[26] M. Langenbruch, *Surjective partial differential operators on spaces of ultradifferentiable functions of Roumieu type*, Results Math. 29 (1996), 254–275.

[27] —, *Continuation of Gevrey regularity for solutions of partial differential operators*, in: S. Dierolf, S. Dineen and P. Domański (eds.), Functional Analysis, Proc. First Workshop at Trier University, de Gruyter, 1996, 249–280.

[28] —, *Surjective partial differential operators on Gevrey classes and their localizations at infinity*, Linear Topol. Spaces Complex Anal. 3 (1997), 95–111,

[29] —, *Surjectivity of partial differential operators in Gevrey classes and extension of regularity*, Math. Nachr. 196 (1998), 103–140.

[30] —, *Extension of analyticity for solutions of partial differential operators*, Note Mat. 17 (1997), 29–59.

[31] —, *Surjective partial differential operators on spaces of real analytic functions*, preprint.

[32] P. Laubin, *Propagation des singularités analytiques pour des opérateurs à caractéristiques involutives de multiplicité variable*, Portugal. Math. 41 (1982), 83–90.

[33] —, *Analyse microlocale des singularités analytiques*, Bull. Soc. Roy. Sci. Liège 52 (1983), 103–212.

[34] O. Liess, *Necessary and sufficient conditions for propagation of singularities for systems of partial differential equations with constant coefficients*, Comm. Partial Differential Equations 8 (1983), 89–198.

[35] R. Meise und D. Vogt, *Einführung in die Funktionalanalysis*, Vieweg, Braunschweig, 1992.

[36] T. Miwa, *On the global existence of real analytic solutions of systems of linear differential equations with constant coefficients*, Proc. Japan Acad. 49 (1973), 500–502.

[37] L. C. Piccinini, *Non surjectivity of the Cauchy–Riemann operator on the space of the analytic functions on* $\mathbb{R}^n$, Boll. Un. Mat. Ital. (4) 7 (1973), 12–28.

[38] J. Sjöstrand, *Singularités Analytiques Microlocales*, Astérisque 95 (1982), 1–166.

[39] G. Zampieri, *Operatori differenziali a coefficienti costanti di tipo iperbolico-(ipo) ellittico*, Rend. Sem. Mat. Univ. Padova 72 (1984), 27–44.

[40] —, *Propagation of singularity and existence of real analytic solutions of locally hyperbolic equations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1984), 373–390.

[41] —, *An application of the fundamental principle of Ehrenpreis to the existence of global Gevrey solutions of linear differential equations*, Boll. Un. Mat. Ital. B (6) 5 (1986), 361–392.

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An asymptotic expansion for the distribution of the supremum of a random walk

by

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Abstract. Let $\{S_n\}$ be a random walk drifting to $-\infty$. We obtain an asymptotic expansion for the distribution of the supremum of $\{S_n\}$ which takes into account the influence of the roots of the equation $1-\int_{\mathbb{R}} e^{sx}\,F(dx)=0$, F being the underlying distribution. An estimate, of considerable generality, is given for the remainder term by means of submultiplicative weight functions. A similar problem for the stationary distribution of an oscillating random walk is also considered. The proofs rely on two general theorems for Laplace transforms.

1. Introduction. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random variables with a common nonarithmetic distribution F. Define $S_0=0$, $S_n=X_1+\ldots+X_n$, $n\ge 1$. Suppose the random walk $\{S_n\}$ drifts to $-\infty$, i.e., with probability one $S_n\to-\infty$ as $n\to\infty$. We set $M_\infty:=\sup_{n\ge 0}S_n$.

Properties of the distribution of M_∞ have been studied by many authors for various reasons. First, the problems involving M_∞ are of interest in their own right, since the supremum is one of the underlying functionals in random walk theory. Second, the distribution of M_∞ appears in some applications; for example, it coincides with the limiting distribution of the waiting time process in the theory of queues [7, Sections XII.5 and VI.9]. The existence of moments of the form $Ef(M_\infty)$ was considered for various choices of the function $f(x)$ by Kiefer and Wolfowitz [12], Tweedie [19], Janson [10], Alsmeyer [1], and Sgibnev [16]. Note that although Theorem 5 of Tweedie [19] concerns moments of the form $\int f(x)\,\pi(dx)$ for the stationary distribution π of the Markov chain $Z_{n+1}=\max(Z_n+X_{n+1},0)$, it is, however, well known

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(see, e.g., Feller [7, Sections VI.9 and XVIII.5]) that π coincides with the distribution of M_∞. In Borovkov [3], Veraverbeke [20], Sgibnev [15], Embrechts and Veraverbeke [6], Lotov [13], Bertoin and Doney [2], and Borovkov and Korshunov [5], the main emphasis in studying the distribution of the supremum was laid upon the asymptotic behavior of $P(M_\infty > x)$.

The absolutely continuous component of an arbitrary distribution G will be denoted by G_c, and its singular component by G_s: $G_s = G - G_c$. For a complex-valued measure κ, we denote by $\widehat{\kappa}(s)$ its Laplace transform: $\widehat{\kappa}(s) := \int_{\mathbb{R}} \exp(sx)\,\kappa(dx)$ for appropriate values of s. In particular, $(F^{m*})^\wedge_s(r)$ will stand for the Laplace transform at the point r of the singular component $(F^{m*})_s$ of the m-fold convolution F^{m*}.

The present paper deals with the distribution of M_∞ when the underlying distribution F satisfies the following conditions:

(a) $\int_{\mathbb{R}} \varphi(x)\,F(dx) < \infty$, where $\varphi(x)$ is some *submultiplicative* function, i.e., $\varphi(x+y) \le \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}$;

(b) the *characteristic equation* $1 - \widehat{F}(s) = 0$ has nonzero roots in the strip $\Pi(r) := \{s \in \mathbb{C} : 0 \le \Re s \le r\}$ for some $r > 0$;

(c) for some integer $m \ge 1$, $(F^{m*})^\wedge_s(r) < 1$.

Notice that condition (c) is automatically fulfilled if the distribution F has a density. Also, we will show that condition (c) is *necessary* in order to obtain a desired estimate of the remainder term in an expansion for the distribution of M_∞ (see Theorem 4 and Remark 2).

The approach used in the present paper is based on Banach algebra techniques and two general theorems on Laplace transforms which are proved in Section 2 (Theorems 2 and 3). Their application allows us to make substantial progress in two directions. First, the remainder term is estimated by means of submultiplicative weight functions. The notion of submultiplicativity is a very general concept, covering a wide range of useful specific functions (see Section 2). Second, we do not exclude from this general treatment the situation when some of the roots of $1 - \widehat{F}(s) = 0$ lie on the boundary of the strip of analyticity of $\widehat{F}(s)$.

It is known that the distribution of the supremum appears as a convolution factor in the stationary distributions π of some Markovian random walks (see [4] and [5]). This fact allows us to apply the techniques of the present paper to obtain, under conditions (a)–(c), asymptotic expansions for the π with rather general estimates of the remainder terms. We limit ourselves to the case of oscillating random walk (Section 4). Here an interesting phenomenon can be observed: coincidence of some roots of the characteristic equations corresponding to the governing distributions of the walk can modify or even cancel the respective terms of the expansion for π.

2. Preliminaries. Let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function, i.e., $\varphi(x)$ is a finite, positive, Borel measurable function with the following properties:

$$\varphi(0) = 1, \quad \varphi(x+y) \le \varphi(x)\varphi(y) \quad \text{for all } x, y \in \mathbb{R}.$$

It is well known [9, Section 7.6] that

$$(1) \qquad -\infty < r_-(\varphi) := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} = \sup_{x<0} \frac{\log\varphi(x)}{x} \le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_+(\varphi) < \infty.$$

Here are some examples of such functions on $[0,\infty)$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\alpha)$ with $c > 0$ and $\alpha \in (0,1)$; $\varphi(x) = \exp(\gamma x)$ for γ real. In the first two cases $r_+(\varphi) = 0$ while in the last case $r_+(\varphi) = \gamma$. Moreover, if $R(x)$, $x \in \mathbb{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent α (i.e., $R(tx)/R(x) \to t^\alpha$ for $t > 0$ as $x \to \infty$ [7, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0,\infty)$ such that $c_1 R(x) \le \varphi(x) \le c_2 R(x)$ for all $x \ge x_0$, where c_1 and c_2 are some positive constants [16, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

Consider the collection $S(\varphi)$ of all complex-valued measures κ defined on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ and such that

$$\|\kappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\,|\kappa|(dx) < \infty.$$

Here $|\kappa|$ stands for the total variation of κ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\kappa\|_\varphi$ under the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ being their convolution $\nu * \kappa$ [9, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ, i.e., the atomic measure of unit mass at the origin. Relation (1) implies that the Laplace transform $\widehat{\kappa}(s) = \int_{\mathbb{R}} \exp(sx)\,\kappa(dx)$ of $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip

$$\Pi(\varphi) = \{s \in \mathbb{C} : r_-(\varphi) \le \Re s \le r_+(\varphi)\}.$$

The following theorem of [14] describes the structure of homomorphisms of $S(\varphi)$ onto $\mathbb{C}$.

THEOREM 1. *Let $m : S(\varphi) \to \mathbb{C}$ be an arbitrary homomorphism. Then the following representation holds:*

$$m(\nu) = \int \chi(x,\nu)\exp(\alpha x)\,\nu(dx), \qquad \nu \in S(\varphi),$$

where α is a real number such that $r_-(\varphi) \le \alpha \le r_+(\varphi)$ and the function $\chi(x,\nu)$ of $x \in \mathbb{R}$ and $\nu \in S(\varphi)$ is a generalized character.

We shall not give a complete definition of a generalized character here; in what follows only one property of a generalized character will be used:

$$\nu\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}} |\chi(x,\nu)| \le 1.$$

THEOREM 2. *Let $\varphi(x)$, $x\in\mathbb{R}$, be a submultiplicative function such that $r_-(\varphi)<r_+(\varphi)$. Suppose the function $\varphi(x)/\exp[r_+(\varphi)x]$, $x\ge 0$, is nondecreasing and $\varphi(x)/\exp[r_-(\varphi)x]$, $x\le 0$, is nonincreasing. Assume $\nu\in S(\varphi)$ and let s_0 be an interior point of $\Pi(\varphi)$. Then $\widehat{\kappa}(s):=[\widehat{\nu}(s)-\widehat{\nu}(s_0)]/(s-s_0)$, $s\in\Pi(\varphi)$, is the Laplace transform of some measure $\kappa\in S(\varphi)$.*

Proof. Consider the absolutely continuous measure κ with the density $k(x)$, $x\in\mathbb{R}$, given by

$$k(x):=\begin{cases}\int_x^\infty e^{s_0(y-x)}\,\nu(dy) & \text{if } x\ge 0,\\ -\int_{-\infty}^x e^{s_0(y-x)}\,\nu(dy) & \text{if } x<0.\end{cases}$$

We show that $\kappa\in S(\varphi)$. Set $\varrho:=\Re s_0$. We have

$$(2)\qquad \begin{aligned} I &:= \int_0^\infty \varphi(x)|k(x)|\,dx \le \int_0^\infty \varphi(x)\int_x^\infty e^{-\varrho(x-y)}\,|\nu|(dy)\,dx\\ &= \int_0^\infty e^{\varrho y}\int_0^y \varphi(x)\exp[-r_+(\varphi)x]\exp\{[r_+(\varphi)-\varrho]x\}\,dx\,|\nu|(dy)\\ &\le \int_0^\infty \varphi(y)\exp\{[\varrho-r_+(\varphi)]y\}\int_0^y \exp\{[r_+(\varphi)-\varrho]x\}\,dx\,|\nu|(dy)\\ &\le \frac{1}{r_+(\varphi)-\varrho}\int_0^\infty \varphi(y)\,|\nu|(dy)<\infty.\end{aligned}$$

Similarly,

$$(3)\qquad J:=\int_{-\infty}^0 \varphi(x)|k(x)|\,dx \le \frac{1}{\varrho-r_-(\varphi)}\int_{-\infty}^0 \varphi(y)\,|\nu|(dy)<\infty.$$

The equality $\widehat{\kappa}(s)=[\widehat{\nu}(s)-\widehat{\nu}(s_0)]/(s-s_0)$, $s\in\Pi(\varphi)$, is directly verified by integration. Indeed, suppose $s\in\Pi(\varphi)$. We have

$$\begin{aligned}\widehat{\kappa}(s) &= -\int_{-\infty}^0 e^{sx}\int_{-\infty}^x e^{s_0(y-x)}\,\nu(dy)\,dx\\ &\quad+\int_0^\infty e^{sx}\int_x^\infty e^{s_0(y-x)}\,\nu(dy)\,dx =: -I_1+I_2.\end{aligned}$$

Let $s\ne s_0$. By Fubini's theorem, $I_1=\int_{-\infty}^0 (e^{s_0y}-e^{sy})\,\nu(dy)/(s-s_0)$. Similarly, $I_2=\int_0^\infty (e^{sy}-e^{s_0y})\,\nu(dy)/(s-s_0)$. Finally, we have $\widehat{\kappa}(s)=[\widehat{\nu}(s)-\widehat{\nu}(s_0)]/(s-s_0)$.

Let now $s=s_0$. Just as before, we obtain $\widehat{\kappa}(s)=\int_{-\infty}^\infty ye^{s_0y}\,\nu(dy)$. It is clear that this integral is the limit of the ratio $[\widehat{\nu}(s)-\widehat{\nu}(s_0)]/(s-s_0)$ as $s\to s_0$, which completes the proof of the theorem.

Theorem 2 of [3, Appendix 2] may be regarded as a particular case of our Theorem 2 for $\varphi(x)=e^{rx}$ for $x\ge 0$ and $\varphi(x)=e^{qx}$ for $x<0$, where $q<r$.

In case s_0 lies on the boundary of the strip $\Pi(\varphi)$, the situation becomes more involved. Nevertheless, the following theorem holds (for the sake of definiteness we consider the case $\Re s_0=r_+(\varphi)$).

THEOREM 3. *Let $\varphi(x)$, $x\in\mathbb{R}$, be a submultiplicative function such that $r_-(\varphi)\le r_+(\varphi)$. Suppose the function $\varphi(x)/\exp[r_+(\varphi)x]$, $x\ge 0$, is nondecreasing and $\varphi(x)/\exp[r_-(\varphi)x]$, $x\le 0$, is nonincreasing. Assume that*

$$(4)\qquad \int_0^\infty (1+x)\varphi(x)\,|\nu|(dx)<\infty \quad \text{or} \quad \int_{\mathbb{R}} (1+|x|)\varphi(x)\,|\nu|(dx)<\infty,$$

depending on whether $r_-(\varphi)<r_+(\varphi)$ or $r_-(\varphi)=r_+(\varphi)$. Let $\Re s_0=r_+(\varphi)$. Then $\widehat{\kappa}(s):=[\widehat{\nu}(s)-\widehat{\nu}(s_0)]/(s-s_0)$, $s\in\Pi(\varphi)$, is the Laplace transform of some measure $\kappa\in S(\varphi)$.

Proof. Suppose $r_-(\varphi)<r_+(\varphi)$. Then the estimate (3) for the integral J remains unchanged. As for the integral I, we use in (2) the first of the inequalities (4) to obtain

$$(5)\qquad I\le \int_0^\infty y\varphi(y)\,|\nu|(dy)<\infty.$$

Let now $r_-(\varphi)=r_+(\varphi)$. Then (5) remains true, and J satisfies a similar inequality: $J\le\int_{-\infty}^0 |y|\varphi(y)\,|\nu|(dy)<\infty$. The theorem is proved.

The measure κ that appears in Theorems 2 and 3 will be denoted by $T(s_0)\nu$, i.e., $\kappa=:T(s_0)\nu$. In the particular case $s_0=0$ we shall write $\kappa=T\nu$ instead of $\kappa=T(0)\nu$.

3. Supremum. Define $D(x)=P(M_\infty\le x)$. For the Laplace transform of D the following representation holds [7, Section XVIII.5, Theorem 2]:

$$(6)\qquad \widehat{D}(s)=\exp\Big\{\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty (e^{sx}-1)\,F^{n*}(dx)\Big\},\qquad \Re s\le 0.$$

Denote by $G(x)$ the distribution function of the first negative sum S_{N_-}, $N_- := \inf\{k \ge 1 : S_k < 0\}$. Then [7, Section XVIII.3, Lemma 1]

$$1-\widehat{G}(s) = \exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{0} e^{sx}\,F^{n*}(dx)\Big\},\qquad \Re s \ge 0. \tag{7}$$

Put $a = \exp\{\sum_{n=1}^{\infty} P(S_n \ge 0)/n\}$. By the factorization theorem [7, Section XVIII.3],

$$1-\widehat{F}(s) = \big[a\widehat{D}(s)\big]^{-1}\cdot\big[1-\widehat{G}(s)\big],\qquad \Re s = 0. \tag{8}$$

Let now conditions (a)–(c) be fulfilled, and suppose the set $\mathcal{Z}$ of nonzero roots of the characteristic equation $1-\widehat{F}(s) = 0$ which lie in the strip $\Pi(r)$ is *finite*. (In case $\widehat{F}(s) \ne 1$ on $\{\Re s = r\}$, analyticity of $\widehat{F}(s)$ and condition (c) imply the finiteness of $\mathcal{Z}$; see, e.g., [8].) Among the elements of $\mathcal{Z}$ there exists one real, say $q \in (0, r]$. Denote the remaining elements of $\mathcal{Z}$ by $s_1, \ldots, s_l$. The *multiplicity* of the root s_j is an integer m_j such that $1-\widehat{F}(s) = (s-s_j)^{m_j}F_j(s)$, where $F_j(s_j) \ne 0$. If $s \in \mathcal{Z}$ and $s \ne q$, then $\overline{s} \in \mathcal{Z}$ and the root $\overline{s}$ has the same multiplicity as s. Put

$$f(s) := \frac{1-\widehat{G}(s)}{a[1-\widehat{F}(s)]},\qquad s \in \Pi(r)\setminus(\mathcal{Z}\cup\{0\}).$$

Let the coefficients $B_{jk}, k = 1, \ldots, m_j$, be defined by the asymptotic expansion

$$f(s) = \sum_{k=1}^{m_j}(-1)^k B_{jk}/(s-s_j)^k + o(1/(s-s_j))\qquad \text{as } s \to s_j, \tag{9}$$

provided $\int_{\mathbb{R}} |x|^{m_j} e^{\Re s_j x}\,F(dx) < \infty$. Similarly, define B_q by the asymptotic expansion $f(s) = -B_q/(s-q) + o(1/(s-q))$ as $s \to q$, provided $\int_{\mathbb{R}} |x| e^{qx}\,F(dx) < \infty$.

Denote by $\mathcal{E}_j$ the complex-valued measure with density $\mathbf{1}_{(0,\infty)}(x)e^{-s_j x}$ ($\mathbf{1}_A(x)$ is the indicator function of the set A); the Laplace transform of this measure is $-1/(s-s_j)$, $\Re(s-s_j) < 0$. Further, let $\mathcal{E}_q$ be the measure with density $\mathbf{1}_{(0,\infty)}(x)e^{-qx}$.

THEOREM 4. *Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random variables with a common nonarithmetical distribution F, and let $\{S_n\}_{n=0}^{\infty}$ be the corresponding random walk generated by its partial sums such that with probability one, $S_n \to -\infty$ as $n \to \infty$.*

Let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function such that $\varphi(x) \equiv 1$ for $x < 0$, $r := r_+(\varphi) > 0$ and the function $\varphi(x)e^{-rx}$, $x \ge 0$, is nondecreasing.

Let q and s_j be the nonzero roots of the equation $1-\widehat{F}(s) = 0$ lying in the strip $\Pi(r)$ with multiplicities 1 and m_j respectively, $j = 1, \ldots, l$. Denote by N the maximal multiplicity of those roots which lie on $\{\Re s = r\}$ ($N = 0$ means that there are no such roots). Suppose that $\int_0^{\infty}(1+x)^{2N}\varphi(x)\,F(dx) < \infty$ and $(F^{m*})_{\mathrm{s}}^{\wedge}(r) < 1$ for some integer $m \ge 1$. Then

$$D = B_q\mathcal{E}_q + \sum_{j=1}^{l}\sum_{k=1}^{m_j} B_{jk}\mathcal{E}_j^{k*} + R, \tag{10}$$

where the remainder R satisfies the inequality $\int_0^{\infty}\varphi(x)\,|R|(dx) < \infty$.

Proof. We form the following submultiplicative functions: $\varphi_k(x) := (1+x)^k\varphi(x)$ for $x \ge 0$ and $\varphi_k(x) := \exp(r'x)$ for $x < 0$, where $r' \in (0, q)$ and the integer parameter k runs from 0 to $2N$. Obviously, $r_+(\varphi_k) = r$ and $r_-(\varphi_k) = r'$ for all $k = 0, \ldots, 2N$. Let $p = \sum_{j=1}^{l} m_j + 1$. Consider the function

$$v(s) := \frac{[1-\widehat{F}(s)](s-r-1)^p}{(s-q)\prod_{j=1}^{l}(s-s_j)^{m_j}},\qquad s \in \{r' \le \Re s \le r\}\setminus\mathcal{Z}.$$

Define the function $v(s)$ at the points q, s_j, $j = 1, \ldots, l$, by continuity. We will show that $v(s)$ is the Laplace transform $\widehat{V}(s)$ of some real-valued measure $V \in S(\varphi_N)$ and, moreover, $1/v(s)$ is the Laplace transform $\widehat{W}(s)$ of some $W \in S(\varphi_N)$, i.e., V is invertible in $S(\varphi_N)$.

Representing a rational function as a sum of partial fractions, we have

$$v(s) = [1-\widehat{F}(s)]\Big[1 + \frac{b_q}{s-q} + \sum_{j=1}^{l}\sum_{k=1}^{m_j}\frac{b_{jk}}{(s-s_j)^k}\Big], \tag{11}$$

where b_q, b_{jk} are some constants.

By the hypotheses of the theorem, $F \in S(\varphi_{2N})$. Consider the functions $f_{jk}(s) := [\widehat{F}(s)-1]/(s-s_j)^k$, $k = 1, \ldots, m_j$, $j = 1, \ldots, l$. We now establish that if $\Re s_j < r$, then $f_{jk}(s)$ is the Laplace transform of some measure in $S(\varphi_{2N})$, and if $\Re s_j = r$, then $f_{jk}(s)$ is the Laplace transform of some measure in $S(\varphi_{2N-k})$.

Let $\nu \in S(\varphi_m)$. If $\Re s_j < r$, then by Theorem 2, $T(s_j)\nu \in S(\varphi_m)$, and if $\Re s_j = r$ and $m > 0$, then by Theorem 3, $T(s_j)\nu \in S(\varphi_{m-1})$ (the operator $T(s_j)$ was introduced at the end of Section 2). Therefore, $f_{jk}(s) = [T(s_j)^k F]^{\wedge}(s)$, $k = 1, \ldots, m_j$, $j = 1, \ldots, l$, are the Laplace transforms of some measures in $S(\varphi_{2N})$ or $S(\varphi_{2N-k})$, respectively. Thus, by (11), $v(s)$ is the Laplace transform $\widehat{V}(s)$ of some $V \in S(\varphi_N)$.

Let $\mathcal{M}$ be the space of maximal ideals of the Banach algebra $S(\varphi_N)$. The following facts are well known from the theory of Banach algebras. Each maximal ideal $M \in \mathcal{M}$ induces a homomorphism of the Banach algebra $S(\varphi_N)$ onto the field of complex numbers $\mathbb{C}$; moreover, M is the kernel of this homomorphism. Denote by $\nu(M)$ the value of this homomorphism at $\nu \in S(\varphi_N)$. An element $\nu \in S(\varphi_N)$ has an inverse if and only if ν does not belong to any maximal ideal $M \in \mathcal{M}$ (in other words, ν is invertible if and only if $\nu(M) \neq 0$ for every $M \in \mathcal{M}$).

The space $\mathcal{M}$ is split into two sets: $\mathcal{M}_1$ is the set of those maximal ideals which do not contain the collection $L(\varphi_N)$ of absolutely continuous measures from $S(\varphi_N)$, and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism $S(\varphi_N) \to \mathbb{C}$ induced by M is of the form $\nu \mapsto \widehat{\nu}(s_0)$, $\nu \in S(\varphi_N)$, where s_0 is some complex number such that $r' \leq \Re s_0 \leq r$. In this case, $M = \{\mu \in S(\varphi_N) : \widehat{\mu}(s_0) = 0\}$ [9, Chap. IV, Section 4]. If $M \in \mathcal{M}_2$, then $\nu(M) = 0$ for each absolutely continuous measure $\nu \in S(\varphi_N)$.

We now show that $V(M) \neq 0$ for each $M \in \mathcal{M}$; this yields the existence of $V^{-1} \in S(\varphi_N)$. Indeed, if $M \in \mathcal{M}_1$, then, for some $s_0 \in \{r' \leq \Re s \leq r\}$, we have $V(M) = \widehat{V}(s_0) \neq 0$. Let $M \in \mathcal{M}_2$. First, note that the condition $(F^{m*})^{\wedge}_{\mathrm{s}}(r) < 1$ implies $(F^{m*})^{\wedge}_{\mathrm{s}}(\alpha) < 1$ for all $\alpha \in [0,r]$. In fact, the function $(F^{m*})^{\wedge}_{\mathrm{s}}(\alpha)$, $\alpha \in [0,r]$, is convex and clearly $(F^{m*})^{\wedge}_{\mathrm{s}}(0) < 1$, whence the desired assertion follows. Applying Theorem 1, we have, for some $\alpha \in [r', r]$,

$$\begin{aligned} |F(M)|^m &= |F^{m*}(M)| = |(F^{m*})_{\mathrm{s}}(M)| \\ &= \Big|\int \chi(x, (F^{m*})_{\mathrm{s}}) \exp(\alpha x)\, (F^{m*})_{\mathrm{s}}(dx)\Big| \\ &\leq \int \exp(\alpha x)\, (F^{m*})_{\mathrm{s}}(dx) < 1. \end{aligned}$$

Replacing in (11) the Laplace transforms by the corresponding measures, we find that $V = \delta - F + V^*$, where $V^* \in L(\varphi_N)$. Hence $|V(M)| = |1 - F(M)| > 0$. This means that there exists an inverse element $W = V^{-1}$, and that the function $\widehat{W}(s) = 1/\widehat{V}(s)$, $r' \leq \Re s \leq r$, is the Laplace transform of the measure W.

Put $u(s) = [1 - \widehat{G}(s)]\widehat{W}(s)/a$. The distribution G is concentrated on the negative half-axis and is clearly an element of $S(\varphi_N)$. Hence $u(s)$ is the Laplace transform $\widehat{U}(s)$ of some $U \in S(\varphi_N)$. We have (see (8))

$$\widehat{D}(s) = u(s) \frac{(s - r - 1)^p}{(s - q)\prod_{j=1}^{l}(s - s_j)^{m_j}}, \quad \Re s = 0. \tag{12}$$

Again the decomposition into partial fractions gives

$$\widehat{D}(s) = u(s) + \frac{b_q u(s)}{s - q} + \sum_{j=1}^{l}\sum_{k=1}^{m_j} \frac{b_{jk} u(s)}{(s - s_j)^k}. \tag{13}$$

We transform each summand of the double sum:

$$\frac{b_{jk} u(s)}{(s - s_j)^k} = b_{jk} \sum_{i=0}^{k-1} \frac{u_{j,i}(s_j)}{(s - s_j)^{k-i}} + b_{jk} u_{j,k}(s), \tag{14}$$

where $u_{j,0}(s) := u(s)$, $u_{j,i}(s) := [u_{j,i-1}(s) - u_{j,i-1}(s_j)]/(s - s_j)$, $i = 1, \ldots, k$. Applying step by step either Theorem 2 or Theorem 3, we establish that the measures $U_{j,k}$ with Laplace transforms $\widehat{U}_{j,k}(s) = u_{j,k}(s)$, $k = 1, \ldots, m_j$, $j = 1, \ldots, l$, belong to $S(\varphi_N)$ or $S(\varphi_{N-k})$, depending on whether $\Re s_j$ is less than or equal to r. Finally,

$$\frac{b_q u(s)}{s - q} = \frac{b_q u(q)}{s - q} + \frac{b_q[u(s) - u(q)]}{s - q} = -\frac{B_q}{s - q} + b_q u_q(s), \tag{15}$$

where by the same Theorems 2 and 3, $u_q(s)$ is the Laplace transform $\widehat{U}_q(s)$ of some $U_q \in S(\varphi_N)$ or $U_q \in S(\varphi_{N-1})$, depending on whether $q < r$ or $q = r$. Put

$$R = U + b_q U_q + \sum_{j=1}^{l}\sum_{k=1}^{m_j} b_{jk} U_{j,k}.$$

By the above, $R \in S(\varphi_0)$. Substituting (14) and (15) into (13) and collecting similar terms, we obtain

$$\widehat{D}(s) = -B_q/(s - q) + \sum_{j=1}^{l}\sum_{k=1}^{m_j} (-1)^k B_{jk}/(s - s_j)^k + \widehat{R}(s). \tag{16}$$

The fact that the coefficient of $(s - s_j)^{-k}$ is precisely $(-1)^k B_{jk}$ follows from uniqueness of the expansion (9). To complete the proof of Theorem 4, it remains to go over from the Laplace transforms to the corresponding measures.

COROLLARY. *Let the hypotheses of Theorem 4 be satisfied. Then*

$$P(M_\infty > x) = B_q e^{-qx}/q + \sum_{j=1}^{l}\sum_{k=1}^{m_j} B_{jk} \mathcal{E}_j^{k*}((x, \infty)) + r(x), \tag{17}$$

where $|r(x)| \leq |R|((x, \infty)) = o(1/\varphi(x))$ *as* $x \to \infty$.

REMARK 1. The particular case of the Corollary when $\varphi(x) = e^{r\max(0,x)}$ yields the estimate $o(e^{-rx})$ of the remainder $r(x)$ as $x \to \infty$, which improves upon the estimate $o(x^N e^{-rx})$ of [13, Theorem 2], where $N = \max_{1 \leq j \leq l} m_j - 1$; also, in [13] it was assumed that $\widehat{F}(s) \neq 1$ on $\{\Re s = r\}$.

REMARK 2. Condition (c) is also necessary for the remainder term R in Theorem 4 to satisfy $\int_0^\infty \varphi(x)\,|R|(dx) < \infty$. Indeed, suppose that (c) is not fulfilled. Then $[(F_+)^{n*}]^{\wedge}_{\mathrm{s}}(r) \geq 1$ for all $n \geq 1$ [17], where F_+ is the defective distribution of the *first positive sum*, i.e., $F_+(A) := P(S_\eta \in A;\ \eta < \infty)$ and $\eta := \inf\{n \geq 1 : S_n > 0\}$. It is known that D coincides, up to a constant factor c, with the renewal measure generated by F_+: $D = c\sum_{n=0}^{\infty}(F_+)^{n*}$. This can be derived from the factorization identities (8) and $1 - \widehat{F}(s) = a_0[1 - \widehat{G}(s)][1 - \widehat{F}_+(s)]$, where $a_0 = \exp\{-\sum_{n=1}^{\infty} P(S_n = 0)/n\}$ [7, Section XVIII.6, (6.5)]. Choose sets A_n of Lebesgue measure zero such that $\int_{A_n} e^{rx}\,[(F_+)^{n*})]_s(dx) \geq 1$. Put $A = \bigcup_{n=1}^{\infty} A_n$. Then the Lebesgue measure of A is zero and

$$\int_A e^{rx}\,D(dx) = c\sum_{n=0}^{\infty}\int_A e^{rx}\,[(F_+)^{n*})]_s(dx) = \infty.$$

But, on the other hand, it follows from (10) and (1) that

$$\int_A e^{rx}\,D(dx) = \int_A e^{rx}\,R(dx) \leq \int_0^{\infty} \varphi(x)\,|R|(dx) < \infty.$$

This contradiction shows that (c) is necessary for $\int_0^{\infty}\varphi(x)\,|R|(dx) < \infty$ to hold.

4. Oscillating random walk

DEFINITION ([11]). Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of independent identically distributed random variables. A sequence $\{Z_n\}$ of random variables is called an *oscillating random walk* with governing sequences $\{X_n\}$ and $\{Y_n\}$ if the random variables $\{Z_n\}$ are defined by an initial value Z_0 and the relations

$$Z_{n+1} = \begin{cases} Z_n + X_n & \text{if } Z_n > 0,\\ Z_n + Y_n & \text{if } Z_n < 0,\\ Z_n + W_n & \text{if } Z_n = 0,\end{cases}$$

where $W_n = X_n$ with probability p and $W_n = Y_n$ with probability $1-p$.

Denote by R_X and R_Y the sets of possible values of X_1 and Y_1. We assume that $R_X \cup R_Y$ is *nonarithmetic*, i.e., its elements cannot be represented in the form kd, where d is a fixed positive number and $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers). Let F and F^* be the distributions of X_1 and Y_1 with finite expectations $\mu = EX_1 < 0$ and $\nu = EY_1 > 0$. Theorem 2 of [4] gives an explicit expression for the stationary distribution π of the oscillating random walk $\{Z_n\}$. In this section we investigate the influence of the roots of the characteristic equations $1-\widehat{F}(s) = 0$ and $1-\widehat{F}^*(s) = 0$ on the properties of π. To this end we introduce some new notation.

Put $S_0^* = 0$, $S_k^* = \sum_{i=1}^{k} Y_i$, $k \geq 1$. As before, $S_k = \sum_{i=1}^{k} X_i$. Let $L_\infty^* = \inf\{S_0^*, S_1^*, \ldots\}$. Denote the distribution of L_∞^* by D^*, and let F_- and let F_+^* denote respectively the distributions of the first nonpositive sum S_N of $\{S_n\}$ and the first nonnegative sum $S_{N^*}^*$ of $\{S_n^*\}$; here $N = \inf\{k \geq 1 : S_k \leq 0\}$ and $N^*(x) = \inf\{k \geq 1 : S_k^* \geq 0\}$.

Define probability distributions as follows: $H := TF_-/ES_N$ and $H^* := TF_+^*/ES_{N^*}^*$ (the operator T was introduced at the end of Section 2).

In this notation the representation for the stationary distribution π of the oscillating random walk $\{Z_n\}$ derived in [4] takes the following form:

$$\pi = \frac{\nu}{\nu-\mu}\,D * H^* + \frac{-\mu}{\nu-\mu}\,D^* * H =: \pi_1 + \pi_2. \tag{18}$$

Since the restriction $\pi|_{(0,\infty)}$ coincides with π_1 (and, similarly, $\pi|_{(-\infty,0)} = \pi_2$), the problem of studying the influence of the roots of $1-\widehat{F}(s) = 0$ and $1-\widehat{F}^*(s) = 0$ on the properties of π reduces to similar problems for π_1 and π_2.

Suppose $\widehat{F}(r) \geq 1$ and $\widehat{F}^*(r)$ are finite for some $r > 0$. Along with $\mathcal{Z}$ consider the set Ξ^* of roots of $1-\widehat{F}^*(s) = 0$ lying in $\Pi(r)$. If Ξ^* is nonvoid, denote its elements by ξ_j^*, and their multiplicities by m_j^*, $j = 1,\ldots,l^*$.

Notice that the zeros of $\widehat{H}^*(s)$ in $\Pi(r)$ coincide with the roots of $1-\widehat{F}^*(s) = 0$; moreover, the multiplicities of the zeros of $\widehat{H}^*(s)$ and those of the corresponding roots of $1-\widehat{F}^*(s) = 0$ are the same. Indeed, this follows from the factorization identity

$$1-\widehat{F}^*(s) = [b^*\widehat{D}^*(s)]^{-1}[1-\widehat{F}_+^*(s)], \quad 0 \leq \Re s \leq r, \tag{19}$$

where $b^* = \exp\{\sum_{n=1}^{\infty} P(S_n^* < 0)/n\}$; identity (19) is a consequence of the factorization theorem, Lemma 2 of [7, Section XVIII.3] and the expression for $\widehat{D}^*(s)$, symmetric to (6). Dividing both sides of (19) by $-s$, we obtain

$$\frac{\widehat{F}^*(s)-1}{s} = [b^*\widehat{D}^*(s)]^{-1} ES_{N^*}^* \cdot \widehat{H}^*(s), \quad 0 \leq \Re s \leq r, \tag{20}$$

whence the assertion about the zeros of $\widehat{H}^*(s)$ immediately follows.

Let $\psi(x)$ be a submultiplicative function such that $0 < r_-(\psi) \leq r_+(\psi)$. We have $b^*\widehat{D}^*(s) = \exp\{\sum_{n=1}^{\infty}\int_{-\infty}^{0} e^{sx}(F^*)^{n*}(dx)/n\}$, $\Re s \geq 0$, or $b^*\widehat{D}^*(s) = \exp[\widehat{\kappa}(s)]$, where the measure $\kappa(A) := \sum_{n=1}^{\infty}(F^*)^{n*}(A \cap (-\infty,0))/n$ belongs to the Banach algebra of finite measures since $\int_{\mathbb{R}} x\,F^*(dx) > 0$. Hence $[b^*\widehat{D}^*(s)]^{-1} = \exp[-\widehat{\kappa}(s)]$ is the Laplace transform of the finite measure $\sum_{n=0}^{\infty}(-1)^n\kappa^{n*}/n!$ concentrated on $(-\infty,0]$. By (1), this measure is obviously in $S(\psi)$. Now, it follows from (20) that $H^* \in S(\psi) \Leftrightarrow TF^* \in S(\psi)$. In particular,

$$\int_{\mathbb{R}} |x|^k e^{rx}\,H^*(dx) < \infty \Leftrightarrow \int_{\mathbb{R}} |x|^k e^{rx}[1-F^*(x)]\,dx < \infty$$

(here we have taken $\psi(x) := (1+|x|)^k e^{rx}$). Put

$$f_1(s) := \frac{1-\widehat{G}(s)}{a[1-\widehat{F}(s)]}\,\widehat{H}^*(s), \quad s \in \Pi(r) \setminus (\mathcal{Z} \cup \{0\}).$$

Define the coefficients C_{jk}, $k=1,\dots,n_j$, by the asymptotic expansion

(21) $$f_1(s)=\sum_{k=1}^{n_j}(-1)^k C_{jk}/(s-s_j)^k+o(1/(s-s_j))\quad \text{as } s\to s_j,$$

provided $\int_{\mathbb R}|x|^{m_j}e^{\Re s_j x}F(dx)<\infty$ and $\int_{\mathbb R}|x|^{m_j}e^{\Re s_j x}[1-F^*(x)]\,dx<\infty$; here and in what follows $n_j=m_j$ if $s_j\notin\Xi^*$, and $n_j=\max(0,m_j-m_i^*)$ if $s_j=\xi_i^*\in\Xi^*$. Similarly, define C_q by the asymptotic expansion

(22) $$f_1(s)=-C_q/(s-q)+o(1/(s-q))\quad \text{as } s\to q,$$

provided $\int_{\mathbb R}|x|e^{qx}F(dx)<\infty$ and $\int_{\mathbb R}|x|e^{qx}[1-F^*(x)]\,dx<\infty$.

THEOREM 5. *Let $\{Z_n\}_{n=0}^\infty$ be an oscillating random walk and π be the stationary distribution defined by formula* (18).

Let $\varphi(x)$, $x\in\mathbb R$, be a submultiplicative function such that $\varphi(x)\equiv1$ for $x<0$, $r:=r_+(\varphi)>0$ and the function $\varphi(x)e^{-rx}$, $x\ge0$, does not decrease.

Denote by N the maximal multiplicity of the roots of the equation $1-\widehat F(s)=0$ lying on the line $\{\Re s=r\}$, and by Q the maximal n_j corresponding to such roots (if $r=q$, then $N=Q=1$).

Suppose $\int_0^\infty x^{N+Q}\varphi(x)\,F(dx)<\infty$, $\int_0^\infty x^N\varphi(x)[1-F^*(x)]\,dx<\infty$, and $(F^{m*})_s^\wedge(r)<1$ for some integer $m\ge1$. Then

$$\pi_1=\frac{\nu}{\nu-\mu}\Big[C_q\mathcal E_q+\sum_{j=1}^{l}\sum_{k=1}^{n_j}C_{jk}\mathcal E_j^{k*}+R_1\Big],$$

where the remainder R_1 satisfies the inequality $\int_0^\infty\varphi(x)\,|R_1|(dx)<\infty$.

Proof. According to (18) and (12), we have

$$\frac{\nu-\mu}{\nu}\widehat\pi_1(s)=\widehat D(s)\widehat H^*(s)=c(s)\frac{(s-r-1)^p}{(s-q)\prod_{j=1}^{l}(s-s_j)^{m_j}},$$

where $c(s)$ is the Laplace transform of the measure $K:=U*H^*$. Clearly, $TF^*\in S(\varphi_N)$. As pointed out above, this is equivalent to $H^*\in S(\varphi_N)$. It follows from the proof of Theorem 4 that $U\in S(\varphi_Q)$, and hence $K\in S(\varphi_Q)$.

By decomposition into partial fractions, we obtain

$$\frac{\nu-\mu}{\nu}\widehat\pi_1(s)=c(s)+\frac{b_qc(s)}{s-q}+\sum_{j=1}^{l}\sum_{k=1}^{m_j}\frac{b_{jk}c(s)}{(s-s_j)^k}.$$

The further reasoning is, on the whole, the same as in the proof of Theorem 4; the main difference is in the need to take into account the fact that some of the roots s_j and ξ_i^* of $1-\widehat F(s)=0$ and $1-\widehat F^*(s)=0$ may coincide, especially when these roots lie on the line $\{\Re s=r\}$. If $s_j\notin\Xi^*$, then $n_j=m_j$ and

$$\sum_{k=1}^{m_j}\frac{b_{jk}\,c(s)}{(s-s_j)^k}=\sum_{k=1}^{m_j}b_{jk}\sum_{i=0}^{k-1}\frac{k_{j,i}(s_j)}{(s-s_j)^{k-i}}+\sum_{k=1}^{m_j}b_{jk}k_{j,i}(s),$$

where $k_{j,0}(s):=c(s)$, $k_{j,i}(s):=[k_{j,i-1}(s)-k_{j,i-1}(s_j)]/(s-s_j)$, $i=1,\dots,m_j$. Depending on whether $\Re s_j<r$ or $\Re s_j=r$, we apply Theorem 2 or Theorem 3 to establish that the measures $T(s_j)^kK$ with Laplace transforms $k_{j,k}(s)$ are in $S(\varphi_Q)$ or $S(\varphi_{Q-k})$ respectively, $k=1,\dots,m_j$. Let now $s_j=\xi_i^*$ and $n_j=0$. Then, by Theorem 2 or Theorem 3, the function $\sum_{k=1}^{m_j}b_{jk}c(s)/(s-s_j)^k$ is the Laplace transform of the measure $\sum_{k=1}^{m_j}b_{jk}U*[T(s_j)^kH^*]$ belonging to $S(\varphi_Q)$ or to $S(\varphi_Q)\cap S(\varphi_{N-m_j})\subset S(\varphi_0)$, depending on whether $\Re s_j<r$ or $\Re s_j=r$. Finally, let $s_j=\xi_i^*$ and $n_j>0$. Then

$$\sum_{k=1}^{m_j}\frac{b_{jk}\,c(s)}{(s-s_j)^k}=\sum_{k=1}^{m_i^*}b_{jk}\widehat K_{jk}(s)+\sum_{k=1}^{n_j}b_{j,k+m_i^*}\frac{\widehat K_{jm_i^*}(s)}{(s-s_j)^k}=:\Sigma_{j1}+\Sigma_{j2},$$

where $K_{jk}=U*[T(s_j)^kH^*]$ is in $S(\varphi_Q)$ or $S(\varphi_Q)\cap S(\varphi_{N-k})$, depending on whether $\Re s_j<r$ or $\Re s_j=r$, $k=1,\dots,m_i^*$. Thus, Σ_{j1} is the Laplace transform of some measure in $S(\varphi_Q)\cap S(\varphi_{N-m_i^*})\subset S(\varphi_0)$. Applying the already known procedure to the expression $\widehat K_{jm_i^*}(s)/(s-s_j)^k$, we obtain

$$\Sigma_{j2}=\sum_{k=1}^{n_j}b_{j,k+m_i^*}\sum_{p=0}^{k-1}\frac{[T(s_j)^pK_{jm_i^*}]^\wedge(s_j)}{(s-s_j)^{k-p}}+\sum_{k=1}^{n_j}b_{j,k+m_i^*}[T(s_j)^kK_{jm_i^*}]^\wedge(s).$$

Clearly, $T(s_j)^kK_{jm_i^*}=T(s_j)^k\{U*[T(s_j)^{m_i^*}H^*]\}$ is in $S(\varphi_Q)$ or $S(\varphi_{Q-k})\cap S(\varphi_{N-m_i^*-k})\subset S(\varphi_0)$, depending on whether $\Re s_j<r$ or $\Re s_j=r$; here $k=1,\dots,n_j$. Hence $Q-k\ge Q-n_j\ge0$ and $N-m_i^*-k\ge N-m_i^*-n_j=N-m_j\ge0$. Applying the same procedure to the term $b_qc(s)/(s-q)$, summing up over j from 1 to l, and collecting similar terms, we obtain (by uniqueness of the asymptotic expansions (21) and (22))

$$\frac{\nu-\mu}{\nu}\widehat\pi_1(s)=\frac{-C_q}{s-q}+\sum_{j=1}^{l}\sum_{k=1}^{n_j}(-1)^k\frac{C_{jk}}{(s-s_j)^k}+\widehat R_1(s),$$

where $\widehat R_1(s)$ is the Laplace transform of some measure $R_1\in S(\varphi_0)$. The theorem is proved.

The picture on the negative half-axis is quite symmetric.

REMARK 3. Theorem 5 allows us to derive an asymptotic expansion for the tail $\pi((x,\infty))$ as $x\to\infty$ (see the Corollary of Theorem 4).

REMARK 4. Particular cases of the results of this paper have been considered in [18]. They correspond to $\varphi(x)=\exp(rx)$ for $x\ge0$ and $\varphi(x)\equiv1$

for $x < 0$. Moreover in [18] it is assumed that $EX_1 < 0$ is finite and that on the line $\{\Re s = r\}$ there are no roots of the characteristic equation.

REMARK 5. In Theorems 2–5 the monotonicity conditions on $\varphi(x)$ may be replaced by the following: $\varphi(y)/\exp[r_+(\varphi)y] \le C\varphi(x)/\exp[r_+(\varphi)x]$ for all $0 \le y \le x$ and $\varphi(y)/\exp[r_-(\varphi)y] \le C\varphi(x)/\exp[r_-(\varphi)x]$ for all $x \le y \le 0$, where $C \ge 1$ is a constant.

REMARK 6. Let $\varphi(k)$, $k \in \mathbb{Z}$, be a submultiplicative function defined on the set $\mathbb{Z}$ of all integers. If in the proofs of Theorems 2–5 we replace the Laplace transforms by the generating functions and use the Banach algebras of complex sequences $\{\nu_k\}$ such that $\sum_{k=-\infty}^{\infty} \varphi(k)|\nu_k| < \infty$ instead of the Banach algebras $S(\varphi)$, then we obtain analogous theorems for measures and distributions concentrated on $\mathbb{Z}$. In this case the measures $\mathcal{E}_j$, $\mathcal{E}_q$, $\underline{\mathcal{E}}_j$ and $\underline{\mathcal{E}}_{q^*}$ must be replaced by their discrete counterparts. Moreover, in the arithmetic case there is no need for condition (c).

References

[1] G. Alsmeyer, *On generalized renewal measures and certain first passage times*, Ann. Probab. 20 (1991), 1229–1247.

[2] J. Bertoin and R. A. Doney, *Some asymptotic results for transient random walks*, J. Appl. Probab. 28 (1996), 206–226.

[3] A. A. Borovkov, *Stochastic Processes in Queueing Theory*, Springer, New York, 1976.

[4] —, *A limit distribution for an oscillating random walk*, Theory Probab. Appl. 25 (1980), 649–651.

[5] A. A. Borovkov and D. A. Korshunov, *Probabilities of large deviations for one-dimensional Markov chains. I. Stationary distributions*, Theory Probab. Appl. 41 (1996), 1–24.

[6] P. Embrechts and N. Veraverbeke, *Estimates for the probability of ruin with special emphasis on the possibility of large claims*, Insurance Math. Econom. 1 (1982), 55–72.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications II*, Wiley, New York, 1966.

[8] B. B. van der Genugten, *Asymptotic expansions in renewal theory*, Compositio Math. 21 (1969), 331–342.

[9] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Colloq. Publ. 31, Amer. Math. Soc., Providence, 1957.

[10] S. Janson, *Moments for first passage times, the minimum, and related quantities for random walks with positive drift*, Adv. Appl. Probab. 18 (1986), 865–879.

[11] J. H. B. Kemperman, *The oscillating random walk*, Stochastic Process. Appl. 2 (1974), 1–29.

[12] J. Kiefer and J. Wolfowitz, *On the characteristics of the general queueing process, with applications to random walk*, Ann. Math. Statist. 27 (1956), 147–161.

[13] V. I. Lotov, *Asymptotics of the distribution of the supremum of consecutive sums*, Mat. Zametki 38 (1985), 668–678 (in Russian).

[14] B. A. Rogozin and M. S. Sgibnev, *Banach algebras of measures on the line*, Siberian Math. J. 21 (1980), 265–273.

[15] M. S. Sgibnev, *On the distribution of the supremum of sums of independent summands with negative drift*, Mat. Zametki 22 (1977), 763–770 (in Russian).

[16] —, *Submultiplicative moments of the supremum of a random walk with negative drift*, Statist. Probab. Lett. 32 (1997), 377–383.

[17] —, *Equivalence of two conditions on singular components*, ibid. 40 (1998), 127–131.

[18] —, *On the distribution of the supremum in the presence of roots of the characteristic equation*, Teor. Veroyatnost. i Primenen. 43 (1998), 383–390 (in Russian).

[19] R. L. Tweedie, *The existence of moments for stationary Markov chains*, J. Appl. Probab. 20 (1983), 191–196.

[20] N. Veraverbeke, *Asymptotic behavior of Wiener–Hopf factors of a random walk*, Stochastic Process. Appl. 5 (1977), 27–37.

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