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A splitting theory for the space of distributions

by

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Abstract. The splitting problem is studied for short exact sequences consisting of countable projective limits of DFN-spaces

$$(*) \qquad 0 \to F \to X \to G \to 0,$$

where F or G are isomorphic to the space of distributions $\mathcal{D}'$. It is proved that every sequence $(*)$ splits for $F \simeq \mathcal{D}'$ iff G is a subspace of $\mathcal{D}'$ and that, for ultrabornological F, every sequence $(*)$ splits for $G \simeq \mathcal{D}'$ iff F is a quotient of $\mathcal{D}'$.

0. Introduction. The main aim of the paper is to give a complete splitting theory for the space of distributions $\mathcal{D}'$ in a spirit similar to the previous splitting theory for power series spaces (see [V1] or [MV, Sections 30, 31]), and especially for the space s of rapidly decreasing sequences.

We will consider the whole theory in the categories of PLS-spaces and PLN-spaces, i.e., projective limits of countable spectra of LS-spaces and LN-spaces, the latter meaning inductive limits of sequences of Banach spaces where the linking maps are compact or nuclear, respectively. So we will consider short topologically exact sequences of PLS-spaces

$$(**) \qquad 0 \to F \xrightarrow{j} X \xrightarrow{q} G \to 0,$$

i.e., j is an embedding onto the kernel of the surjective map q, and "topologically" means that both j and q are open onto their images. Throughout the paper *algebraically exact* means only that each map in the diagram is surjective onto the kernel of the next map, while just *exact* means algebraically *and* topologically exact. The sequence $(**)$ *splits* whenever $j(F)$

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is complemented in X or, equivalently, q has a linear and continuous right inverse. The splitting of all such sequences $(**)$ (where F and G are fixed and all the spaces are PLS-spaces) is denoted by $\mathrm{Ext}^1_{\mathrm{PLS}}(G,F)=0$.

We show that a PLN-space G is a subspace of $\mathcal{D}'$ iff $\mathrm{Ext}^1_{\mathrm{PLS}}(G,\mathcal{D}')=0$. On the other hand a PLN-space F satisfies $\mathrm{Ext}^1_{\mathrm{PLS}}(\mathcal{D}',F)=0$ if and only if F is a continuous image of $\mathcal{D}'$ satisfying some condition which means exactly that $\mathrm{Ext}^1_{\mathrm{PLS}}(\omega,F)=0$. If we assume that G and F are ultrabornological (or, equivalently, $\mathrm{Proj}^1 F=\mathrm{Proj}^1 G=0$, see below), then $\mathrm{Ext}^1_{\mathrm{PLS}}(G,\mathcal{D}')=0$ is equivalent to the fact that there exists a short exact sequence

$$0\to G\to\mathcal{D}'\to\mathcal{D}'\to 0$$

while $\mathrm{Ext}^1_{\mathrm{PLS}}(\mathcal{D}',F)=0$ is equivalent to the fact that F is a topological quotient of $\mathcal{D}'$ or even there exists a short exact sequence

$$0\to\mathcal{D}'\to\mathcal{D}'\to F\to 0.$$

As we have seen the results obtained are completely analogous to that for s, the more so since a PLS-space F is isomorphic to a complemented subspace of $\mathcal{D}'$ if and only if it is isomorphic to a subspace and isomorphic to a quotient of $\mathcal{D}'$, as proved in [DV2]. The results of [DV2] were applied to the problem of splitting of differential complexes appearing naturally in the theory of systems of partial differential equations (see [P] and [T]). Let us mention that a similar (but essentially different) theory for the space of smooth functions $C^\infty(\Omega)$ has also been developed (see [DV1]). Finally, let us recall that the isomorphic type of $\mathcal{D}'=\mathcal{D}'(\Omega)$ does not depend on the open set $\Omega\subseteq\mathbb{R}^n$ (see e.g. [V5]) and $\mathcal{D}'\simeq(s')^{\mathbb{N}}$ where s' denotes the strong dual of the space s.

1. Notation and preliminaries. For basic properties of LS-spaces see [F1] (comp. also [F2]). We will frequently use the fact that a closed subspace or a Hausdorff quotient of an LS-space (LN-space) is an LS-space (LN-space, resp.).

Usually, we will denote by F a PLS-space which is a limit of a projective spectrum $\mathcal{F}=(F_K,i_K^{K+1})$, where $F_K=\mathrm{Ind}_l F_{Kl}$, F_{Kl} are Banach spaces with closed unit balls B_{Kl} and the linking maps in the inductive spectrum are compact. The notation concerning the linking maps $i_K^{K+1}:F_{K+1}\to F_K$, $i_K:F\to F_K$ will be used also for spectra other than $\mathcal{F}$. We write $F=\mathrm{Proj}\,\mathcal{F}$. Generally, we denote by script capital letters projective spectra and by corresponding Roman capital letters their limits. A spectrum $\mathcal{F}$ is called *reduced* if

$$\forall k\ \exists l\ \forall m\ge l\quad \overline{i_k^m F_m}=\overline{i_k^l F_l}$$

and, it is called *strict* if

$$\forall k\ \exists l\ \forall m\ge l\quad i_k^m F_m=i_k^l F_l$$

or, equivalently, F has a representing spectrum with surjective linking maps. Without loss of generality we consider from now on only reduced spectra representing PLS- or PLN-spaces. For a countable spectrum $\mathcal{F}$ (or the corresponding limit F) we can construct the so-called *canonical resolution*:

$$0\to F\to\prod_{K\in\mathbb{N}}F_K\xrightarrow{\sigma_{\mathcal{F}}}\prod_{K\in\mathbb{N}}F_K\to 0$$

where

$$j(f)=(i_Kf)_{K\in\mathbb{N}},\qquad \sigma_{\mathcal{F}}((f_K)_{K\in\mathbb{N}})=(i_K^{K+1}f_{K+1}-f_K)_{K\in\mathbb{N}}\in\prod_{K\in\mathbb{N}}F_K.$$

We denote $\prod F_K/\mathrm{Im}\,\sigma_{\mathcal{F}}$ by $\mathrm{Proj}^1\mathcal{F}$ (or $\mathrm{Proj}^1 F$, since for PLS-spaces that quotient does not depend on the reduced representing spectrum, see [V2] or [V3]). Clearly, by the Open Mapping Theorem, $\mathrm{Proj}^1 F=0$ means that the fundamental resolution is exact. If every short exact sequence $(**)$ of LS-spaces splits we say that $\mathrm{Ext}^1_{\mathrm{LS}}(G,F)=0$. If we consider splitting in the category of Fréchet spaces, we analogously write $\mathrm{Ext}^1(\cdot,\cdot)=0$. The above notation has its source in Homological Algebra. An interested reader can find more on the "homological theory" of lcs in [P1], [P2], [Re1], [Re2], [V1]–[V4].

If $B\subseteq C$ are closed absolutely convex bounded sets in an lcs X, then X_B denotes $\mathrm{lin}\,B$ equipped with the gauge functional of B as its norm and $i_{BC}:X_B\to X_C$ denotes the identity embedding. We call $B\subseteq X$ a *Banach disc* if X_B is a Banach space. If $Y\subseteq X$, then $\overline{Y}^B:=\bigcap_{\varepsilon>0}\varepsilon B+Y$. The condition $\mathrm{Proj}^1 F=0$ for PLS-spaces was characterized by Retakh [Re1, Th. 3] as follows: $\mathrm{Proj}^1 F=0$ iff there is a sequence $\mathcal{B}=(B_K)$ of Banach discs, $B_K\subseteq F_K$, such that

$$i_K^{K+1}B_{K+1}\subseteq B_K\quad\text{and}\quad\forall K\ \exists L_0\ \forall L\ge L_0\quad i_K^L F_L\subseteq\overline{i_K(F)}^{B_K}.$$

It turns out that this condition also has some topological consequences as shown by Vogt (necessity part, [V2, 5.7], [V3, 3.4]) and Wengenroth (sufficiency part, [W]).

THEOREM 1.1. *A PLS-space X satisfies* $\mathrm{Proj}^1 X=0$ *if and only if X is ultrabornological.*

By F^{ub} we denote the ultrabornological space associated with F.

A family of sets $(W_{(n_1,\dots,n_k)})$ indexed by all finite sequences of natural numbers is called a *strict web* in E if the following conditions are satisfied (if $\varphi\in\mathbb{N}^{\mathbb{N}}$, $k\in\mathbb{N}$, then $W_{\varphi,k}:=W_{(\varphi(1),\dots,\varphi(k))}$) (see [J, Ch. 5] or [DW], comp. [MV]):

(W1) $W_{\varphi,k}$ are balanced convex sets;
(W2) $\bigcup\{W_{\varphi,1}:\varphi\in\mathbb{N}^{\mathbb{N}}\}$ is absorbing in E;

(W3) for all φ, k, $\bigcup\{W_{\psi,k+1} : \psi \in \mathbb{N}^{\mathbb{N}},\ \psi(i) = \varphi(i),\ i \le k\}$ is absorbing in $\operatorname{lin} W_{\varphi,k}$;

(W4) $W_{\varphi,k+1} + W_{\varphi,k+1} \subseteq W_{\varphi,k}$;

(W5) for every 0-neighbourhood U in E and every φ there is n such that $W_{\varphi,n} \subseteq U$;

(W6) if $(y_n) \subseteq E$, $y_n \in W_{\varphi,n}$ for some fixed φ, then for every $k \in \mathbb{N}$ the series $\sum_{n=k+1}^{\infty} y_n$ is convergent to an element of $W_{\varphi,k}$.

We call an lcs space E a *strictly webbed space* if it has a strict web. The class is closed with respect to taking closed subspaces, quotients, countable projective or inductive limits and contains all Banach spaces; in particular, all PLS-spaces are strictly webbed.

Now, we need some permanence properties of PLS- and PLN-spaces.

PROPOSITION 1.2. *Every closed subspace of a PLS-space* (*PLN-space*) *is a PLS-space* (*PLN-space*) *as well.*

Proof. Let $F = \operatorname{Proj} \mathcal{F}$, where $\mathcal{F} = (F_K, i_K^{K+1})$ is a spectrum of LS-spaces. Let $X \subseteq F$ be a closed subspace and let $X_K := \overline{i_K(X)}$. Obviously, X_K is an LS-space, $j_K^{K+1} := i_K^{K+1}|_{X_{K+1}}$ acts from X_{K+1} into X_K. It is enough to observe that X is equal to the projective limit of the constructed spectrum $\mathcal{X} = (X_K, j_K^{K+1})$. Indeed, topologically we have

$$X \subseteq \operatorname{Proj} \mathcal{X} \subseteq \operatorname{Proj} \mathcal{F}.$$

On the other hand, if $f = (f_K) \in \operatorname{Proj} \mathcal{X}$ and $f_K \in X_K$, then for every K and every 0-neighbourhood U in F_K there is $g_K^U \in X$ such that $i_K(g_K^U) - f_K \in U$. That means that the net (g_K^U) tends to f and, by completeness of X, $f \in X$.

For PLN-spaces the proof is exactly the same.

Unfortunately, there are quotients of $\mathcal{D}'$ which are not complete [F2, 5.2], and thus not PLS-spaces. Nevertheless, we have:

THEOREM 1.3. *Every complete Hausdorff quotient of a PLS-space* (*PLN-space, respectively*) *is a PLS-space* (*PLN-space, respectively*).

Proof. Let $F = \operatorname{Proj} F_K$ be a PLS-space and let A be its closed subspace. We define $A_K := \overline{i_K(A)} \subseteq F_K$, $Y := F/A$, $Y_K := F_K/A_K$. We have obtained a projective spectrum (Y_K, I_K^{K+1}), where $I_K^{K+1} : Y_{K+1} \to Y_K$ is induced by $i_K^{K+1} : F_{K+1} \to F_K$. We get a "projective spectrum" of short exact sequences:

$$0 \to A_K \to F_K \to F_K/A_K \to 0.$$

The following sequence is the limit of this spectrum:

$$0 \to A \xrightarrow{j} F \xrightarrow{q} \operatorname{Proj} F_K/A_K \to 0,$$

where $\ker q = j(A)$, j is a topological embedding and $\operatorname{Im} q = F/A$ (topologically) is a dense topological subspace of $\operatorname{Proj} F_K/A_K$. Since $\operatorname{Im} q$ is complete, $F/A = \operatorname{Proj} F_K/A_K$.

Of course, if F_K are LS-spaces, then F_K/A_K are LS-spaces as well.

For PLN-spaces the proof is exactly the same as for PLS-spaces.

COROLLARY 1.4. *Let F be a PLS-space satisfying* $\operatorname{Proj}^1 F = 0$ *and let A be a closed subspace of F. The following assertions are equivalent*:

(1) *F/A is a PLS-space*;

(2) *F/A is complete*;

(3) $\operatorname{Proj}^1 A = 0$.

Proof. (1)⇔(2) follows from 1.3.

(3)⇔(2). By the proof of 1.3, F/A is a PLS-space iff q is onto. Now, (3)⇔(2) follows from [V3, Th. 5.1].

PLS-spaces have nice and useful properties.

LEMMA 1.5. *Let X be a PLS-space, Y an lcs and let $q : X \to Y$ be a continuous map. If $T : U \to Y$, $T(U) \subseteq q(X)$, is an operator with a Banach domain U, then there is a compact Banach disc C in X such that $q(C) \supseteq T(B_U)$, B_U the unit ball of U.*

Proof. By the Webbed Closed Graph Theorem [MV, 24.31], we can assume without loss of generality that q is a quotient map, thus if $(W_{\varphi,k})$ is a strict web in X then $q(W_{\varphi,k})$ is a strict web in Y.

First, we observe that Y is *webbed Schwartz*, i.e., for every φ there is ψ such that

$$\forall k\ \exists n\ \forall \varepsilon > 0 \quad q(W_{\varphi,n}) \subseteq \bigcup_{i=1}^{p} (x_i + \varepsilon q(W_{\psi,k}))$$

for some finite set $\{x_1, \dots, x_p\} \subseteq Y$. Indeed, LS-spaces are webbed Schwartz and the class is obviously closed with respect to quotients, closed subspaces and countable products. Using the Localization Theorem [J, 5.6.3] we find φ such that for every k,

$$B := T(B_U) \subseteq \bigcup_{i=1}^{n} (x_i + q(W_{\varphi,k}))$$

(i.e., B is *webbed compact*).

The family $(W_{\varphi,k})_{k\in\mathbb{N}}$ of absolutely convex sets forms a 0-neighbourhood basis for some metrizable complete group topology τ on X such that B is precompact in the corresponding quotient topology τ_q on Y. Since B is absolutely convex it is easily seen that

$$\overline{B}^{\tau_q} \subseteq \check{Y}_q,$$

where $Y_q := (Y, \tau_q)$ and $\check{Y}_q$ denotes the largest subspace of Y where τ_q is a linear topology (i.e., $\check{Y}_q := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} nq(W_{\varphi,k})$). On the other hand, $\check{Y}_q$ is a Fréchet space as a closed subspace of Y_q. Hence, there is a τ_q-null sequence (y_n) such that

$$B \subseteq \overline{\operatorname{absconv}\{y_n : n\in\mathbb{N}\}}^{\tau_q}.$$

We can lift (y_n) to a τ-null sequence $(x_n) \subseteq X$. Obviously, (x_n) is also a null sequence in the original topology of X. Finally $C := \overline{\operatorname{absconv}\{x_n : n\in\mathbb{N}\}}$ is the set we are looking for.

The easy proof of the next result is left to the reader (it is based on the fact that classes of co-Schwartz and co-nuclear spaces are closed with respect to taking closed subspaces and countable products).

LEMMA 1.6. *Every PLS-space (PLN-space) X is semi-Montel and co-Schwartz (co-nuclear), i.e., for every Banach disc $B \subseteq X$ there is another Banach disc $C \subseteq X$ such that $i_{BC} : X_B \to X_C$ is compact (nuclear).*

The first result concerning short exact sequences is well known (comp. [D1, Cor. 3.2]), we give it only to make our presentation self-contained. We will use it several times without any reference.

PROPOSITION 1.7. *Let X, Y, Z, U be lcs and let*

$$0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$$

be a short exact sequence.

(a) *If $T : Y \to U$ is an operator, then there is a unique (up to equivalence) commutative diagram with both rows exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{j_U} & X_U & \xrightarrow{q_U} & Z & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle T_1} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0
\end{array}$$

where

$$X_U := (U\times X)/A, \quad A := \{(Ty, -jy) \in U \times X : y \in Y\}.$$

The operator T has an extension onto X iff the upper row in the above diagram splits. If T is a topological embedding, then T_1 is a topological embedding as well and $X_U/\operatorname{Im} T_1 \simeq U/\operatorname{Im} T$.

If X, U, Z are LS-spaces (PLS-spaces, PLN-spaces), then X_U is an LS-space (PLS-space, PLN-space) as well. Finally, if U is a PLS-space, X is an LB-space and Z is an LS-space, then X_U is a PLS-space.

(b) *If $T : U \to Z$ is an operator, then there is a unique (up to equivalence) commutative diagram with both rows exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle T_1} & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & Y & \xrightarrow{j^U} & X^U & \xrightarrow{q^U} & U & \longrightarrow & 0
\end{array}$$

where $X^U := \{(x,u) \in X \times U : qx = Tu\}$ is a closed subspace of $X\times U$. The operator T has a lifting to X iff the lower row in the diagram above splits. If T is a topological quotient map, then T_1 is a topological quotient map as well and $\ker T \simeq \ker T_1$. If X, U, Z are LS-spaces (PLS-spaces, PLN-spaces), then X^U is an LS-space (PLS-space, PLN-space) as well.

(c) *Let W, V be lcs and let the following diagram with exact rows commute:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{j_0} & W & \xrightarrow{q_0} & V & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle Q} & & \uparrow{\scriptstyle R} & & \\
0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0
\end{array}$$

Then P extends onto X iff R lifts with respect to q_0.

REMARK. The procedures described in (a) and (b) are called *push-out* and *pull-back*, respectively. Let us mention that in part (a), X_U is complete because completeness is a *three space property*, i.e., if in (**), F and G have this property, then X has it as well (see [G, Th. 17] or [RD, Th. 1.3]).

Proof of Proposition 1.7. We only prove the last sentence of part (a).

We apply an analogous proof as for 1.3. In the present case we have for $U = \operatorname{Proj} U_K$ a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_K & \longrightarrow & W_K & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

where $W_K := (U_K \times X)/A_K$ is a Hausdorff complete LB-space (see the Remark above). As easily seen, it is an LS-space. Using the proof of 1.3 we immediately get $X_U = \operatorname{Proj} W_K$.

Let D be an LB-space. It is known [Gr, Lemme 2] that for each sequence $(p_n)_{n\in\mathbb{N}}$ of continuous seminorms on D there is a continuous seminorm p stronger than all p_n. This implies that every operator between PLS-spaces is a projective limit of operators between steps. In particular, the above mentioned fact implies that all the reduced projective spectra of LS-spaces representing a given PLB-space are equivalent.

Now, assume that $G = \operatorname{Proj} G_K$, where (G_K) is a reduced spectrum of LS-spaces, and assume that

$$0 \to E \xrightarrow{J} G \xrightarrow{Q} F \to 0$$

is a short exact sequence of PLS-spaces. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{E}_0 & \xrightarrow{J_0} & \widetilde{G}_0 & \xrightarrow{Q_0} & \widetilde{F}_0 & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widetilde{E}_n & \xrightarrow{J_n} & \widetilde{G}_n & \xrightarrow{Q_n} & \widetilde{F}_n & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widetilde{E}_{n+1} & \xrightarrow{J_{n+1}} & \widetilde{G}_{n+1} & \xrightarrow{Q_{n+1}} & \widetilde{F}_{n+1} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \xrightarrow{J} & G & \xrightarrow{Q} & F & \longrightarrow & 0
\end{array}$$

where $(\widetilde{E}_n)$, $(\widetilde{G}_n)$, $(\widetilde{F}_n)$ are reduced spectra of LS-spaces and

$$E = \operatorname{Proj} \widetilde{E}_n, \quad F = \operatorname{Proj} \widetilde{F}_n, \quad G = \operatorname{Proj} \widetilde{G}_n,$$
$$J = \operatorname{Proj} J_n, \quad Q = \operatorname{Proj} Q_n.$$

Indeed, it is enough to take $\widetilde{G}_n := G_n$ and as $\widetilde{E}_n$ the closure of E in the topology of G_n, while $\widetilde{F}_n := \widetilde{G}_n/\widetilde{E}_n$. Using push-out and pull-back, we may assume analogously that $(\widetilde{E}_n)$ and $(\widetilde{F}_n)$ are subsequences of the corresponding given spectra (E_n) and (F_n); then $(\widetilde{G}_n)$ is only equivalent to (G_n).

Later on we will use the following two special short exact sequences.

PROPOSITION 1.8. *If D is an LN-space, then there exists a short exact sequence*

$$0 \to \bigoplus_{n\in\mathbb{N}} l_1 \to \bigoplus_{n\in\mathbb{N}} l_1 \to D \to 0.$$

Proof. This follows easily from [V2, Th. 3.8] or, by use of [MV, 26.24], from [MV, 26.16] with $E_k \cong c_0$.

PROPOSITION 1.9. *If an LN-space D satisfies $\operatorname{Ext}^1_{\mathrm{LS}}(s', D) = 0$ (or, equivalently, is a quotient of s'), then there exists a short exact sequence of the form*

$$0 \to s' \to s' \to D \to 0.$$

Proof. See [DV1, Prop. 1.3].

We will also need some known splitting results. The first two come from [DV2].

LEMMA 1.10. *If $\operatorname{Ext}^1_{\mathrm{LS}}(D, D) = 0$, then $\operatorname{Ext}^1_{\mathrm{PLS}}(D^{\mathbb{N}}, D^{\mathbb{N}}) = 0$. In particular,*

$$\operatorname{Ext}^1_{\mathrm{PLS}}(\mathcal{D}', \mathcal{D}') = 0.$$

LEMMA 1.11. *If G is a strict PLS-space, D_1, D_2 are LS-spaces such that*

$$\operatorname{Ext}^1_{\mathrm{LS}}(D_1, D_2) = 0$$

and H is a PLS-space, then in the exact sequence

$$0 \to G \to D_2^{\mathbb{N}} \to H \xrightarrow{Q} D_1^{\mathbb{N}} \to 0$$

the map Q has a right inverse.

The next splitting result is a three space property type theorem:

PROPOSITION 1.12. *Let W be an ultrabornological PLS-space and let*

$$0 \to F \xrightarrow{j} X \xrightarrow{q} G \to 0$$

be a short exact sequence of PLS-spaces. If

$$\operatorname{Ext}^1_{\mathrm{PLS}}(W, F) = 0 \quad \text{and} \quad \operatorname{Ext}^1_{\mathrm{PLS}}(W, G) = 0,$$

then $\operatorname{Ext}^1_{\mathrm{PLS}}(W, X) = 0$ as well.

Proof. Let

$$0 \to X \to Y \xrightarrow{q_Y} W \to 0$$

be a short exact sequence of PLS-spaces. By 1.7, we obtain a commutative diagram of PLS-spaces:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & Y_1 & \longrightarrow & W & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle q} & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{q_Y} & W & \longrightarrow & 0
\end{array}$$

Since the upper sequence splits, the map q extends to a map $Q : Y \to G$ (use Prop. 1.7(c)). It is easily seen that

$$0 \to \ker q \to \ker Q \xrightarrow{q_Y} W \to 0$$

is an algebraically exact sequence and, by 1.2, it consists of PLS-spaces. The sequence is also topologically exact because W is ultrabornological (use the Webbed Open Mapping Theorem). We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \scriptstyle{\mathrm{id}} & & \\ 0 & \longrightarrow & \ker q & \longrightarrow & \ker Q & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

with a splitting lower row because $F \simeq \ker q$. Hence the upper sequence splits as well.

We finish with a characterization of complemented subspaces of $\mathcal{D}'$ given in [DV2].

THEOREM 1.13. *A PLN-space F is isomorphic to a complemented subspace of $\mathcal{D}'$ if and only if it is isomorphic to a subspace and to a quotient of $\mathcal{D}'$.*

For notions and facts not explained here we refer to [J] or [MV].

2. Quotients of $\mathcal{D}'$ and the splitting. Let $F = \operatorname{Proj} F_N$ be a PLS-space. We say that $F \in (P_\omega)$ if and only if for every sequence $\mathcal{B} = (B_K)$ of Banach discs, $B_K \subseteq F_K$, such that $i_K^{K+1} B_{K+1} \subseteq B_K$ we have

$$\forall K\ \exists L \quad i_K^L(\overline{i_L(F)}^{B_L}) \subseteq i_K(F).$$

The main aim of this section is to prove the following result:

THEOREM 2.1. *For a PLN-space F the following assertions are equivalent*:

(1) $\operatorname{Ext}^1_{\mathrm{PLS}}(\mathcal{D}', F) = 0$;
(2) $F \in (P_\omega)$ *and F is a continuous image of $\mathcal{D}'$*;

and, under the additional assumption that F is ultrabornological,

(3) *F is isomorphic to a quotient of $\mathcal{D}'$*;
(4) *there is an exact sequence*

$$0 \to \mathcal{D}' \to \mathcal{D}' \to F \to 0.$$

In order to prove the main result we need some lemmas.

LEMMA 2.2. *For a PLS-space F the following assertions are equivalent*:

(1) $\operatorname{Ext}^1_{\mathrm{PLS}}(\omega, F) = 0$;
(2) $F \in (P_\omega)$.

Proof. We first prove that (2) implies (1): Let

$$0 \to F \xrightarrow{j} G \xrightarrow{q} \omega \to 0$$

be a short exact sequence of PLS-spaces. Let $G = \operatorname{Proj} G_K$ and $F = \operatorname{Proj} F_K$, $F_K \subseteq G_K$ topologically. Moreover, $G_K = \operatorname{Ind}_l G_{Kl}$ and B_{Kl} is the unit ball in the Banach space G_{Kl}.

By the De Wilde Localization Theorem [J, 5.6.3], using the standard strict web in G, we find a sequence (n_K) of natural numbers such that for every L there is M such that

$$q(\widetilde{B}_{1n_1} \cap \ldots \cap \widetilde{B}_{Ln_L}) \supseteq \{x \in \omega : x_1 = \ldots = x_{M-1} = 0\},$$

where $\widetilde{B}_{Kl} := i_K^{-1} B_{Kl}$. We define Banach discs:

$$D_L := (i_1^L)^{-1} B_{1n_1} \cap (i_2^L)^{-1} B_{2n_2} \cap \ldots \cap B_{Ln_L}, \qquad B_L := D_L \cap F_L$$

in G_L and F_L, respectively.

We denote by g_n an arbitrary fixed element of $q^{-1}(e_n)$, where e_n is the nth unit vector in ω. By the above arguments, for $m \ge M$ and every $\varepsilon \ge 0$,

$$g_m \in \varepsilon(\widetilde{B}_{1n_1} \cap \ldots \cap \widetilde{B}_{Ln_L}) + F \quad \text{and} \quad i_L g_m \in \varepsilon D_L + i_L F.$$

Hence

$$i_L g_m \in \overline{i_L F}^{D_L} \subseteq \overline{i_L F} = F_L$$

and

$$i_L g_m \in \overline{i_L F}^{B_L} \quad \text{for } m \ge M.$$

Now, we apply (P_ω) to the nested sequence $(B_L)_{L \in \mathbb{N}}$ of Banach discs. For every K we find L such that

$$i_K^L(\overline{i_L F}^{B_L}) \subseteq i_K F.$$

Summarizing, we find an increasing sequence $M(K)$ such that $i_K g_m \in i_K F$ for $m \ge M(K)$. Then for $M(K) \le m < M(K+1)$ we find $h_m \in F$ such that $i_K g_m = i_K h_m$. We define $h_m := 0$ for $m < M(1)$ and

$$R : \omega \to G, \qquad R((x_n)_{n \in \mathbb{N}}) := \sum_{n \in \mathbb{N}} x_n (g_n - h_n);$$

it is easily seen that R is a continuous right inverse for q.

For the reverse direction of the proof assume that (P_ω) is not satisfied, thus there is a nested sequence (B_K) of Banach discs and K_0 such that there exists $g_L \in \overline{i_L F}^{B_L}$ but for each $L > K_0$ we have $i_{K_0}^L g_L \notin i_{K_0} F$. It is easily seen that $Z := \overline{\operatorname{lin}}_{L \ge K_0} \widetilde{g}_L \subseteq \prod_{M \in \mathbb{N}} F_M$ is isomorphic to ω, where $\widetilde{g}_L := (0, \ldots, 0, g_L, 0, \ldots)$, g_L at the Lth place.

Let $(a_L)_{L \in \mathbb{N}} \in \omega$ be a given sequence. Then

$$a_L g_L = i_L v_L + u_L \quad \text{for } L \ge 1$$

where $v_L \in F$ and $u_L \in 2^{-L} B_L$. Thus

$$(a_L g_L) = \sigma_{\mathcal{F}}((x_L)_{L \in \mathbb{N}})$$

for

$$x_M := -a_M g_M + \sum_{L=1}^{M} i_M v_L - \sum_{L=M+1}^{\infty} i_M^L u_L$$

(series convergent in $(F_L)_{B_L}$). Here $\sigma_{\mathcal{F}}$ denotes as usual the quotient map in the canonical resolution of F. We have proved that $Z \subseteq \operatorname{Im} \sigma_{\mathcal{F}}$. Finally, we have a short exact sequence

$$0 \to F \to X \xrightarrow{q} Z \to 0,$$

where $X := \sigma_{\mathcal{F}}^{-1}(Z)$ and $q = \sigma_{\mathcal{F}}|_X$. By 1.2, X is a PLS-space. If the above sequence splits, then there is an operator $B : Z \to X \subseteq \prod_{L\in\mathbb{N}} F_L$ such that $q \circ B = \mathrm{id}_Z$. Let $B(\widetilde{g}_L) := (b_{1,L}, b_{2,L}, \ldots)$, $b_{M,L} \in F_M$; then

(i) $i_M^{M+1} b_{M+1,L} - b_{M,L} = 0$ for $M \neq L$;
(ii) $i_L^{L+1} b_{L+1,L} - b_{L,L} = g_L$;
(iii) for each K we have $L_K > K$ such that $b_{K,L} = 0$ for $L \geq L_K$.

Condition (i) implies that there is $b_L \in F$ such that $i_M b_L = b_{M,L}$ for all $M > L$. If we choose $L \geq L_{K_0}$ we get

$$i_{K_0}^L g_L = i_{K_0} b_L - i_{K_0}^L b_{L,L} = i_{K_0} b_L - b_{K_0,L} = i_{K_0} b_L \in i_{K_0}(F),$$

a contradiction.

The method of proof of 2.2 cannot be used for more complicated spaces $D^{\mathbb{N}}$ (except $l_1^{\mathbb{N}}$) since in the proof we use specific properties of one-dimensional spaces (for example, projectivity) and since the topology of ω is Fréchet, unlike the case of general spaces $D^{\mathbb{N}}$, where D is an LN-space.

COROLLARY 2.3. *For an ultrabornological PLS-space F the following are equivalent*:

(1) $\operatorname{Ext}^1_{\mathrm{PLS}}(\omega, F) = 0$;
(2) *F is strict.*

Proof. Clearly, strictness implies (P_ω). On the other hand, if F is ultrabornological, then $\operatorname{Proj}^1 F = 0$ and there is a sequence of Banach discs $\mathcal{B} = (B_L)$, $B_L \subseteq F_L$, $i_L^{L+1} B_{L+1} \subseteq B_L$, so that for every L there is M with

$$i_L^M(F_M) \subseteq \overline{i_L F}^{B_L}$$

(see Section 1). For given K we find L according to (P_ω) and then M according to the above. We obtain

$$i_K^M(F_M) \subseteq i_K^L(\overline{i_L F}^{B_L}) \subseteq i_K F,$$

which means strictness.

Now, we show that (P_ω) has strong consequences for the structure of the space F.

LEMMA 2.4. *Let F be a PLS-space (PLN-space) satisfying (P_ω). Then F^{ub} is a strict projective limit of a sequence of LS-spaces (LN-spaces, respectively) H_K, where $H_K := (i_K(F))^{\mathrm{ub}}$.*

Proof. By 1.5, Banach discs in $i_K(F)$ are exactly images of compact Banach discs in F. Of course, by 1.6, $i_K(F)$ is then co-Schwartz (or co-nuclear). Now, it suffices to show that there is a fundamental sequence of Banach discs in $i_K(F)$.

Let B be an arbitrary Banach disc in F. There is a sequence $(n_J)_{J\in\mathbb{N}}$ of natural numbers such that

$$B \subseteq \bigcap_{K\in\mathbb{N}} i_K^{-1}(B_{K,n_K}).$$

If $C_K := \bigcap_{J\leq K} (i_J^K)^{-1}(B_{J,n_J})$ and $D_K := \overline{C_K \cap i_K(F)}^{C_K}$, then

$$i_K(B) \subseteq i_K^L(D_L) \quad \text{for every } L \geq K.$$

Moreover, we can apply (P_ω) to the nested sequence $(C_K)_{K\in\mathbb{N}}$ and then

$$\forall K\ \exists L_0\ \forall L \geq L_0 \quad i_K^L(D_L) \subseteq i_K(F).$$

By 1.5, $i_K^L(D_L) \subseteq i_K(D)$ for some compact Banach disc D in F. In particular,

$$B_{L,K,(n_J)} := \overline{i_K^L(D_L)}^{F_K}$$

is a Banach disc in $i_K(F)$! Since $B_{L,K,(n_J)}$ depends only on $(n_J)_{J\leq L}$, we have only countably many distinct sets of that form and they form a fundamental sequence of Banach discs in $i_K(F)$.

REMARK. The above proof gives a useful description of a fundamental sequence of Banach discs in $i_K(F)$ and a fundamental family of Banach discs in F whenever $F \in (P_\omega)$.

The property (P_ω) also has some surprising "splitting" consequences.

LEMMA 2.5. *Let F, G be PLS-spaces and let $F \in (P_\omega)$, G be ultrabornological.*

(a) *If* $\operatorname{Ext}^1_{\mathrm{PLS}}(G, F) = 0$, *then* $\operatorname{Ext}^1_{\mathrm{PLS}}(G, F^{\mathrm{ub}}) = 0$.
(b) *If* $G \in (P_\omega)$ *and* $\operatorname{Ext}^1_{\mathrm{PLS}}(G, F^{\mathrm{ub}}) = 0$, *then* $\operatorname{Ext}^1_{\mathrm{PLS}}(G, F) = 0$.

REMARK. Of course, for any LN-space D we have $D^{\mathbb{N}} \in (P_\omega)$ and it is ultrabornological.

Proof of Lemma 2.5. (a) Consider an arbitrary short exact sequence of PLS-spaces (use 2.4)

$$0 \to F^{\mathrm{ub}} \to X \xrightarrow{q} G \to 0.$$

By 1.7(a), we obtain the following commutative diagram, where rows are exact and all spaces are PLS-spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & (X,\tau) & \xrightarrow{q_1} & G & \longrightarrow & 0\\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & F^{\mathrm{ub}} & \longrightarrow & X & \xrightarrow{q} & G & \longrightarrow & 0 \end{array}$$

By the assumption, q_1 has a continuous linear section $S : G \to (X,\tau)$. By 1.5, X and (X,τ) have the same families of Banach discs. Since G is ultrabornological, the map S is continuous also when we equip X with the topology τ^{ub}. This completes the proof because the topology of X is weaker than τ^{ub}.

(b) Consider an arbitrary short exact sequence of PLS-spaces

$$0 \to F \to X \to G \to 0.$$

By 1.12 and 2.2, $X \in (P_\omega)$, thus using 2.4 we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0\\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & (F,\tau) & \longrightarrow & X^{\mathrm{ub}} & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

where τ is the topology on F induced from X^{ub}. By 2.4, X^{ub} is a strict PLS-space and, by 1.2, (F,τ) is a PLS-space. Of course, $\mathrm{Proj}^1 X^{\mathrm{ub}} = 0$ and $\mathrm{Proj}^1(F,\tau) = 0$ because of Cor. 1.4 (comp. [V3, 5.4]). By Th. 1.1, (F,τ) is ultrabornological and therefore it must be the ultrabornological space associated with F. By the assumption, the lower sequence in the above diagram splits and hence the same holds for the upper one (see 1.7(c)).

LEMMA 2.6. *If F is a strict PLS-space, $\mathrm{Ext}^1_{\mathrm{PLS}}(\mathcal{D}', F) = 0$, then there exists a strict spectrum $(F_K)_{K\in\mathbb{N}}$, $F = \mathrm{Proj}\, F_K$, so that $\mathrm{Ext}^1_{\mathrm{LS}}(s', F_K) = 0$ for all K.*

Proof. Let $(F_K)_{K\in\mathbb{N}}$ be any projective spectrum representing F with surjective linking maps. Let

$$0 \to F_K \to X \xrightarrow{Q} s' \to 0$$

be an arbitrary short exact sequence of LS-spaces. On the other hand, by 1.8, we have a short exact sequence

$$0 \to \bigoplus l_1 \to \bigoplus l_1 \xrightarrow{q} s' \to 0.$$

By the lifting property of l_1 and by the Grothendieck Factorization Theorem,

we can lift q with respect to Q and we obtain the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_K & \longrightarrow & X & \xrightarrow{Q} & s' & \longrightarrow & 0\\ & & \uparrow{\scriptstyle R} & & \uparrow{\scriptstyle S} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \bigoplus l_1 & \longrightarrow & \bigoplus l_1 & \xrightarrow{q} & s' & \longrightarrow & 0 \end{array}$$

Similarly, by Lemma 1.5, we can lift R to F with respect to the surjective map $i_K : F \to F_K$ and we get another commutative diagram, $S = S_1 \circ S_2$, where X_1 is a PLS-space (apply 1.7(a)):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_K & \longrightarrow & X & \xrightarrow{Q} & s' & \longrightarrow & 0\\ & & \uparrow{\scriptstyle i_K} & & \uparrow{\scriptstyle S_1} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & F & \longrightarrow & X_1 & \longrightarrow & s' & \longrightarrow & 0\\ & & \uparrow{\scriptstyle R_1} & & \uparrow{\scriptstyle S_2} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \bigoplus l_1 & \longrightarrow & \bigoplus l_1 & \xrightarrow{q} & s' & \longrightarrow & 0 \end{array}$$

Since the middle row splits, so does the upper one (use 1.7(c)).

LEMMA 2.7. *For every PLN-space F there exists a short exact sequence of PLN-spaces of the form*

$$0 \to \mathcal{D}' \to K \to F \to 0,$$

where K is a subspace of $\mathcal{D}'$.

Proof. Every nuclear Fréchet space G is a subspace of $s^{\mathbb{N}}$ [J, 21.7.1], the latter space is a quotient of s ([V0, 1.6]). This implies that G is a quotient of a subspace of s. By duality, every nuclear DF-space is a subspace of a quotient of s'. We have proved that F is a subspace of $\prod Q_n$, where Q_n are quotients of s'. By 1.9, for every n there exists a short exact sequence of the form

$$0 \to s' \to s' \to Q_n \to 0.$$

Taking products we get

$$0 \to \mathcal{D}' \to \mathcal{D}' \xrightarrow{q} \prod Q_n \to 0$$

and $K := q^{-1}(F)$ gives the sequence we are looking for (use 1.2).

Proof of Theorem 2.1. First we consider the case of ultrabornological F (i.e., $\mathrm{Proj}^1 F = 0$). While (4)⇒(3) is obvious, we prove (1)⇒(3), (3)⇒(4) and (4)⇒(1).

(1)⇒(3). Because of Cor. 2.3, Lemma 2.6 and Prop. 1.9, we may choose a surjective spectrum $(F_K)_{K\in\mathbb{N}}$ of F such that for any K there is a short exact sequence

$$0 \to s' \to s' \to F_K \to 0.$$

The canonical resolution gives the first line of the following diagram, the product of the above sequences its right column, the rest is obtained via 1.7:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & F & \to & \prod_K F_K & \to & \prod_K F_K & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F & \to & H & \to & (s')^{\mathbb{N}} & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & (s')^{\mathbb{N}} & \to & (s')^{\mathbb{N}} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The middle row splits by assumption, hence $H \simeq F \oplus (s')^{\mathbb{N}}$. The left column gives the first line of the following commutative diagram, the right column is constructed as previously:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & (s')^{\mathbb{N}} & \to & F \oplus (s')^{\mathbb{N}} & \to & \prod_K F_K & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & (s')^{\mathbb{N}} & \to & G & \to & (s')^{\mathbb{N}} & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & (s')^{\mathbb{N}} & \to & (s')^{\mathbb{N}} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The middle row splits (see 1.10), hence $G \simeq (s')^{\mathbb{N}}$. So we have an exact sequence

$$0 \to (s')^{\mathbb{N}} \to (s')^{\mathbb{N}} \to F \oplus (s')^{\mathbb{N}} \to 0.$$

Since $\mathrm{Ext}^1_{\mathrm{PLS}}((s')^{\mathbb{N}}, (s')^{\mathbb{N}}) = 0$ (again by 1.10), we can lift the canonical injection $(s')^{\mathbb{N}} \to F \oplus (s')^{\mathbb{N}}$ and this yields a complemented subspace Y of $(s')^{\mathbb{N}}$ and an exact sequence

$$0 \to (s')^{\mathbb{N}} \to Y \to F \to 0.$$

We add to this sequence the following trivial exact sequence:

$$0 \to (s')^{\mathbb{N}} \xrightarrow{\mathrm{id}} (s')^{\mathbb{N}} \to 0 \to 0.$$

The Pełczyński Decomposition in the form of [V5, Prop. 1.2] implies $Y \oplus (s')^{\mathbb{N}} \simeq (s')^{\mathbb{N}}$ and we have obtained the required sequence.

(3)⇒(4). By use of Lemma 2.7, we find an exact sequence

$$0 \to (s')^{\mathbb{N}} \to K \to F \to 0,$$

where $K \subseteq (s')^{\mathbb{N}}$. Let $Q : (s')^{\mathbb{N}} \to F$ be a topological quotient map. Then we obtain the following commutative diagram (use 1.7) with q a topological quotient map:

$$\begin{array}{ccccccccc} 0 & \to & (s')^{\mathbb{N}} & \to & K & \to & F & \to & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle q} & & \uparrow{\scriptstyle Q} & & \\ 0 & \to & (s')^{\mathbb{N}} & \to & X & \to & (s')^{\mathbb{N}} & \to & 0 \end{array}$$

The lower row splits by 1.10. By 1.13, K is a complemented subspace of $(s')^{\mathbb{N}}$. This completes the proof by the same argument as in the previous part.

(4)⇒(1). Let

$$0 \to F \to X \xrightarrow{q} \mathcal{D}' \to 0$$

be an exact sequence of PLS-spaces. Combining it with the sequence given in the assumption, we obtain an exact sequence

$$0 \to (s')^{\mathbb{N}} \to (s')^{\mathbb{N}} \to X \xrightarrow{q} (s')^{\mathbb{N}} \to 0.$$

Lemma 1.11 yields a right inverse for q.

Now, let us consider the general case.

(1)⇒(2). By Lemma 2.2, $F \in (P_\omega)$. Lemma 2.5(a) and (1)⇒(3) imply that F^{ub} is a quotient of $\mathcal{D}'$.

(2)⇒(1). By Lemma 1.5, F^{ub} is a topological quotient of $\mathcal{D}'$. Thus, by (3)⇒(1), $\mathrm{Ext}^1_{\mathrm{PLS}}(\mathcal{D}', F^{\mathrm{ub}}) = 0$. Now, apply Lemma 2.5(b).

The example below shows that (P_ω) does not imply ultrabornologicity.

Example 2.8. We give an example of a non-ultrabornological Köthe PLN-space F with a continuous norm and dense Banach discs in the steps such that F satisfies (P_ω) but there is no representing strict projective spectrum of LS-spaces.

Remark. In view of Lemma 2.4, the associated ultrabornological topology of F is the topology of an LS-space.

CONSTRUCTION. We define

$$a_{n,k,L,m} := \begin{cases} n^{-2(m-k)} 2^{nk-k^2-2^n k} & \text{for } k < m, L-1, \\ n^{-2(m-k)} 2^{n(L-1)-k^2-2^n k} & \text{for } L-1 \le k < m, \\ 2^{2^{2^n}+nk-mk-2^n m} & \text{for } m \le k < L-1, \\ 2^{2^{2^n}+n(L-1)-mk-2^n m} & \text{for } m, L-1 \le k. \end{cases}$$

It is easily seen that

$$a_{n,k,L,m} \le a_{n,k,L+1,m} \quad \text{and} \quad a_{n,k,L,m} \ge a_{n,k,L,m+1}.$$

Thus we can define

$$F := \{(x_{n,k}) : \text{for all } L \text{ there is } m \text{ with } \sup_{n,k} |x_{n,k}| a_{n,k,L,m} < \infty\}$$

and it is the projective limit of the spaces

$$F_L := \{(x_{n,k}) : \text{there is } m \text{ with } \|(x_{n,k})\|_{L,m} := \sup_{n,k} |x_{n,k}| a_{n,k,L,m} < \infty\}.$$

The spaces F_L are LN-spaces because for all m, L with $m > L$ the series

$$\sum_n \sum_k \frac{a_{n,k,L,m+1}}{a_{n,k,L,m}} = \sum_n \Big((m-1) n^{-2} + 2^{-2^{2^n}} n^{-2} + \sum_{k=m+1}^{\infty} 2^{-k} 2^{-2^n} \Big)$$

converges. Moreover, for $m > L+1$ we have

$$\frac{a_{n,L,L+1,m}}{a_{n,L,L,L+1}} = 2^n n^{-2(m-L-1)} \xrightarrow[n\to\infty]{} \infty,$$

hence $i_L^{L+1}(F_{L+1})$ is a proper subspace of F_L. In particular, there is no equivalent strict spectrum representing F.

Finally, we show that for $2 \le L < m$ and $S(L) := 1 + \sup_n 2^{nL-2^{2^n}}$ we have

$$a_{n,k,L+1,m+1} \le S(L) \max(a_{n,k,1,L}; a_{n,k,L,m}) \quad \text{for every } n, k \in \mathbb{N}.$$

Indeed,

$$a_{n,k,L+1,m+1} \le \begin{cases} a_{n,k,L,m} & \text{for } k \le L-1, \\ S(L) a_{n,k,1,L} & \text{for } L \le k \le m, \\ a_{n,k,L,m} & \text{for } m+1 \le k. \end{cases}$$

Let $B_{L,m}$ be the unit ball of the norm $\|\cdot\|_{L,m}$ in F_L. The above inequality means that

$$B_{1,L} \cap B_{L,m} \subseteq S(L) B_{L+1,m+1}.$$

Hence

$$B_{1,L} \cap B_{L,m} \subseteq S(L)(B_{L+1,m+1} \cap B_{1,L+1}) \subseteq S(L) S(L+1) B_{L+2,m+2}.$$

Repeating this inductively we get

$$B_{1,L} \cap B_{L,m} \subseteq F.$$

Since every nested sequence $\mathcal{B} = (B_L)$ satisfies for some sequence (n_L) the inclusion

$$B_L \subseteq \bigcap_{J \le L} B_{J,n_J},$$

we see that $B_{n_1} \subseteq F$ and obviously (P_ω) is satisfied.

3. Subspaces of $\mathcal{D}'$ and splitting. Now, we prove an analogue of 2.1 for subspaces.

THEOREM 3.1. *For a PLN-space F the following are equivalent:*

(1) $\operatorname{Ext}^1_{\mathrm{PLS}}(F, \mathcal{D}') = 0$;

(2) *F is isomorphic to a subspace of $\mathcal{D}'$;*

and, under the additional assumption that F is ultrabornological,

(3) *there is an exact sequence*

$$0 \to F \to \mathcal{D}' \to \mathcal{D}' \to 0.$$

Proof. (1)⇒(2). By Lemma 2.7, we obtain a short exact sequence

$$0 \to \mathcal{D}' \to K \to F \to 0$$

with $K \subseteq \mathcal{D}'$. Hence F is isomorphic to a complemented subspace of K by (1).

(2)⇒(1). It suffices to show that every short exact sequence of PLS-spaces of the form

$$(1) \qquad 0 \to s' \to Y \to F \to 0$$

splits. We may choose spectra (Y_N) and (F_N) for Y and F respectively so that

$$(2) \qquad 0 \to s' \to Y_N \to F_N \to 0$$

are exact and the "projective limit" of these sequences is equal to the sequence (1) (see Section 1). Moreover, we may assume that F_N are subspaces of s' (i.e., step spaces of $(s')^{\mathbb{N}} \simeq \mathcal{D}'$). By [V6] (see [MV, 31.5, 31.6, 30.1]), all the sequences (2) split and there is a projection $P_N : Y_N \to s'$. Now, $P_N \circ i_N$ is the required projection for (1).

Now, assume that F is ultrabornological, i.e., $\operatorname{Proj}^1 F = 0$. We show (2)⇒(3).

If $F \subseteq \mathcal{D}'$, $Q = \mathcal{D}'/F$, then $\operatorname{Proj}^1 F = 0$ implies (Cor. 1.4) that Q is a PLN-space. We apply Lemma 2.7 in order to get the last column of the following diagram, the rest is obtained via 1.7:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & \mathcal{D}' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & H & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{D}' & \longrightarrow & \mathcal{D}' & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $K \subseteq \mathcal{D}'$. The first column splits by 1.10. So we obtain an exact sequence

$$0 \to F \to \mathcal{D}' \to K \to 0. \tag{3}$$

By 1.13, K is isomorphic to a complemented subspace of $\mathcal{D}'$. We multiply (3) by

$$0 \to 0 \to \mathcal{D}' \xrightarrow{\text{id}} \mathcal{D}' \to 0$$

and using the Pełczyński Decomposition as at the end of the proof of (1)⇒(3) of Th. 2.1, we get the required sequence.

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