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = (\xi_i\xi_j\Delta^{-1}(\xi))^\vee * F = \int_{\mathbb{R}^n} K_{ij}(x-y)F(y)\,dy$$

where $K_{ij}(x) : \mathbf{V} \to \mathbf{V}$ are Calderón–Zygmund type singular integrands. The L^p-theory yields

$$\left\|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\|_p \le c_p\|F\|_p \quad \text{for } 1 < p < \infty. \tag{3.5}$$

Next observe that for every vector $v \in \mathbf{V}$, we have

$$\begin{aligned}\langle\Delta(\xi)v, v\rangle &= \sum_{j,k}\xi_j\xi_k\langle A_jA_k^*v, v\rangle + \sum_{j,k}\xi_j\xi_k\langle B_j^*B_k^v, v\rangle\\ &= \sum_{j,k}\xi_j\xi_k[\langle A_k^*v, A_j^*v\rangle + \langle B_kv, B_jv\rangle]\\ &= \Big|\sum_j \xi_jA_j^*v\Big|^2 + \Big|\sum_j\xi_jB_jv\Big|^2 = |\mathcal{P}^*(\xi)v|^2 + |\mathcal{Q}(\xi)v|^2 \ge 0.\end{aligned}$$

It is important to realize that equality occurs if and only if $v = 0$. Indeed,

$$\{\mathcal{P}^*(\xi)v = 0 \text{ and } \mathcal{Q}(\xi)v = 0\} \Leftrightarrow \{v \in \ker\mathcal{Q}(\xi) \text{ and } v \in \ker\mathcal{P}^*(\xi)\}.$$

By ellipticity of the complex (3.1), $\ker\mathcal{Q}(\xi) = \operatorname{Im}\mathcal{P}(\xi)$. It is well known in algebra that $\operatorname{Im}\mathcal{P}(\xi)$ is orthogonal to $\ker\mathcal{P}^*(\xi)$, therefore the vector v, being orthogonal to itself, is zero.

Summarizing, the operator $\Delta(\xi) : \mathbf{V} \to \mathbf{V}$ is positive for $\xi \ne 0$.

4. Hodge decomposition and Hilbert transform. Consider a vector field $F \in L^p(\mathbb{R}^n, \mathbf{V})$. The Poisson equation $F = \Delta\varphi$ for $\varphi \in W^{2,p}(\mathbb{R}^n, \mathbf{V})$, $1 < p < \infty$, yields a decomposition of F:

$$F = \mathcal{P}u + \mathcal{Q}^*w \tag{4.1}$$

where $u = \mathcal{P}^*\varphi \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $w = \mathcal{Q}\varphi \in W^{1,p}(\mathbb{R}^n, \mathbf{W})$. In view of (3.5) we have the estimate

$$\|\nabla u\|_p + \|\nabla w\|_p \le c_p\|F\|_p.$$

LEMMA 4.1 (orthogonality property). *For $\alpha \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $\beta \in W^{1,q}(\mathbb{R}^n, \mathbf{W})$, $1/p + 1/q = 1$, the vector fields $\mathcal{P}\alpha \in L^p(\mathbb{R}^n, \mathbf{V})$ and $\mathcal{Q}^*\beta \in L^q(\mathbb{R}^n, \mathbf{V})$ are orthogonal.*

Proof. Using the equality $\operatorname{Im}\mathcal{P} = \ker\mathcal{Q}$, we have

$$\int\langle\mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle = \int\langle\mathcal{Q}\mathcal{P}\alpha, \beta\rangle = 0$$

whenever $\alpha \in W^{2,p}(\mathbb{R}^n, \mathbf{U})$ and $\beta \in W^{1,q}(\mathbb{R}^n, \mathbf{W})$ with $1/p + 1/q = 1$. Since $W^{2,p}(\mathbb{R}^n, \mathbf{U})$ is dense in $W^{1,p}(\mathbb{R}^n, \mathbf{U})$, the lemma follows by approximation. ■

By this lemma, we are able to prove the uniqueness of the components $\mathcal{P}u$ and $\mathcal{Q}^*w$ in the decomposition (4.1) with $u \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $w \in W^{1,p}(\mathbb{R}^n, \mathbf{W})$. Indeed, consider a decomposition of the zero vector field, say $0 = \mathcal{P}\alpha + \mathcal{Q}^*\beta$, where $\alpha \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $\beta \in W^{1,p}(\mathbb{R}^n, \mathbf{V})$, so $\mathcal{P}\alpha = -\mathcal{Q}^*\beta$. Take an arbitrary $\Phi \in L^q(\mathbb{R}^n, \mathbf{V})$, where q is Hölder conjugate to p. By applying the decomposition (4.1) we can write $\Phi = \mathcal{P}a + \mathcal{Q}^*b$ for some (not necessarily unique) $a \in W^{1,q}(\mathbb{R}^n, \mathbf{U})$ and $b \in W^{1,q}(\mathbb{R}^n, \mathbf{W})$. Hence

$$\begin{aligned}\int_{\mathbb{R}^n}\langle\mathcal{P}\alpha, \Phi\rangle &= \int_{\mathbb{R}^n}\langle\mathcal{P}\alpha, \mathcal{P}a\rangle + \int_{\mathbb{R}^n}\langle\mathcal{P}\alpha, \mathcal{Q}^*b\rangle\\ &= \int_{\mathbb{R}^n}\langle\mathcal{P}\alpha, \mathcal{P}a\rangle = -\int_{\mathbb{R}^n}\langle\mathcal{Q}^*\beta, \mathcal{P}a\rangle = 0\end{aligned}$$

Therefore $\mathcal{P}\alpha = 0$ and $\mathcal{Q}^*\beta = 0$, as desired.

THEOREM 4.2. *Each vector field $F \in L^p(\mathbb{R}^n, \mathbf{V})$, $1 < p < \infty$, admits a unique decomposition*

$$F = \mathcal{P}u + \mathcal{Q}^*w \tag{4.2}$$

with $u \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $w \in W^{1,p}(\mathbb{R}^n, \mathbf{W})$. In symbols,

$$L^p(\mathbb{R}^n, \mathbf{V}) = \mathcal{P}W^{1,p}(\mathbb{R}^n, \mathbf{U}) \oplus \mathcal{Q}^*W^{1,p}(\mathbb{R}^n, \mathbf{W}).$$

We also have a uniform bound for the components:

$$\|\mathcal{P}u\|_p + \|\mathcal{Q}^*w\|_p \le C_p\|F\|_p. \tag{4.3}$$

REMARK. Let us emphasize explicitly that u, w need not be unique, only their images $\mathcal{P}u$ and $\mathcal{Q}^*w$ are unique.

It is also possible to develop a theory of Hodge decomposition on domains $\Omega \subset \mathbb{R}^n$. But this requires some regularity of Ω if one wants to go beyond L^2-theory. The interested reader can consult [GT] and the references given there.

In the case of the elliptic complex $\mathcal{D}'(\mathbb{R}^n, \Lambda) \xrightarrow{d} \mathcal{D}'(\mathbb{R}^n, \Lambda) \xrightarrow{d} \mathcal{D}'(\mathbb{R}^n, \Lambda)$ formula (4.2) provides us with the familiar decomposition of a differential form as a sum of an exact and coexact form (no harmonic fields in $\mathbb{R}^n$). Because of this analogy we call (4.2) the *Hodge decomposition* associated with the given elliptic complex.

Associated with the Hodge decomposition (4.2) is a singular integral operator

$$S : L^p(\mathbb{R}^n, \mathbf{V}) \to L^p(\mathbb{R}^n, \mathbf{V}), \quad 1 < p < \infty,$$

acting on a function $F = \mathcal{P}u + \mathcal{Q}^*w$ by the rule

$$SF = \mathcal{P}u - \mathcal{Q}^*w.$$

Thus S acts as identity on the range of the operator $\mathcal{P}$ and minus identity on the kernel of $\mathcal{P}^*$. It is self-adjoint. Indeed, if $F = \mathcal{P}u + \mathcal{Q}^*w$ and $G = \mathcal{P}\alpha + \mathcal{Q}^*\beta$, then

$$\begin{aligned}\int_{\mathbb{R}^n} \langle SF, G\rangle &= \int_{\mathbb{R}^n} \langle \mathcal{P}u - \mathcal{Q}^*w, \mathcal{P}\alpha + \mathcal{Q}^*\beta\rangle \\ &= \int_{\mathbb{R}^n} [\langle \mathcal{P}u, \mathcal{P}\alpha\rangle - \langle \mathcal{Q}^*w, \mathcal{Q}^*\beta\rangle] \\ &= \int_{\mathbb{R}^n} \langle \mathcal{P}u + \mathcal{Q}^*w, \mathcal{P}\alpha - \mathcal{Q}^*\beta\rangle = \int_{\mathbb{R}^n} \langle F, SG\rangle\end{aligned}$$

as claimed. Since S is an involution, that is, $S \circ S = \mathrm{Id}$, it then follows that S is an isometry in $L^2(\mathbb{R}^n, \mathbf{V})$,

$$\int_{\mathbb{R}^n} |SF|^2 = \int_{\mathbb{R}^n} \langle SF, SF\rangle = \int_{\mathbb{R}^n} \langle F, F\rangle = \int_{\mathbb{R}^n} |F|^2.$$

A fundamental question of interest in the L^p-theory of PDEs concerns the sharp constant in the inequality

$$\|SF\|_p \le A_p \|F\|_p, \quad 1 < p < \infty. \tag{4.4}$$

As in the div-curl setting [IS3] we conjecture that

$$A_p = \max\left\{p - 1, \frac{1}{p-1}\right\}. \tag{4.5}$$

We now refer to Burkholder's work [Bu] and the subsequent developments [BM-S], [A], [I] to recall that inequality (4.4) with constant (4.5) would follow if one proves that

$$\mathcal{E}[F] = \int [A_p|SF| - |F|][|SF| + |F|]^{p-1} \ge 0. \tag{4.6}$$

A reason for preferring (4.6) to the inequality (4.4) is that the functional $\mathcal{E}$ is convex in the so-called singular directions (see Section 7 for the definition). The proof of this fact is much the same as in [I], so it is left to the reader. In light of the conjecture (4.5) it may very well be that $\mathcal{E}$ is also quasiconvex and, consequently, inequality (4.6) would follow.

5. Elliptic couples. Following the definitions in [IS3] we study a dramatic extension of the notion of div-curl couples.

An *elliptic couple* is a pair

$$\mathcal{F} = [A, B] = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$$

where $\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{U})$ and $\beta \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{W})$. Here Ω is any domain in $\mathbb{R}^n$, $n \ge 2$. Furthermore, we introduce the *norm*

$$|\mathcal{F}(x)|^2 = |A(x)|^2 + |B(x)|^2 \tag{5.2}$$

and the *Jacobian*

$$J(x, \mathcal{F}) = \langle A(x), B(x)\rangle_{\mathbf{V}} = \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle$$

for $x \in \Omega$. On analogy with the complex Cauchy–Riemann operators, we introduce the so-called $\pm$ *components* of $\mathcal{F}$:

$$\mathcal{F}^+ = \tfrac{1}{2}(A + B) \quad \text{and} \quad \mathcal{F}^- = \tfrac{1}{2}(A - B)$$

Thus

$$\tfrac{1}{2}|\mathcal{F}|^2 = |\mathcal{F}^+|^2 + |\mathcal{F}^-|^2 \quad \text{and} \quad J(x, \mathcal{F}) = |\mathcal{F}^+|^2 - |\mathcal{F}^-|^2.$$

6. Poincaré type inequalities. In what follows we shall exploit the following inequalities which allow us to improve regularity of some distributions without affecting their image under the operator $\mathcal{Q}$ or $\mathcal{P}^*$.

LEMMA 6.1. *For each distribution $F \in \mathcal{D}'(\mathbb{R}^n, \mathbf{V})$ with $\mathcal{Q}F \in L^2(\mathbb{R}^n, \mathbf{W})$, there exists $F_0 \in \ker \mathcal{Q}$ such that $F - F_0 \in W^{1,2}(\mathbb{R}^n, \mathbf{V})$ and we have a uniform bound*

$$\|F - F_0\|_{1,2} \le C\|\mathcal{Q}F\|_2.$$

We shall argue similarly for the dual statement:

LEMMA 6.2. *For each distribution $F \in \mathcal{D}'(\mathbb{R}^n, \mathbf{V})$ with $\mathcal{P}^*F \in L^2(\mathbb{R}^n, \mathbf{W})$, there exists F_0 such that $\mathcal{P}^*F_0 = 0$ and $F - F_0 \in W^{1,2}(\mathbb{R}^n, \mathbf{V})$ and we have a uniform bound*

$$\|F - F_0\|_{1,2} \le C\|\mathcal{P}^*F\|_2. \tag{6.1}$$

Proof of Lemma 6.1. By Hodge decomposition

$$F = \mathcal{P}\mathcal{P}^*\varphi + \mathcal{Q}^*\mathcal{Q}\varphi.$$

Consider $F_0 = F - \mathcal{Q}^*\mathcal{Q}\varphi$. Then $\mathcal{Q}F_0 = 0$ and $F - F_0 = \mathcal{Q}^*\mathcal{Q}\varphi \in L^2(\mathbb{R}^n, \mathbf{V})$. Hence $\mathcal{Q}(F - F_0) = \mathcal{Q}F \in L^2$. Computing the Fourier transforms we find that $\mathcal{Q}(\xi)\widehat{\Phi}(\xi) \in L^2$ and $\mathcal{P}^*(\xi)\widehat{\Phi}(\xi) \in L^2$, where we have set $\Phi = F - F_0$.

Let us observe the following inequality:

$$|\mathcal{Q}(\xi)y| + |\mathcal{P}^*(\xi)y| \geq c_0|\xi| \cdot |y|$$

with a positive constant c_0. In fact, suppose that $|\xi| = 1$, $|y| = 1$ (by homogeneity). If $\mathcal{Q}(\xi)y = 0$ and $\mathcal{P}^*(\xi)y = 0$, then $y \in \ker \mathcal{Q}(\xi) \cap \ker \mathcal{P}^*(\xi)$. This implies that $y = 0$, contradicting the assumption that y is a unit vector.

Applying the above inequality to $\widehat{\Phi}(\xi)$ we have

$$c_0|\xi| \cdot |\widehat{\Phi}(\xi)| \leq |\mathcal{Q}(\xi)\widehat{\Phi}(\xi)| + |\mathcal{P}^*(\xi)\widehat{\Phi}(\xi)|.$$

This implies $|\xi|\widehat{\Phi}(\xi) \in L^2$. Hence $\Phi \in W^{1,2}(\mathbb{R}^n, \mathbf{V})$ and

$$\|\Phi\|_{1,2} \leq c_2(n)\|\mathcal{Q}F\|_2. \ \blacksquare$$

Let us mention that L^p-variants of the above inequalities are also available.

7. Variational integrals and convexity concepts. This section is concerned with variational integrals defined on elliptic couples. The integrals in question take the form

$$I[\mathcal{F}] = \int_{\mathbb{R}^n} f(X, Y) \quad \text{for } \mathcal{F} = [X, Y] \in L^p(\mathbb{R}^n, \mathbf{V} \times \mathbf{V}).$$

We assume here that the integrand $f : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is at least continuous. For the duration of this paper, $W_\bullet^{1,\infty}(\Omega, \mathbf{V})$ will denote the space of Lipschitz ($\mathbf{V}$-valued) functions with compact support in $\Omega \subset \mathbb{R}^n$.

Here are three basic definitions adopted from the Calculus of Variations (see for example [D]).

Definition 7.1. f is said to be *quasiconvex* if for any constant vectors $A, B \in \mathbf{V}$ we have

$$\int_{\mathbb{R}^n} [f(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta) - f(A, B)]\, dx \geq 0$$

whenever $\alpha \in W_\bullet^{1,\infty}(\mathbb{R}^n, \mathbf{V})$ and $\beta \in W_\bullet^{1,\infty}(\mathbb{R}^n, \mathbf{W})$.

The next notion seems to be an excellent extension of rank-one convexity.

Definition 7.2. We say that f is *convex in singular directions* if the real-variable function

$$t \mapsto f(A + tX, B + tY)$$

is convex whenever $A, B, X, Y \in \mathbf{V}$ and X is orthogonal to Y in $\mathbf{V}$.

Finally, we notice that the only null Lagrangians in this setting are $\mathcal{P}\alpha$, $\mathcal{Q}^*\beta$ and the Jacobian $\langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle$, hence the following definition:

Definition 7.3. f is said to be *polyconvex* if it can be expressed as

$$f(X, Y) = g(X, Y, \langle X, Y\rangle)$$

where $g : \mathbf{V} \times \mathbf{V} \times \mathbb{R} \to \mathbb{R}$ is convex.

In the recent years a fairly large amount of work has been done trying to understand all possible connections between these notions of convexity. We will be able to recover, in this generality, most of the results which are known for $\mathcal{P} = \nabla$ and $\mathcal{Q} = \text{curl}$.

It is not difficult to see that polyconvexity implies quasiconvexity. Indeed, given $A, B \in \mathbf{V}$ and arbitrary functions $\alpha \in W_\bullet^{1,\infty}(D, \mathbf{U})$ and $\beta \in W_\bullet^{1,\infty}(D, \mathbf{W})$, supported in a bounded domain D, we can use Jensen's inequality to obtain

$$\begin{aligned}
&\frac{1}{|D|}\int_{\mathbb{R}^n} [f(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta) - f(A, B)]\, dx \\
&= \fint_D [g(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta, \langle A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta\rangle) - g(A, B, \langle A, B\rangle)]\, dx \\
&\geq g\Big[\fint_D (A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta, \langle A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta\rangle)\Big] - g(A, B, \langle A, B\rangle) \\
&= g\Big(A + \fint_D \mathcal{P}\alpha, B + \fint_D \mathcal{Q}^*\beta, \langle A, B\rangle + \fint_D \langle A, \mathcal{Q}^*\beta\rangle + \fint_D \langle \mathcal{P}\alpha, B\rangle + \fint_D \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle\Big) \\
&\quad - g(A, B, \langle A, B\rangle) = 0;
\end{aligned}$$

the first four integral averages vanish by the Divergence Theorem, the latter vanishes due to L^2-orthogonality of $\mathcal{P}\alpha$ and $\mathcal{Q}^*\beta$ (cf. Lemma 4.1). Thus f is quasiconvex.

It is more or less clear that without additional hypotheses about the elliptic complex the further analogy with the classical setting will fail. Our next result addresses this issue.

Theorem 7.1. *Suppose that the elliptic complex* (3.1) *satisfies the condition*

$$\bigcup_{|\xi|=1} \ker Q(\xi) = \mathbf{V}. \tag{7.1}$$

Then every quasiconvex function is convex in singular directions.

For the proof we need to show the inequality

$$f(\lambda\Phi + (1 - \lambda)\Psi) \leq \lambda f(\Phi) + (1 - \lambda)f(\Psi)$$

whenever $0 < \lambda < 1$ and $\Phi - \Psi = [X, Y]$ with X orthogonal to Y in $\mathbf{V}$. We shall argue with the aid of the following

LEMMA 7.2. *There exist $u \in W^{1,\infty}(\mathbb{R}^n, \mathbf{U})$ and $w \in W^{1,\infty}(\mathbb{R}^n, \mathbf{W})$ and a partition $\mathbb{R}^n = \Omega \cup \Omega'$ into disjoint measurable subsets such that*

$$[\mathcal{P}u, \mathcal{Q}^* w] = [(1-\lambda)\chi_\Omega - \lambda\chi_{\Omega'}](\Phi - \Psi), \tag{7.2}$$

$$\lim_{R\to\infty} \frac{|\Omega \cap B_R|}{|B_R|} = \lambda, \tag{7.3}$$

and therefore

$$\lim_{R\to\infty} \frac{|\Omega' \cap B_R|}{|B_R|} = 1 - \lambda. \tag{7.4}$$

Proof. We begin with the following periodic function on the real line:

$$h = h(t) = \begin{cases} (1-\lambda)t & \text{if } l \le t \le \lambda + l, \\ (1-t)\lambda & \text{if } \lambda + l \le t \le 1 + l, \end{cases}$$

where $l = 0, \pm1, \pm2, \ldots$ Thus

$$h'(t) = \begin{cases} 1-\lambda & \text{if } l \le t \le \lambda + l, \\ -\lambda & \text{if } \lambda + l \le t \le 1 + l. \end{cases} \tag{7.5}$$

Next, by the hypothesis (7.1), $X \in \ker \mathcal{Q}(\xi)$ for some unit vector ξ, which we fix for the rest of this proof. By exactness of the sequence (3.2), $X \in \operatorname{Im} \mathcal{P}(\xi)$, say $X = \mathcal{P}(\xi)U$ for some $U \in \mathbf{U}$. Since Y is orthogonal to X, by basic algebra, we infer that $Y \in \operatorname{Im} \mathcal{Q}^*(\xi)$, say $Y = \mathcal{Q}^*(\xi)W$ for some $W \in \mathbf{W}$.

Now we are in a position to define

$$u(x) = h(\langle x, \xi\rangle)U \in W^{1,\infty}(\mathbb{R}^n, \mathbf{U}),$$
$$w(x) = h(\langle x, \xi\rangle)W \in W^{1,\infty}(\mathbb{R}^n, \mathbf{W}).$$

The partition in question is in fact a lamination of $\mathbb{R}^n$ given by

$$\begin{aligned} \Omega &= \bigcup_{l=-\infty}^{\infty} \{x : l \le \langle x, \xi\rangle \le \lambda + l\}, \\ \Omega' &= \bigcup_{l=-\infty}^{\infty} \{x : \lambda + l \le \langle x, \xi\rangle \le 1 + l\} \end{aligned} \tag{7.6}$$

We then compute

$$\mathcal{P}u = h'(\langle x, \xi\rangle)\mathcal{P}(\xi)U = h'(\langle x, \xi\rangle)X,$$
$$\mathcal{Q}^* w = h'(\langle x, \xi\rangle)\mathcal{Q}^*(\xi)W = h'(\langle x, \xi\rangle)Y.$$

Hence $[\mathcal{P}u, \mathcal{Q}^* w] = h'(\langle x, \xi\rangle)[\Phi - \Psi]$ and formula (7.2) is immediate from (7.5) and (7.6). The density relations (7.3) and (7.4) follow by simple geometric considerations, completing the proof of the lemma. ∎

Proof of Theorem 7.1. Consider concentric balls $B_R \subset B_{R+1}$ and a cut-off function $\eta \in C_0^\infty(B_{R+1})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|\nabla\eta(x)| \le 2$ in $\mathbb{R}^n$. The functions $\alpha = \eta u$ and $\beta = \eta w$ are Lipschitz with support in B_{R+1},

and therefore can be used as test functions in the definition of quasiconvexity. Accordingly,

$$|B_{R+1}| f(\lambda\Phi + (1-\lambda)\Psi) \le \int_{B_{R+1}} f(\lambda\Phi + (1-\lambda)\Psi + \mathcal{F})$$

where $\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$ is an elliptic couple. We split the integral over three subdomains:

$$\int_{B_{R+1}} = \int_{\Omega\cap B_R} + \int_{\Omega'\cap B_R} + \int_{B_{R+1}-B_R}.$$

It is important to observe that

$$\mathcal{F} = \begin{cases} (1-\lambda)(\Phi - \Psi) & \text{on } \Omega \cap B_R, \\ -\lambda(\Phi - \Psi) & \text{on } \Omega' \cap B_R, \end{cases}$$

and

$$\|\mathcal{F}\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Hence, we obtain

$$|B_{R+1}| f(\lambda\Phi + (1-\lambda)\Psi) \le |\Omega \cap B_R| f(\Phi) + |\Omega' \cap B_R| f(\Psi) + c|B_{R+1} - B_R|$$

where c is a constant independent of R. Finally, dividing by $|B_R|$ and letting R go to infinity, we get the desired inequality

$$f(\lambda\Phi + (1-\lambda)\Psi) \le \lambda f(\Phi) + (1-\lambda) f(\Psi)$$

by the density relations (7.3) and (7.4). ∎

8. Quasiharmonic fields associated with an elliptic complex.

We denote by $\mathcal{E}^p(\Omega, \mathbf{V} \times \mathbf{V})$, $1 < p < \infty$, the L^p-space of elliptic couples $[\mathcal{P}\alpha, \mathcal{Q}^*\beta]$, with $\alpha \in W^{1,p}(\Omega, \mathbf{U})$ and $\beta \in W^{1,p}(\Omega, \mathbf{W})$. This space is a closed subspace of $L^p(\Omega, \mathbf{V} \times \mathbf{V})$. The proof goes as follows:

Let $\mathcal{P}\alpha_j \to A$ in $L^p(\Omega, \mathbf{V})$ and $\mathcal{Q}^*\beta_j \to B$ in $L^p(\Omega, \mathbf{V})$. We want to show that

$$A = \mathcal{P}\alpha \quad \text{for some } \alpha \in W^{1,p}(\Omega, \mathbf{U}),$$
$$B = \mathcal{Q}^*\beta \quad \text{for some } \beta \in W^{1,p}(\Omega, \mathbf{W}).$$

Let φ_j be the $W_0^{2,p}(\Omega)$-solution of the Poisson equation

$$\mathcal{P}\alpha_j = \Delta\varphi_j.$$

Then

$$\|\nabla^2\varphi_j\|_p \le c_p \|\mathcal{P}\alpha_j\|_p \le C$$

with a constant c_p independent of j. We may assume that φ_j converges to a function φ, weakly in $W_0^{2,p}(\Omega)$. Then the functions $\mathcal{P}\alpha_j = \mathcal{P}\mathcal{P}^*\varphi_j + \mathcal{Q}^*\mathcal{Q}\varphi_j$ converge to $A = \mathcal{P}\mathcal{P}^*\varphi + \mathcal{Q}^*\mathcal{Q}\varphi$, weakly in $L^p(\Omega, \mathbf{V})$. It remains to prove that $\mathcal{Q}^*\mathcal{Q}\varphi = 0$: we then have $A = \mathcal{P}\alpha$ with $\alpha = \mathcal{P}^*\varphi$, as desired.

Consider an arbitrary $\Psi \in L^q(\Omega,\mathbf{V})$, and decompose it as

$$\Psi = \mathcal{P}\mathcal{P}^*\psi_0 + \mathcal{Q}^*\mathcal{Q}\psi_0$$

with some $\psi_0 \in W_0^{2,q}(\Omega,\mathbf{V})$. By density, for each $\varepsilon > 0$, there exists $\psi \in C_0^\infty(\Omega,\mathbf{V})$ such that

$$\|\psi - \psi_0\|_{W^{2,q}} \le \varepsilon.$$

Then

$$\Psi = (\mathcal{P}\mathcal{P}^*\psi + \mathcal{Q}^*\mathcal{Q}\psi) + \mathcal{P}\mathcal{P}^*(\psi-\psi_0) + \mathcal{Q}^*\mathcal{Q}(\psi-\psi_0).$$

Therefore,

$$\int_\Omega \langle \Psi, \mathcal{Q}^*\mathcal{Q}\varphi\rangle = \int_\Omega \langle \mathcal{P}\mathcal{P}^*(\psi-\psi_0), \mathcal{Q}^*\mathcal{Q}\varphi\rangle + \int_\Omega \langle \mathcal{Q}^*\mathcal{Q}(\psi-\psi_0), \mathcal{Q}^*\mathcal{Q}\varphi\rangle.$$

The first integral vanishes, while the second one can be made as small as we wish by choosing ε small enough. Hence

$$\langle \Psi, \mathcal{Q}^*\mathcal{Q}\varphi\rangle = 0 \quad \text{for every } \Psi \in L^q(\Omega,\mathbf{V}),$$

which means that $\mathcal{Q}^*\mathcal{Q}\varphi = 0$.

The remainder of this section is reserved to some definitions. For $\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta] \in \mathcal{E}^p(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$ we can introduce the norm

$$|\mathcal{F}(x)| = (|\mathcal{P}\alpha|^2 + |\mathcal{Q}^*\beta|^2)^{1/2}.$$

Then the following, rather obvious, relation can be viewed as an analogue of the Hadamard inequality for determinants:

$$2J(x,\mathcal{F}) \le |\mathcal{F}(x)|^2.$$

DEFINITION 8.1. An elliptic couple $\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$ is called *K-quasiharmonic* with $1 \le K = K(x) < \infty$ if

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x) J(x,\mathcal{F}) \tag{8.1}$$

where $\mathcal{K}(x) = K(x) + 1/K(x) \ge 2$.

Precisely, the inequality (8.1) yields

$$|\mathcal{F}^-(x)| \le \frac{K(x)-1}{K(x)+1} |\mathcal{F}^+(x)|$$

where the $\pm$ components of $\mathcal{F}$ are defined by the rules

$$\mathcal{F}^- = \tfrac12(\mathcal{P}\alpha - \mathcal{Q}^*\beta) \quad \text{and} \quad \mathcal{F}^+ = \tfrac12(\mathcal{P}\alpha + \mathcal{Q}^*\beta).$$

9. $\mathcal{H}^1$-theory of the Jacobian. One special feature of the Jacobian we shall use here is its higher regularity, already recognized by H. Wente [W] and developed by S. Müller [M] (see also [IS1], [IV1,2]). The book of E. Stein [St2] is particularly useful here.

THEOREM 9.1. *Let $\mathcal{F} \in L^2(\Omega, \mathbf{V}\times\mathbf{V})$ be an elliptic couple. Then $J(x,\mathcal{F}) \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$.*

We only sketch the proof as it is similar to one in [CLMS].

Proof. Fix an arbitrary subdomain Ω' compactly contained in Ω, and fix an arbitrary $\eta \in C_0^\infty(\Omega)$ which is equal to 1 on Ω'. For each test function $\varphi \in C_0^\infty(\Omega')$, we shall estimate the integral

$$\int_\Omega \varphi(x) J(x,\mathcal{F})\,dx = \int_\Omega \langle \varphi\mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle = \int_{\mathbb{R}^n} \langle \varphi\mathcal{P}\eta\alpha, \mathcal{Q}^*\eta\beta\rangle$$

(η equals 1 on the support of φ).

We use the Hodge decomposition in the entire space $\mathbb{R}^n$ to write

$$\varphi\mathcal{P}\eta\alpha = \mathcal{P}\alpha' + \mathcal{Q}^*\beta'.$$

Observe that the component $\mathcal{Q}^*\beta'$ can be expressed as a singular integral of $\varphi\mathcal{P}\eta\alpha$, say $\mathcal{Q}^*\beta' = \mathbf{B}(\varphi\mathcal{P}\eta\alpha)$. The singular integral operator $\mathbf{B} : L^p(\mathbb{R}^n,\mathbf{V}) \to L^p(\mathbb{R}^n,\mathbf{V})$, projection onto $\mathcal{Q}^*W^{1,p}(\mathbb{R}^n,\mathbf{W}) \subset L^p(\mathbb{R}^n,\mathbf{V})$, is bounded for all $1 < p < \infty$. It is also important to observe that $\mathbf{B}$ vanishes on the subspace $\mathcal{P}W^{1,p}(\mathbb{R}^n,\mathbf{U})$. Therefore, we can look at $\mathcal{Q}^*\beta'$ as the image of $\mathcal{P}\eta\alpha$ under the commutator of $\mathbf{B}$ with the multiplication by φ:

$$\mathcal{Q}^*\beta' = (\mathbf{B}\varphi - \varphi\mathbf{B})(\mathcal{P}\eta\alpha).$$

Next, we apply the celebrated commutator result of R. Coifman, R. Rochberg and G. Weiss [CRW] which implies that

$$\|\mathcal{Q}^*\beta'\|_2 \le C(n)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{P}(\eta\alpha)\|_2.$$

Since $\mathcal{P}\alpha'$ is orthogonal to $\mathcal{Q}^*\eta\beta$, by Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x)J(x,\mathcal{F})\,dx &= \int_{\mathbb{R}^n}\langle \mathcal{P}\alpha', \mathcal{Q}^*\eta\beta'\rangle + \int_{\mathbb{R}^n}\langle \mathcal{Q}^*\beta', \mathcal{Q}^*\eta\beta\rangle \\ &\le \|\mathcal{Q}^*\beta'\|_2\|\mathcal{Q}^*\eta\beta\|_2 \\ &\le C(n)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{P}\eta\alpha\|_2\|\mathcal{Q}^*\eta\beta\|_2 \\ &\le c(n,\eta)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{F}\|_2^2.\end{aligned}$$

In conclusion,

$$\int_\Omega \varphi(x)J(x,\mathcal{F})\,dx \le C(n,\eta)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{F}\|_2^2.$$

In view of the BMO-$\mathcal{H}^1$ duality it follows that $J(x,\mathcal{F}) \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$. We also have the local bounds

$$\|J(x,\mathcal{F})\|_{\mathcal{H}^1(\Omega')} \le C_{\Omega'}\|\mathcal{F}\|_2^2. \ ■$$

Further, if $J(x,\mathcal{F}) \ge 0$, by Theorem 2 of E. Stein [St1] we find that it belongs to the Zygmund class $L\log L_{\mathrm{loc}}(\Omega)$.

For later use we record the inequality

$$\|J(x,\mathcal{F})\|_{L\log L(\Omega')} \le C_{\Omega'}\|J(x,\mathcal{F})\|_{\mathcal{H}^1(\Omega')} \le C_{\Omega'}\|\mathcal{F}\|_2^2. \tag{9.1}$$

The study of Jacobians in Hardy spaces was originated in [CLMS] and then pursued in [IV2].

10. Limit theorems. Just as in the theory of quasiconformal mappings, constructions of quasiharmonic fields rely on limiting processes. At this point of development it is of interest to know that such fields are closed under weak convergence. The following theorem addresses this issue.

THEOREM 10.1. *Let $\mathcal{F}_\nu$ be a sequence of quasiharmonic fields converging to $\mathcal{F}$ weakly in $L^2(\Omega,\mathbf{V}\times\mathbf{V})$ and suppose that the distortion functions $\mathcal{K}_\nu$ converge to $\mathcal{K}$ weakly in $L^1(\Omega)$. Then $\mathcal{F}$ is a quasiharmonic field of distortion $\mathcal{K}$.*

We will need the following two lemmas.

LEMMA 10.2 (lower semicontinuity of the norm). *For every $\eta\in L^\infty(\Omega)$, $\eta\ge 0$ and $\mathcal{F}_\nu$ converging to $\mathcal{F}$ weakly in $L^2(\Omega,\mathbf{V}\times\mathbf{V})$, we have*

$$\int_\Omega \eta(x)|\mathcal{F}(x)|\,dx \le \liminf_{\nu\to\infty}\int_\Omega \eta(x)|\mathcal{F}_\nu(x)|\,dx. \tag{10.1}$$

Proof. We have

$$\int_\Omega \eta(x)(|\mathcal{F}_\nu(x)|-|\mathcal{F}(x)|)\,dx \ge \int_\Omega \langle \eta(\mathcal{F}_\nu-\mathcal{F}),\mathcal{F}/|\mathcal{F}|\rangle\,dx$$

where we have defined $\mathcal{F}/|\mathcal{F}|=0$ at the points where $|\mathcal{F}|=0$. Since $\eta\mathcal{F}/|\mathcal{F}|\in L^\infty(\Omega)$ and $\mathcal{F}_\nu$ converges to $\mathcal{F}$ weakly in $L^2(\Omega,\mathbf{V}\times\mathbf{V})$, inequality (10.1) follows. ■

Our next prerequisite is the weak continuity property of the Jacobian. The proof presented here relies on the additional assumption that the elliptic complex can be prolonged either from the left or from the right, say

$$\mathcal{D}'(\mathbb{R}^n,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n,\mathbf{W}) \xrightarrow{\mathcal{R}} \mathcal{D}'(\mathbb{R}^n,\mathbf{Z}).$$

Assuming ellipticity of this longer complex we can apply the Poincaré inequality (6.1) to conclude that

LEMMA 10.3. *For each distribution $\beta\in\mathcal{D}'(\mathbb{R}^n,\mathbf{W})$ with $\mathcal{Q}^*\beta\in L^2(\mathbb{R}^n,\mathbf{V})$ there exists $\beta^0\in\ker\mathcal{Q}^*$ such that*

$$\|\beta-\beta^0\|_{W^{1,2}(\mathbb{R}^n,\mathbf{W})} \le C\|\mathcal{Q}^*\beta\|_{L^2(\mathbb{R}^n,\mathbf{V})}.$$

For notational convenience we denote by $L^\infty_\bullet(\Omega)$ the space of functions in $L^\infty(\Omega)$ with compact support.

LEMMA 10.4. (weak continuity of the Jacobian). *Under the assumptions of Theorem 10.1, for every $\lambda\in L^\infty_\bullet(\Omega)$ we have*

$$\int_\Omega \lambda(x)J(x,\mathcal{F})\,dx = \lim_{\nu\to\infty}\int_\Omega \lambda(x)J(x,\mathcal{F}_\nu)\,dx. \tag{10.2}$$

Proof. Let $\mathcal{F}_\nu=[\mathcal{P}\alpha_\nu,\mathcal{Q}^*\beta_\nu]$, where $\mathcal{P}\alpha_\nu\rightharpoonup\mathcal{P}\alpha$ and $\mathcal{Q}^*\beta_\nu\rightharpoonup\mathcal{Q}^*\beta$, weakly in $L^2(\Omega,\mathbf{W})$. The field $\mathcal{Q}^*\beta_\nu$ is not affected if we subtract from β_ν a function in $\ker\mathcal{Q}^*$. With the aid of Lemma 10.3 we can modify the sequence $\{\beta_\nu\}$ to ensure that it stays bounded in $W^{1,2}(\Omega,\mathbf{W})$ (details being left to the reader). Therefore, we can extract a subsequence $\{\beta_{\nu_k}\}$ converging strongly in $L^2(\Omega,\mathbf{W})$ to β. With these preliminary adjustments we proceed to the proof of (10.2).

First assume that $\lambda\in C_0^\infty(\Omega)$. Integration by parts yields

$$\begin{aligned}\int_\Omega \lambda(x)J(x,\mathcal{F})\,dx &= \int_\Omega \lambda\langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle\,dx = \int_\Omega\langle\mathcal{Q}\lambda\mathcal{P}\alpha,\beta\rangle\,dx\\ &= \int_\Omega\langle\nabla\lambda\otimes\mathcal{P}\alpha,\beta\rangle\,dx = \lim_{k\to\infty}\int_\Omega\langle\nabla\lambda\otimes\mathcal{P}\alpha_{\nu_k},\beta_{\nu_k}\rangle\,dx\\ &= \lim_{k\to\infty}\int_\Omega\lambda(x)J(x,\mathcal{F}_{\nu_k})\,dx\end{aligned}$$

as $\mathcal{P}\alpha_{\nu_k}$ converges weakly in $L^2(\Omega)$ to $\mathcal{P}\alpha$ and β_{ν_k} converges strongly in $L^2(\Omega,\mathbf{W})$ to β.

Notice that the limit function does not depend on the subsequence we have extracted. Therefore, we also have the convergence in $\mathcal{D}'(\Omega)$ of the entire sequence

$$\lim_{\nu\to\infty}\int_\Omega\lambda(x)J(x,\mathcal{F}_\nu)\,dx = \int_\Omega\lambda(x)J(x,\mathcal{F})\,dx.$$

To get rid of the redundant assumption that $\lambda\in C_0^\infty(\Omega)$ we now fix any $\lambda\in L^\infty(\Omega)$ supported in a compact subdomain, say $\Omega'\subset\Omega$. We then approximate λ a.e. by a sequence $\lambda_j\in C_0^\infty(\Omega)$ of functions supported in Ω' such that $\|\lambda_j\|\ \le\|\lambda\|_\infty$. We can write

$$\begin{aligned}\Big|\int_\Omega\lambda(x)[J(x,\mathcal{F}_\nu)-J(x,\mathcal{F})]\,dx\Big| &\le \int_\Omega|\lambda(x)-\lambda_j(x)|\cdot|J(x,\mathcal{F}_\nu)|\,dx\\ &\quad+\int_\Omega|\lambda(x)-\lambda_j(x)|\cdot|J(x,\mathcal{F})|\,dx\\ &\quad+\Big|\int_\Omega\lambda_j(x)[J(x,\mathcal{F}_\nu)-J(x,\mathcal{F})]\,dx\Big|.\end{aligned}$$

Hence, using the elementary inequality

$$ab\le\varepsilon\Big[a\log\Big(e+\frac{a}{N}\Big)+N(e^{b/\varepsilon}-1)\Big]$$

for positive numbers a, b, ε and N, we obtain

$$\begin{aligned}
\limsup_{\nu\to\infty} \Big| \int_\Omega \lambda(x)[J(x,\mathcal{F}_\nu) - J(x,\mathcal{F})]\,dx \Big| &\le \int_\Omega |\lambda(x) - \lambda_j(x)| \cdot |J(x,\mathcal{F})|\,dx \\
&\quad + \varepsilon \limsup_{\nu\to\infty} \bigg[\int_{\Omega'} J(x,\mathcal{F}_\nu) \log\Big(e + \frac{J(x,\mathcal{F}_\nu)}{\|J(x,\mathcal{F}_\nu)\|_{L^1(\Omega')}}\Big)\,dx \\
&\quad + \|J(x,\mathcal{F}_\nu)\|_{L^1(\Omega)} \int_\Omega (e^{|\lambda-\lambda_j|/\varepsilon} - 1)\,dx \bigg] \\
&\quad + \lim_{\nu\to\infty} \Big| \int_\Omega \lambda_j(x)[J(x,\mathcal{F}_\nu) - J(x,\mathcal{F})]\,dx \Big|.
\end{aligned}$$

As already shown, for $\lambda_j \in C_0^\infty(\Omega)$ the latter term vanishes. The first term can be made as small as we wish, by the Dominated Convergence Theorem. The middle term is bounded by

$$\varepsilon\Big[C + C\int_\Omega (e^{|\lambda-\lambda_j|/\varepsilon} - 1)\,dx\Big]$$

where the constant C depends neither on ε nor on j. This is because the Jacobians $J(x,\mathcal{F}_\nu)$ stay bounded in $L\log L(\Omega')$ (see (9.1)). Therefore this term can also be made as small as we wish by first choosing ε sufficiently small and then taking j sufficiently large. In conclusion,

$$\lim_{\nu\to\infty} \int_\Omega \lambda(x)[J(x,\mathcal{F}_\nu) - J(x,\mathcal{F})]\,dx = 0$$

as claimed.

Proof of Theorem 10.1. Fix $\varepsilon > 0$ and $\delta > 0$. Then

$$\frac{|\mathcal{F}_\nu(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F}_\nu)} \le \mathcal{K}_\nu(x).$$

Algebraic calculations reveal that

$$\begin{aligned}
&\frac{|\mathcal{F}_\nu(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F}_\nu)} - \frac{|\mathcal{F}(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F})} \\
&\qquad \ge \frac{2|\mathcal{F}(x)|(|\mathcal{F}_\nu(x)| - |\mathcal{F}(x)|)}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F})} - \frac{|\mathcal{F}(x)|^2[J(x,\mathcal{F}_\nu) - J(x,\mathcal{F})]}{(\delta + \varepsilon|\mathcal{F}| + J(x,\mathcal{F}))^2}.
\end{aligned}$$

For every nonnegative test function $\varphi \in L^\infty_\bullet(\Omega)$, we can write

$$\begin{aligned}
&\int_\Omega \frac{\varphi(x)|\mathcal{F}_\nu(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F}_\nu)}\,dx - \int_\Omega \frac{\varphi(x)|\mathcal{F}(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F})}\,dx \\
&\qquad \ge \int_\Omega \frac{2\varphi(x)|\mathcal{F}(x)|(|\mathcal{F}_\nu(x)| - |\mathcal{F}(x)|)}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F})}\,dx \\
&\qquad\quad - \int_\Omega \frac{\varphi(x)|\mathcal{F}(x)|^2[J(x,\mathcal{F}_\nu) - J(x,\mathcal{F})]}{(\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F}))^2}\,dx.
\end{aligned}$$

By Lemmas 10.2 and 10.4 this estimate yields

$$\int_\Omega \frac{\varphi(x)|\mathcal{F}(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F})}\,dx \le \liminf_{\nu\to\infty} \int_\Omega \frac{\varphi(x)|\mathcal{F}_\nu(x)|^2}{\delta + \varepsilon|\mathcal{F}(x)| + J(x,\mathcal{F}_\nu)}\,dx,$$

and from distortion inequality this is

$$\le \liminf_{\nu\to\infty} \int_\Omega \varphi(x)\mathcal{K}_\nu(x)\,dx = \int_\Omega \varphi(x)\mathcal{K}(x)\,dx.$$

By the Monotone Convergence Theorem we can pass to the limit as ε goes to zero:

$$\int_\Omega \frac{\varphi(x)|\mathcal{F}(x)|^2}{\delta + J(x,\mathcal{F})}\,dx \le \int_\Omega \varphi(x)\mathcal{K}(x)\,dx.$$

Since φ was arbitrary and nonnegative in $L^\infty_\bullet(\Omega)$, it follows that

$$\frac{|\mathcal{F}(x)|^2}{\delta + J(x,\mathcal{F})} \le \mathcal{K}(x) \quad \text{a.e.}$$

Hence

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x)[\delta + J(x,\mathcal{F})].$$

The last inequality holds for every $\delta > 0$, so for $\delta = 0$ as well:

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x)J(x,\mathcal{F}) \quad \text{a.e.},$$

completing the proof. ■

References

[A] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37–60.

[BM-S] A. Baernstein and S. J. Montgomery-Smith, *Some conjectures about integral means of ∂f and $\overline{\partial} f$*, to appear.

[Bu] D. L. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Astérisque 157–158 (1988), 75–94.

[C] H. Cartan, *Differential Forms*, Hermann, Paris, 1967.

[CLMS] R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. 72 (1993), 247–286.

[CRW] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 569–645.

[D] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer, Berlin, 1989.

[ET] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, New York, 1976.

[FM] I. Fonseca and S. Müller, *A-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal. 30 (1999), 1355–1390.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1977.

[I] T. Iwaniec, *Nonlinear Cauchy–Riemann operators in* $\mathbb{R}^n$, Proc. Amer. Math. Soc., to appear.

[IS1] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. 119 (1992), 129–143.

[IS2] —, —, *Div-curl fields of finite distortion*, C. R. Acad. Sci. Paris Sér. I 327 (1998), 729–734.

[IS3] —, —, *Quasiharmonic fields*, Ann. Inst. H. Poincaré, to appear.

[ISS] T. Iwaniec, C. Scott and B. Stroffolini, *Nonlinear Hodge theory on manifolds with boundary*, Ann. Mat. Pura Appl. 177 (1999), 37–115.

[IV1] T. Iwaniec and A. Verde, *Note on the operator* $\mathcal{L}(f) = f\log|f|$, J. Funct. Anal. 169 (1999), 391–420.

[IV2] —, —, *A study of Jacobians in Hardy–Orlicz spaces*, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 539–570.

[M] S. Müller, *Higher integrability of determinants and weak convergence in* L^1, J. Reine Angew. Math. 412 (1990), 20–34.

[R] Yu. G. Reshetnyak, *Stability theorems for mappings with bounded distortion*, Siberian Math. J. 9 (1968), 499–512.

[Š] V. Šverák, *Rank one convexity does not imply quasiconvexity*, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992), 185–189.

[St1] E. M. Stein, *Note on the class* $L\log L$, Studia Math. 32 (1969), 305–310.

[St2] —, *Harmonic Analysis*, Princeton Univ. Press, 1993.

[W] H. Wente, *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26 (1969), 318–344.

Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
via Cintia
80126 Napoli, Italy
E-mail: giannett@matna2.dma.unina.it
aversde@matna2.dma.unina.it

Received July 26, 1999 (4366)