$\lim_{n\to\infty} A(I-h_nA)^{-1}x_n = Ax$. For each $\tau > 0$ we have

$$\int_0^\tau \|B_n(s)x_n - B(s)x\|\,ds \le \tau \sup\{\|B(t)\|_{Y,X} : t\in[0,\tau+1]\}\|(I-h_nA)^{-1}x_n - x\|_Y + \int_0^\tau \|B(s+h_n)x - B(s)x\|\,ds$$

for $n \ge 1$. Since $B(\cdot)x \in L^1_{\rm loc}([0,\infty);X)$ the last term tends to zero as $n\to\infty$. It follows that condition (iv) of (b_2) is satisfied. Theorem 3.1 therefore asserts that $\lim_{n\to\infty} F_{n,[t/h_n]}x = R(t)x$, which implies in turn that $\lim_{n\to\infty} U_{n,[t/h_n]}x = R(t)x$ for $t\ge 0$ and $x\in X$. ■

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Localizations of partial differential operators and surjectivity on real analytic functions

by

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Abstract. Let $P(D)$ be a partial differential operator with constant coefficients which is surjective on the space $A(\Omega)$ of real analytic functions on an open set $\Omega\subset\mathbb{R}^n$. Then $P(D)$ admits shifted (generalized) elementary solutions which are real analytic on an arbitrary relatively compact open set $\omega\subset\subset\Omega$. This implies that any localization $P_{m,\Theta}$ of the principal part P_m is hyperbolic w.r.t. any normal vector N of $\partial\Omega$ which is noncharacteristic for $P_{m,\Theta}$. Under additional assumptions P_m must be locally hyperbolic.

Surjectivity criteria for partial differential operators have been obtained in most of the classical spaces of (generalized) functions in the fifties and early sixties. However, the basic question of when

$$P(D) : A(\Omega) \to A(\Omega) \quad \text{is surjective,} \tag{0.1}$$

remained open. Here $P(D)$ is a partial differential operator with constant coefficients, $\Omega\subset\mathbb{R}^n$ is an open set and $A(\Omega)$ is the space of real analytic functions on Ω.

Piccinini [37] showed that the heat equation is not surjective on $A(\mathbb{R}^3)$ as was conjectured by Cattabriga–de Giorgi [12]. Then Hörmander [21] characterized (0.1) for convex sets Ω by means of a Phragmén–Lindelöf condition valid on the complex variety of P. Since then Hörmander's method has been adapted by several authors for further studies on this problem (Miwa [36], Andreotti–Nacinovich [3], Zampieri [40], Braun [9]), and on the related surjectivity problem on nonquasianalytic Gevrey classes (Zampieri [41], Braun–Meise–Vogt [10, 11]).

Specifically, (0.1) was proved to hold for operators having a locally hyperbolic principal part P_m if $\Omega=\mathbb{R}^n$ (see Andersson [2] and Hörmander [21]) or if Ω is convex and additional conditions on the local propagation cones of

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P_m are satisfied (Zampieri [40]). Hörmander's method is restricted to convex sets Ω by the use of Fourier theory. However, local hyperbolicity of P_m (combined with some geometrical condition on the local propagation cones of P_m) is also sufficient for (0.1) for certain (not necessarily convex) bounded sets Ω as was shown by Kawai [25]. The assumption of boundedness was removed by Kaneko [23]. Local hyperbolicity is thus a useful, but restrictive, sufficient condition for (0.1).

The aim of the present paper is to show that hyperbolicity of the localizations $P_{m,\Theta}$ of P_m and local hyperbolicity of P_m are in fact necessary for (0.1) in many cases. The proof is based on a new necessary condition for (0.1) which is proved in Section 1 (see Theorem 1.3). It roughly states that $P(D)$ has (generalized) elementary solutions which are real analytic on arbitrary relatively compact open subsets of Ω if (0.1) holds. The elementary solutions used here are harmonic functions (in $n+1$ variables) defined outside thin strips near $\mathbb{R}^n$ and thus can be considered as generalized hyperfunctions. This basic necessary condition is the appropriate extension to the case of real analytic functions of the criterion of Langenbruch [26] for surjective partial differential operators on nonquasianalytic Gevrey classes.

We then state the results on extension of analyticity from Langenbruch [30] in Section 2. These are used in Section 3 to show that the basic necessary condition implies certain bounds of hyperbolic type on the location of zeros of P and P_m. The consequences of these bounds are studied in the second part of Section 3 and in Section 4.

We first consider the localizations $P_{m,\Theta}$ of the principal part P_m of P at a point

$$\Theta \in V_{P_m} := \{x \in \mathbb{R}^n \mid P_m(x) = 0,\ |x| = 1\}.$$

Let $N(\partial\Omega)$ denote the set of unit normal vectors of $\partial\Omega$. Then (0.1) implies that $P_{m,\Theta}$ is hyperbolic w.r.t. $N \in N(\partial\Omega)$ if $\Theta \in V_{P_m}$ and if N is noncharacteristic for $P_{m,\Theta}$.

We mention some interesting consequences of this result: if (0.1) holds for a bounded open set Ω with C^1-boundary, then any localization $P_{m,\Theta}$ is the product of real linear forms (times a complex constant). If P_m is independent of a variable (i.e. if the lineality $\Lambda(P_m) \neq \{0\}$), we get the following characterization:

(i) (0.1) holds for a halfspace $\Omega_N := \{x \in \mathbb{R}^n \mid \langle x, N\rangle > 0\}$ with noncharacteristic N if and only if P_m is hyperbolic w.r.t. N.

(ii) (0.1) holds for some open bounded set Ω with C^1-boundary if and only if P_m is the product of real linear forms (times a complex constant).

The sufficiency of these conditions follows from the results of Hörmander [21] and Zampieri [40].

In Section 4 we show that (0.1) implies that P_m is locally hyperbolic w.r.t. $N \in N(\partial\Omega)$ at $\Theta \in V_{P_m}$ if N is locally noncharacteristic for P_m at Θ (this assumption is a local version of a condition of Hörmander [20], see (4.7) and Definition 4.4). This implies the following characterizations:

(i) Let N be locally noncharacteristic for P_m at any $\Theta \in V_{P_m}$. Then (0.1) holds for the halfspace Ω_N if and only if P_m is hyperbolic-elliptic w.r.t. N (in the sense of Fehrman [14]).

(ii) Assume that for any $\Theta \in V_{P_m}$ in any component of $\mathbb{R}^n \setminus V_{P_m}$ there is N which is locally noncharacteristic for P_m at Θ. Then (0.1) holds for some open bounded set Ω with C^1-boundary if and only if for any $\Theta \in V_{P_m}$, $P_{m,\Theta}$ is the product of real linear forms (times a complex constant) and if P_m is locally hyperbolic at Θ w.r.t. any N which is noncharacteristic for $P_{m,\Theta}$.

We finally notice that for a polynomial P in three variables, N is locally noncharacteristic for P_m at Θ if and only if N is noncharacteristic for $P_{m,\Theta}$ (see [31]).

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1. A new necessary condition. In this section we will prove a new necessary condition for surjective partial differential operators on real analytic functions which will then be evaluated in the subsequent sections of this paper. The condition roughly means that there are (generalized) fundamental solutions which are real analytic on large sets (see Theorem 1.3).

We start with some useful notations and conventions: in this paper, $n \in \mathbb{N}$ is always at least 2. A point in $\mathbb{R}^{n+1}$ is usually written as $(x,y) \in \mathbb{R}^n \times \mathbb{R}$. Open euclidean balls in $\mathbb{R}^n$ (and in $\mathbb{R}^{n+1}$) are denoted by $U_\varepsilon(\xi)$ (and $V_\varepsilon(\eta)$, respectively). Also, $U_\varepsilon := U_\varepsilon(0)$ and $V_\varepsilon := V_\varepsilon(0)$ and

$$S^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

$\Delta = \sum_{k=1}^n (\partial/\partial x_k)^2 + (\partial/\partial y)^2$ is the Laplace operator on $\mathbb{R}^{n+1}$. The harmonic functions on an open set $V \subset \mathbb{R}^{n+1}$ (which are even w.r.t. y) are denoted by $C_\Delta(V)$ (and $\widetilde{C}_\Delta(V)$, respectively, if $(x,-y) \in V$ for any $(x,y) \in V$).

In the following, Ω is always an open set in $\mathbb{R}^n$. The real analytic functions on Ω are denoted by $A(\Omega)$. $P(D) = P(D_x)$ is always a partial differential operator in n variables of degree m with constant coefficients. P_m denotes the principal part of P and

$$V_{P_m} := \{\Theta \in S^{n-1} \mid P_m(\Theta) = 0\}.$$

To state the necessary criterion for surjectivity we need a sufficiently general notion of an elementary solution of $P(D)$. In fact, the elementary solutions used in this paper are taken from the space

$$\widetilde{C}_\Delta(\Omega_c), \quad \Omega_c := \Omega \times (\mathbb{R} \setminus [-c, c]), \qquad c \geq 0,$$

which can be considered as a generalization of hyperfunctions. Indeed, hyperfunctions on Ω can be defined as $\mathcal{B}(\Omega) := \widetilde{C}_\Delta(\Omega_0)/\widetilde{C}_\Delta(\Omega \times \mathbb{R})$ (see Bengel [4] and Hörmander [22, Chapter IX]).

$E \in \widetilde{C}_\Delta(\Omega_c)$ is canonically written as $E(x, y) = E_+(x, |y|)$, $|y| > c$, with $E_+ \in C_\Delta(\Omega \times]c, \infty[)$.

The appropriate notion of a (shifted) elementary solution for $P(D)$ on Ω now is the following:

DEFINITION 1.1. Let $\xi \in \Omega$. $E \in \widetilde{C}_\Delta(\Omega_c)$ is called a *$\{\xi\}$-elementary solution* for $P(D)$ on Ω if $P(D)E$ can be extended to $\Omega \times \mathbb{R}$ as a distribution H such that $\Delta H = \delta_{(\xi,0)}$ where $\delta_{(\xi,0)}$ is the point evaluation in $(\xi, 0)$.

The extension H of $P(D)E$ is unique. $\{0\}$-elementary solutions are called elementary solutions.

Definition 1.1 extends the notion of a distributional elementary solution. Indeed, let F be a distributional elementary solution for $P(D)$ (and $\xi := 0 \in \Omega$). There is $\overline{E} \in D'(\Omega \times \mathbb{R})$ such that $\Delta \overline{E} = F \otimes \delta(y)$. Set $E := \overline{E}|_{\Omega_0}$. Then E is a $\{0\}$-elementary solution in the sense of Definition 1.1 since $P(D)E$ is extended by $H := P(D)\overline{E}$ and $\Delta H = \delta_{(0,0)}$.

It is well known that there is $B_1 \geq 1$ such that for any $\gamma > 0$ there is $C \geq 1$ such that

$$\sup\{|f^{(\alpha)}(0)|(\gamma/B_1)^{|\alpha|}/\alpha! \mid \alpha \in \mathbb{N}_0^n\} \leq C \sup\{|f(\eta)| \mid \eta \in V_\gamma\} \tag{1.1}$$

if $f \in C_\Delta(V_\gamma)$ is bounded on V_γ. With this constant B_1 we now introduce the spaces of real analytic functions with fixed Cauchy estimates which are used in this paper:

DEFINITION 1.2. Let $\nu : \Omega \to]0, \infty[$ satisfy

$$\nu(x) \leq \nu(\xi) + |x - \xi|/(2B_1) \quad \text{for } x, \xi \in \Omega \tag{1.2}$$

with B_1 from (1.1). Then

$$A_\nu(\Omega) := \{f \in A(\Omega) \mid \forall K \subset\subset \Omega : \sup\{|f^{(\alpha)}(x)|\nu(x)^{|\alpha|}/\alpha! \mid x \in K,\ \alpha \in \mathbb{N}_0^n\} < \infty\}.$$

$A_\nu(\Omega)$ is an (F)-space. (1.2) is e.g. satisfied if $\nu(x) = \widetilde{\nu}(\operatorname{dist}(x, \partial\Omega))$ for some $\mathcal{C}^1$-function $\widetilde{\nu} :]0, \infty[\to]0, \infty[$ such that

$$0 \leq \widetilde{\nu}'(t) \leq 1/(2B_1) \quad \text{for any } t > 0. \tag{1.3}$$

The following theorem is the basic result of this paper.

THEOREM 1.3. *Assume that for every $g \in A_\nu(\Omega)$ the equation $P(D)f = g$ has a solution $f \in A(\Omega)$. Then for any open $\omega \subset\subset \Omega$ there is $\delta > 0$ such that for any $\xi \in \Omega$, $P(D)$ has a $\{\xi\}$-elementary solution $E = E_\xi \in \widetilde{C}_\Delta(\Omega_{T_\xi})$, $T_\xi := 2B_1\nu(\xi)$, such that E_+ can be extended as a harmonic function to $\omega \times]T_\xi - \delta, \infty[$.*

P r o o f. a) Fix $\omega \subset\subset \Omega$. We claim that there is $k \in \mathbb{N}$ such that for any $g \in A_\nu(\Omega)$ the equation $P(D)f = g$ can be solved with $f \in \mathcal{C}^\infty(\Omega)$ such that

$$\sup\{|f^{(\alpha)}(x)|/(k^{|\alpha|}\alpha!) \mid x \in \omega,\ \alpha \in \mathbb{N}_0^n\} < \infty. \tag{1.4}$$

Indeed, for $k \in \mathbb{N}$ let F_k be the (F)-space defined by $F_k := \{f \in \mathcal{C}^\infty(\Omega) \mid f$ satisfies (1.4)$\}$ with the topology induced by $\mathcal{C}^\infty(\Omega)$ and the seminorm (1.4) and let $N_k := F_k \cap \ker P(D)$. Then $F := \operatorname{ind}(F_k/N_k)$ is an (LF)-space and $P(D)^{-1} : A_\nu(\Omega) \to F$ is defined by assumption. $P(D)^{-1}$ is continuous by the closed graph theorem for (LF)-spaces (since the inclusion of F into $\mathcal{C}^\infty(\Omega)/(\mathcal{C}^\infty(\Omega) \cap \ker P(D))$ is continuous). By Grothendieck's factorization theorem (Meise–Vogt [35, 24.33]) there is $k \in \mathbb{N}$ such that $P(D)^{-1}(A_\nu(\Omega)) \subset F_k/N_k$. This shows the claim.

b) Let $G(x, y) := -|(x, y)|^{1-n}/((n-1)c_{n+1})$ be the canonical even elementary solution of Δ on $\mathbb{R}^{n+1}$ (since $n + 1 \geq 3$). For $\xi \in \Omega$ let $\chi_+(y)$ be the characteristic function of $[T_\xi, \infty[$. Then

$$\Delta(G(\cdot - \xi,\ )\chi_+) = g_1 \otimes \delta_{T_\xi}(y) + g_2 \otimes \partial_y\delta_{T_\xi}(y) \tag{1.5}$$

where the functions $g_1 := \frac{\partial}{\partial y}G(\cdot - \xi, T_\xi)$ and $g_2 := G(\cdot - \xi, T_\xi)$ are contained in $A_\nu(\Omega)$ by (1.1) and (1.2). By a) there are solutions $f_s \in \mathcal{C}^\infty(\Omega)$ of $P(D)f_s = g_s$ satisfying (1.4) for $s = 1, 2$. Since Δ is elliptic, we can solve the equation

$$\Delta E_1 = f_1 \otimes \delta_{T_\xi}(y) + f_2 \otimes \partial_y\delta_{T_\xi}(y) \tag{1.6}$$

in $D'(\Omega \times \mathbb{R})$. Set $E_2(x, y) := E_1(x, y) + E_1(x, -y)$ for $(x, y) \in \Omega \times \mathbb{R}$ and let χ be the characteristic function of $\mathbb{R} \setminus [-T_\xi, T_\xi]$. Then

$$\Delta(P(D)E_2) = \Delta(G(\cdot - \xi,\ )\chi) \quad \text{on } \Omega \times \mathbb{R}$$

since $\Delta(P(D)E_1) = \Delta(G(\cdot - \xi,\ )\chi_+)$ by the choice of E_1 and (1.5) and since $(G\chi)(x, y) = (G\chi_+)(x, y) + (G\chi_+)(x, -y)$. Thus there is $h \in \widetilde{C}_\Delta(\Omega \times \mathbb{R})$ such that

$$P(D)E_2 = G(\cdot - \xi,\ )\chi + h. \tag{1.7}$$

Set $E := E_2|_{\Omega_{T_\xi}}$. Then by (1.7), $P(D)E$ can be extended to $\Omega \times \mathbb{R}$ by $G(\cdot - \xi,\ ) + h$. Therefore, E is a $\{\xi\}$-elementary solution for $P(D)$.

c) Since f_1 and f_2 satisfy (1.4), we can solve the Cauchy problem

$$\Delta\widetilde{h}(x, y) = 0, \quad \widetilde{h}(x, T_\xi) = f_2(x), \quad \partial_y\widetilde{h}(x, T_\xi) = f_1(x), \qquad x \in \omega,$$

for $|y-T_\xi|<\delta:=1/(2n^{1/2}k)$ by the well known formula

$$(1.8)\qquad \widetilde{h}(x,y)=\sum_{j=0}^{\infty}\frac{(-\Delta_x)^j f_2(x)(y-T_\xi)^{2j}}{(2j)!}+\sum_{j=0}^{\infty}\frac{(-\Delta_x)^j f_1(x)(y-T_\xi)^{2j+1}}{(2j+1)!}.$$

By (1.6) and the argument from (1.5) we get

$$\Delta(\widetilde{h}\chi_+)=\Delta E_1\quad \text{on } \omega\times\,]T_\xi-\delta,T_\xi+\delta[.$$

E_1 is thus extendable as a harmonic function from $\omega\times\,]T_\xi,T_\xi+\delta[$ and $\omega\times\,]T_\xi-\delta,T_\xi[$ to $\omega\times\,]T_\xi-\delta,T_\xi+\delta[$. This shows that $E_+=E_2|_{\Omega\times]T_\xi,\infty[}$ can be extended as a harmonic function to $\omega\times\,]T_\xi-\delta,\infty[$. The theorem is proved.

If $T_\xi=2B_1\nu(\xi)<\delta$, then E_ξ is real analytic on ω, more precisely, $E_{\xi|\omega_{T_\xi}}$ has a real analytic function as boundary value on ω.

A similar condition to the one of Theorem 1.3 is equivalent to the surjectivity of $P(D)$ on nonquasianalytic classes of ultradifferentiable functions of Roumieu type (Langenbruch [26, 28, 29]).

2. Extension of regularity. To evaluate the necessary condition from Theorem 1.3 we will have to improve on the regularity of elementary solutions provided by that theorem. Extension theorems for analyticity and for the complement of the analytic wave front set of zerosolutions have been proved by many authors (usually for operators with variable coefficients). A selection of relevant papers is contained in the references (see Andersson [1], Bony [6, 7], Bony–Schapira [8], Grigis–Schapira–Sjöstrand [16], Hanges [17], Hanges–Sjöstrand [18], Hörmander [19], Kashiwara–Kawai [24], Laubin [32], [33], Liess [34], Sjöstrand [38]; the reader is also referred to the literature cited there). In the present section we will state a quantitative version of such extension theorems (Langenbruch [30, Theorem 3.4]) which leads to better results for the surjectivity question we have in mind (see Remark 3.5). For $\Omega\subset\mathbb{R}^n$ open and $C\geq 1$ let

$$A_{C,\Omega}:=\{(\varphi_k)\in D(\Omega)^{\mathbb{N}}\mid \forall d\in\mathbb{N}\ \exists C_d\geq 1\ \forall k\in\mathbb{N}:\ \|\varphi_k^{(\alpha+\beta)}\|_\infty\leq C_d(kC)^{|\alpha|}\text{ if }|\alpha|\leq k\text{ and }|\beta|\leq d\}.$$

$A_{C,\Omega}$ will serve as "analytic cut-off functions" in the definition of the regularity set (as in the theory of wave front sets for distributions, see e.g. Hörmander [22, Lemma 8.4.4]). The *regularity set* $\operatorname{Reg}_L(f)$ of a C^∞-function f is defined as follows: for $\Theta\in S^{n-1}$ and $b>0$ let

$$\Gamma_b(\Theta):=\{s\in\mathbb{R}^n\mid |s/|s|-\Theta|<b\}.$$

DEFINITION 2.1. Let $\Omega\subset\mathbb{R}^n$ be open, $(x,\Theta)\in\Omega\times S^{n-1}$ and $L\geq 1$. Let $f\in C^\infty(\Omega)$. We say that $\Omega\times\{\Theta\}\subset\operatorname{Reg}_L(f)$ if and only if for any $C\geq 1$ and any $(\varphi_k)\in A_{C,\Omega}$ there is $C_1\geq 1$ such that

$$|(f\varphi_k)^\wedge(s)|\leq C_1(L(C+1)k/(1+|s|))^k\quad \text{if } s\in\Gamma_{1/L}(\Theta).$$

In Langenbruch [30, Definition 1.1], $\operatorname{Reg}_L(f)$ was denoted by $\operatorname{reg}_{(L,L)}(f)$ for technical reasons.

(2.1) The constant C_1 in Definition 1.1 depends on (φ_k) only via the constants C_d from the definition of $A_{C,\Omega}$

(Langenbruch [30, (1.10)]). For the remaining part of this paper,

$$N\in S^{n-1}\quad\text{and}\quad \Theta\in V_{P_m},$$

i.e., Θ is a characteristic unit vector for P.

The *localization* $P_{m,\Theta}$ of P_m at Θ is defined as follows: let

$$q_\Theta:=\min\{k\in\mathbb{N}\mid \exists\beta\in\mathbb{N}^n:|\beta|=k\text{ and }D^\beta P_m(\Theta)\neq 0\}$$

be the order of the zero Θ for P_m. Now,

$$(2.2)\qquad P_{m,\Theta}(\xi):=\sum_{|\alpha|=q_\Theta}P_m^{(\alpha)}(\Theta)\xi^\alpha/\alpha!.$$

Alternatively,

$$(2.3)\qquad P_{m,\Theta}(x)=\lim_{s\to 0}(P_m(\Theta+sx)s^{-q_\Theta}),$$

where s^{q_Θ} is the lowest order term of the expansion of $P_m(\Theta+sx)$.

For $\Theta=e_1$ this means that

$$(2.4)\qquad P_m(x)=P_{m,\Theta}(x')x_1^{m-q_\Theta}+\sum_{k=0}^{m-q_\Theta-1}Q_k(x')x_1^k$$

if $x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1}$ where the Q_k are homogeneous polynomials and $Q_k=0$ or $\deg(Q_k)=m-k$. For a polynomial Q, $x\in\mathbb{C}^n$ and $t>0$ let

$$\widetilde{Q}(x,t):=\Big(\sum_a|Q^{(a)}(x)|^2t^{2|a|}\Big)^{1/2},$$

$$\widetilde{Q}_{\langle N\rangle}(x,t):=\Big(\sum_b|\langle D,N\rangle^bQ(x)|^2t^{2b}\Big)^{1/2},\quad N\in S^{n-1}.$$

The main result on the extension of $\operatorname{Reg}_L(f)$ is the following (Langenbruch [30, Theorem 3.4]):

THEOREM 2.2. (a) *Let $\Theta\in V_{P_m}$ and $P_{m,\Theta}(N)\neq 0$. There are $B\geq 1$ and open cones $K_1\subset K_2\subset\{x\in\mathbb{R}^n\mid\langle x,N\rangle>0\}$ such that $\overline{K}_2\cap\{x\in\mathbb{R}^n\mid \langle x,N\rangle\leq 0\}=\{0\}$ and such that for the truncated cones S_j and Σ_τ defined by*

$$S_1:=\{x\in K_2\mid t_1<\langle x,N\rangle<t_2\},\quad S_2:=\{x\in K_2\mid\langle x,N\rangle<t_2\},$$
$$\textit{and}\quad \Sigma_\tau:=\{x\in K_1\mid\tau<\langle x,N\rangle<(t_1+t_2)/2\}$$

the following holds: for any $0 < t_1 < t_2 \le 2t_1 \le 1$ there is $B_0 \ge 1$ such that for any $L \ge B$ and $0 < \tau \le t_1$, if $f \in \mathcal{C}^\infty(S_2)$, $S_1 \times \{\Theta\} \subset \mathrm{Reg}_L(f)$ and $S_2 \times \{\Theta\} \subset \mathrm{Reg}_L(P(D_x)f)$, then $\Sigma_\tau \times \{\Theta\} \subset \mathrm{Reg}_{h(\tau)L}(f)$ with $h(\tau) := B_0\tau^{-B}$.

(b) *If there is $C \ge 1$ such that for $(x,t) \in \mathbb{R}^{n+1}$,*

$$(2.5)\qquad (P_m)^\sim(x,t) \le C(P_m)^\sim_{\langle N\rangle}(x,t) \quad \text{if } t \in]0,1] \text{ and } |x-\Theta| \le 1/C,$$

then (a) *holds for any $\widehat{\Theta}$ with $|\Theta - \widehat{\Theta}| \le 1/(2C)$ with the cones K_j and the constants B and B_0 independent of $\widehat{\Theta}$.*

Theorem 2.2(a) will be used in Section 3 to show the hyperbolicity of $P_{m,\Theta}$ if $P(D)$ is surjective on $A(\Omega)$. The stronger assumption (2.5) is a local version of a condition of Hörmander ([20], see the remarks in Section 4). (2.5) will be used in Section 4 to deduce the local hyperbolicity of P_m at Θ.

We end the present section by mentioning some useful technical results: There is $B_2 \ge 1$ such that

$$(2.6)\qquad \Omega \times \{\Theta\} \subset \mathrm{Reg}_{B_2L_0}(v) \quad \text{for any } \Theta \in S^{n-1}$$

if $v \in \mathcal{C}^\infty(\Omega)$ satisfies the Cauchy estimates

$$|v^{(\alpha)}(x)| \le C(L_0|\alpha|)^{|\alpha|} \quad \text{on } \Omega.$$

There is $B_3 > 0$ such that for $K \subset\subset \Omega$ and $\delta := \mathrm{dist}(K, \partial\Omega)$,

$$(2.7)\qquad \text{there is } (\varphi_k) \in A_{B_3/\delta,\Omega} \text{ such that } \varphi_k = 1 \text{ near } K \text{ for each } k$$

(see e.g. Langenbruch [30, (1.5)]).

(2.7) easily implies the existence of suitable resolutions of the identity: there is $B_4 \ge 1$ such that the following holds: let Ω, V_j and W_j be open sets such that

$$V_j + U_\varepsilon \subset W_j \quad \text{and} \quad \Omega \subset \bigcup_{j\le d} V_j.$$

Then

$$(2.8)\qquad \text{there are } (\varphi_{k,j})_k \in A_{B_4/\varepsilon,W_j} \text{ such that } \sum_{j=1}^d \varphi_{k,j} = 1 \text{ on } \Omega.$$

Therefore, if for $f \in \mathcal{C}^\infty(\bigcup W_j)$ we have $W_j \times \{\Theta\} \subset \mathrm{Reg}_L(f)$ for $j \le d$, then

$$(2.9)\qquad \Omega \times \{\Theta\} \subset \mathrm{Reg}_{B_5L/\varepsilon}(f)$$

(use also the fact that $(g_kh_k) \in A_{C+D,U}$ if $(g_k) \in A_{C,U}$ and $(h_k) \in A_{D,U}$).

3. Localizations of surjective differential operators. We start the evaluation of the basic necessary condition from Theorem 1.3 using also the results on extension of the regularity set stated in Theorem 2.2(a). We will

show that the existence of regular elementary solutions as in Theorem 1.3 leads to local bounds of hyperbolic type on the location of zeros of P which imply that the localizations $P_{m,\Theta}$ of P_m are hyperbolic. As an illustration we finally study operators whose principal part is independent of some variable since the result is fairly complete for such operators.

We first need a simple lemma already used in Langenbruch [30]:

LEMMA 3.1. *Let $0 \in \Omega$ and let $E \in \widetilde{C}_\Delta(\Omega_T)$ be an elementary solution for $P(D)$ and let H be the distributional extension of $P(D)E$ as in Definition 1.1. For $u \in \widetilde{C}_\Delta(\mathbb{R}^{n+1})$ we have*

$$\begin{aligned} u(0,0) = & -2\int E(\xi,2T)P(D)(h\partial_y u)(-\xi,2T) \\ & + \partial_y E(\xi,2T)P(D)(hu)(-\xi,2T)\,d\xi \\ & + \int_{\Omega\times[-2T,2T]} H(\xi,\eta)\Delta(hu)(-\xi,-\eta)\,d\xi\,d\eta \end{aligned}$$

if $h \in D(-\Omega)$ and $h = 1$ near 0.

Proof. This is a special case of Langenbruch [30, Lemma 3.1] where u is even w.r.t. y (and where $(x,y) := 0$ in loc. cit.).

DEFINITION 3.2. For $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$ and $N \in S^{n-1}$ let

$$\widetilde{\Gamma}_\varepsilon(x_0,N) := x_0 + \{\xi \in \mathbb{R}^n \mid \langle \xi, N\rangle \ge \varepsilon|\xi|\}.$$

Let $N_i(\partial\Omega)$ be the set of unit interior normals of $\partial\Omega$, i.e. the set of all $N \in S^{n-1}$ such that there is $x_0 \in \partial\Omega$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$U_\delta(x_0) \cap \widetilde{\Gamma}_\varepsilon(x_0,N) \subset \Omega \cup \{x_0\}, \qquad N(\partial\Omega) := N_i(\partial\Omega) \cup (-N_i(\partial\Omega)).$$

Notice that $\partial\Omega$ need not be a $\mathcal{C}^1$-manifold near $x_0 \in \partial\Omega$ if there is an interior normal to $\partial\Omega$ at x_0.

THEOREM 3.3. *Let ν satisfy* (1.2) *and let*

$$(3.1)\qquad \nu(\xi) = o(\mathrm{dist}(\xi,\partial\Omega)^{4+B}) \quad \text{for } \xi \to \partial\Omega,\ \xi \in \Omega,$$

for B from Theorem 2.2. Assume that for any $g \in A_\nu(\Omega)$ the equation $P(D)f = g$ has a solution $f \in A(\Omega)$. Then for any $\Theta \in V_{P_m}$ and any $N \in N_i(\partial\Omega)$ with $P_{m,\Theta}(N) \ne 0$ there are $\nu_j, \mu_j > 0$ and $C(j) \ge 1$ with

$$(3.2)\qquad \lim \mu_j = 0 \quad \text{and} \quad \nu_j = o(\mu_j)$$

such that for any j, any $\xi \in \Gamma_{\mu_j}(\Theta)$ with $|\xi| \ge C(j)$ and any $z \in \mathbb{C}$,

$$(3.3)\qquad P(\xi + zN) \ne 0 \quad \text{if } |z| \le \mu_j|\xi| \text{ and } \mathrm{Im}\, z \ge \nu_j|\xi|.$$

Proof. Fix $N \in N_i(\partial\Omega)$ and let $x_0 \in \partial\Omega$ be chosen for N as in Definition 3.2. We can assume that $x_0 = 0$ and that $N = e_n$ and write $x \in \mathbb{C}^n$ as $(x', x_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Fix $\Theta \in V_{P_m}$ with $P_{m,\Theta}(N) \ne 0$.

i) We first use Theorem 2.2 to extend the regularity set of the regular elementary solutions coming from Theorem 1.3: Choose K_1 and K_2 for e_n by Theorem 2.2. We can assume that $K_j = \{x \in \mathbb{R}^n \mid x_n > |x'|/A_j\}$, $j = 1, 2$, for some $A_2 > A_1 > 0$. By Definition 3.2 there is a neighbourhood W of 0 such that $W \cap K_2 \subset \Omega$ and $\partial\Omega \cap W \cap \overline{K}_2 = \{0\}$. We can thus choose $0 < \tilde{t}_1 < \tilde{t}_2 \le 2\tilde{t}_1 \le 1$ and $\delta_0 > 0$ such that $\widetilde{K} := \{x \in \overline{K}_2 \mid \tilde{t}_1 \le x_n \le \tilde{t}_2\}$ satisfiesa

$$\omega := \widetilde{K} + U_{\delta_0} \subset\subset \Omega. \tag{3.4}$$

We now apply Theorem 1.3 to ω and $\xi = \xi_j := (0, 1/j)$ and get $\delta > 0$ and $\{\xi_j\}$-elementary solutions $E_j \in \widetilde{C}_\Delta(\Omega_{T_j})$, $T_j := 2B_1\nu(\xi_j)$, such that

(3.5) $(E_j)_+$ can be extended as a harmonic function to $\omega \times]T_j - \delta, \infty[$

if j is so large that $\xi_j \in \Omega$. We can assume that $\delta = \delta_0$. Define S_1 and S_2 for $N = e_n$ and for $t_1 := \tilde{t}_1$ and $t_2 := (\tilde{t}_1 + \tilde{t}_2)/2$ as in Theorem 2.2 and set $S_{k,x'} := (x', 3/(4j)) + S_k$, $k = 1, 2$. For x' with $|x'| = 3A_2/(4j)$ we have

$$S_{1,x'} \subset \widetilde{K} \quad \text{and} \quad \operatorname{dist}(S_{1,x'}, \partial\omega) \ge \delta \quad \text{for large } j$$

by (3.4). Thus, there is $L_0 \ge 1$ (independent of j) by (3.5), (1.1) and (2.6) such that for $|x'| = 3A_2/(4j) =: \gamma_j$,

$$S_{1,x'} \times \{\Theta\} \subset\subset \operatorname{Reg}_{L_0}(\partial_y^d E_j(\,\cdot\,, 2T_j)), \quad d = 0, 1. \tag{3.6}$$

There is $A_3 \ge 1$ such that $\operatorname{dist}(S_{2,x'}, \xi_j) \ge 1/(A_3 j)$. Since E_j is a $\{\xi_j\}$-elementary solution for $P(D)$ on Ω, $P(D)E_j$ can be extended to $H_j \in \widetilde{C}_\Delta((\Omega \times \mathbb{R}) \setminus \{(\xi_j, 0)\})$. Thus there is $A_4 \ge 1$ such that

$$S_{2,x'} \times \{\Theta\} \subset \operatorname{Reg}_{A_4 j}(P(D)\partial_y^d E_j(\cdot, 2T_j)), \quad d = 0, 1, \quad \text{for } |x'| = \gamma_j \tag{3.7}$$

by (1.1) and (2.6) again. We can assume that $L_0 \le A_4 j$. Since $0 < t_1 < t_2 \le 2t_1 \le 1$, we can apply Theorem 2.2 for $L = A_4 j$ and $\tau = 1/(8j)$ by (3.6) and (3.7) and get, with h from Theorem 2.2,

$$\Sigma_{1/(8j),x'} \times \{\Theta\} \subset \operatorname{Reg}_{L_j}(\partial_y^d E_j(\,\cdot\,, 2T_j)), \quad d = 0, 1, \quad \text{for } |x'| = \gamma_j \tag{3.8}$$

where $L_j := A_4 j h(1/(8j))$ and $\Sigma_{1/(8j),x'} := (x', 3/(4j)) + \Sigma_{1/(8j)}$. Let

$$M_j := \left\{ x \in \mathbb{R}^n \,\middle|\, \left(|x'| < \gamma_j + \frac{A_1}{16j},\ t_1 + \frac{1}{8}(t_2 - t_1) < x_n < t_1 + \frac{1}{4}(t_2 - t_1) \right) \right.$$
$$\left. \text{or } \left(\gamma_j - \frac{A_1}{16j} < |x'| < \gamma_j + \frac{A_1}{16j},\ \frac{15}{16j} < x_n < t_1 + \frac{1}{4}(t_2 - t_1) \right) \right\}$$

Then

$$M_j \subset \Sigma := \bigcup_{|x'| = \gamma_j} \Sigma_{1/(8j),x'} \cup \widetilde{K} \quad \text{for large } j$$

(by the definition of S_1 and Σ_τ in Theorem 2.2) and

$$\operatorname{dist}(M_j, \partial\Sigma) \ge A_1/(16j) \quad \text{for large } j.$$

Using the resolutions of the identity from (2.8) and (2.9) now implies that there is $A_5 \ge 1$ such that for large j,

$$M_j \times \{\Theta\} \subset \operatorname{Reg}_{\Lambda_j}(\partial_y^d E_j(\cdot, 2T_j)), \quad d = 0, 1, \quad \text{for } \Lambda_j := A_5 j^2 h(1/(8j)).$$

We now set $V_j := M_j - (0, 1/j)$ and shift E_j by $(0, -1/j)$. Then E_j is an elementary solution of $P(D)$ and for $\Lambda_j := A_5 j^2 h(1/(8j))$,

$$V_j \times \{\Theta\} \subset \operatorname{Reg}_{\Lambda_j}(\partial_y^d E_j(\cdot, 2T_j)), \quad d = 0, 1. \tag{3.9}$$

ii) We now apply (3.9) and Lemma 3.1 to special harmonic functions u_ζ which are defined to substitute the Fourier transformation in $\mathbb{R}^n$: for $\zeta \in \mathbb{C}^n$ let $\langle\zeta\rangle$ be a square root of $\sum_k \zeta_k^2$ and set $u_\zeta(x, y) := \cosh(\langle\zeta\rangle y)e^{i\langle x,\zeta\rangle}$. Then $u_\zeta \in \widetilde{C}_\Delta(\mathbb{R}^{n+1})$ and

$$u_\zeta(0, 0) = 1 \quad \text{and} \quad Q(D_x)\partial_y^d u_\zeta(x, y) = Q(\zeta)\partial_y^d u_\zeta(x, y), \quad d = 0, 1, \tag{3.10}$$

for any polynomial Q in n (x-) variables if $D_x = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. For $W_j := -\operatorname{conv}(V_j)$ we can choose $(\varphi_k) \in A_{A_6 j, W_j}$ by (2.7) such that $\varphi_k = 1$ near 0 and

$$\begin{aligned} &\operatorname{supp}(\operatorname{grad}(\varphi_k)) \\ &\quad \subset -V_j \cup \{x \in \mathbb{R}^n \mid |x'| < \gamma_j + A_1/(16j),\ 1/(32j) < x_n < 1/(16j)\}. \end{aligned} \tag{3.11}$$

We then have for $\zeta \in \mathbb{C}^n$ with $P(\zeta) = 0$ by (3.10), Lemma 3.1 and Leibniz' rule

$$\begin{aligned}
1 &= u_\zeta(0, 0) \\
&= -2\int \{E_j(\xi, 2T_j)P(D)(\varphi_k \partial_y u_\zeta)(-\xi, 2T_j) \\
&\quad + \partial_y E_j(\xi, 2T_j)P(D)(\varphi_k u_\zeta)(-\xi, 2T_j)\}\, d\xi \\
&\quad + \int_{\Omega \times [-2T_j, 2T_j]} H_j(\xi, \eta)\Delta(\varphi_k u_\zeta)(-\xi, -\eta)\, d\xi\, d\eta \\
&= -2\sum_{a \ne 0} P^{(a)}(\zeta)\int \{E_j(\xi, 2T_j)(D^a\varphi_k \partial_y u_\zeta)(-\xi, 2T_j) \\
&\quad + \partial_y E_j(\xi, 2T_j)(D^a\varphi_k u_\zeta)(-\xi, 2T_j)\}\, d\xi/a! \\
&\quad + \int_{\Omega \times [-2T_j, 2T_j]} H_j(\xi, \eta)(u_\zeta \Delta_x \varphi_k + 2\langle \operatorname{grad}_x \varphi_k, i\zeta\rangle u_\zeta)(-\xi, -\eta)\, d\xi\, d\eta \\
&= -2\sum_{a \ne 0} P^{(a)}(\zeta)\langle\zeta\rangle \sinh(\langle\zeta\rangle 2T_j)(E_j(\ , 2T_j)(D^a\varphi_k)^\vee)^\wedge(\zeta)/a! \\
&\quad - 2\sum_{a \ne 0} P^{(a)}(\zeta)\cosh(\langle\zeta\rangle 2T_j)(\partial_y E_j(\ , 2T_j)(D^a\varphi_k)^\vee)^\wedge(\zeta)/a!
\end{aligned} \tag{3.12}$$

$$+\int_{[-2T_j,2T_j]}(H_j(\,,\eta)((\Delta_x\varphi_k)^\vee+2\langle(\operatorname{grad}\varphi_k)^\vee,i\zeta\rangle)\hat{\ }(\zeta)\cosh(\langle\zeta\rangle\eta)\,d\eta$$
$$\le C_1(1+|\zeta|)^m\exp(2T_j|\zeta|)$$
$$\times\Big[\sup_{d\le 1,0<|a|\le m}\{|(\partial_y^dE_j(\,,2T_j)(D^a\varphi_k)^\vee e^{\langle\operatorname{Im}\zeta,\,\rangle}h_k)\hat{\ }(\operatorname{Re}\zeta)|$$
$$+|(\partial_y^dE_j(\,,2T_j)(D^a\varphi_k)^\vee(1-h_k))\hat{\ }(\zeta)|\}$$
$$+\sup\{|(H_j(\,,\eta)(D^b\varphi_k)^\vee)\hat{\ }(\zeta)|\mid 0\le\eta\le 2T_j,\ 0<|b|\le 2\}\Big]$$

where $h_k=h_k(x_n)\in A_{A_7j,\mathbb{R}}$ is chosen by (2.7) such that

$$h_k(x_n)=1 \text{ for } 1\ge x_n\ge -1/(64j) \text{ and } x_n>-1/(32j) \text{ on } \operatorname{supp}h_k. \tag{3.13}$$

For large j we have $((D^a\varphi_k)^\vee e^{\langle\operatorname{Im}\zeta,\,\rangle}h_ke^{-|\operatorname{Im}\zeta|})\in A_{A_8j,-W_j}$ by (3.13) and (3.11) with uniform constants C_d for $z\in\mathbb{C}^n$ since $t_2\le 1$. We therefore have, by (2.1) and (3.9),

$$\sup_{d\le1,\ 0<|a|\le m}|(\partial_y^dE_j(\,,2T_j)(D^a\varphi_k)^\vee e^{\langle\operatorname{Im}\zeta,\,\rangle}h_k)\hat{\ }(\operatorname{Re}\zeta)| \le C_2(\lambda_jk/(1+|\zeta|))^k\exp(|\operatorname{Im}\zeta|) \tag{3.14}$$

if $\operatorname{Re}\zeta\in\Gamma_{1/\lambda_j}(\Theta)$ and $|\operatorname{Im}\zeta|\le|\operatorname{Re}\zeta|$ where $\lambda_j:=A_9j^3h(1/(8j))$.

For the second term in (3.12) we have by (3.13) the trivial Paley–Wiener estimate

$$\sup_{d\le1,\ 0<|a|\le m}|(\partial_y^dE_j(\,,2T_j)(D^a\varphi_k)^\vee(1-h_k))\hat{\ }(\zeta)| \le C_3\exp(-\operatorname{Im}\zeta_n/(64j)+A_2|\operatorname{Im}\zeta'|/j)\quad\text{if }\operatorname{Im}\zeta_n\ge0. \tag{3.15}$$

Since $H\in C_\Delta(\mathbb{R}^{n+1}\setminus\{0\})$ and since $\operatorname{dist}(\operatorname{supp}(\operatorname{grad}_x(\varphi_k)),0)\ge 1/(32j)$ by (3.11) we can estimate the last term in (3.12) using (1.1):

$$\sup\{|(H_j(\,,\eta)(D^b\varphi_k)^\vee)\hat{\ }(\zeta)|\mid 0\le\eta\le 2T_j,\ 0<|b|\le2\} \le C_4\exp(|\operatorname{Im}\zeta|)(A_{10}kj/(1+|\zeta|))^k \tag{3.16}$$

since $A_2/j+t_2\le1$ for large j. Since $\lambda_j\ge A_{10}j$ for large j we get by (3.12) and (3.14)–(3.16)

$$2e\le\exp(3T_j|\zeta|)\left[\exp(|\operatorname{Im}\zeta|)\left(\frac{\lambda_jk}{1+|\zeta|}\right)^k+\exp\left(\frac{A_2}{j}|\operatorname{Im}\zeta'|-\frac{\operatorname{Im}\zeta_n}{64j}\right)\right]$$

for large j if $P(\zeta)=0$ and if

$$|\zeta|\ge C_5(j),\quad \operatorname{Im}\zeta_n\ge0,\quad |\operatorname{Im}\zeta|\le|\operatorname{Re}\zeta|\quad\text{and}\quad\operatorname{Re}\zeta\in\Gamma_{1/\lambda_j}(\Theta). \tag{3.17}$$

We now set $k:=[(1+|\zeta|)/(e\lambda_j)]$ and get for large j and ζ as in (3.17),

$$2e\le\exp\left(1+|\operatorname{Im}\zeta|-\frac{|\zeta|}{2e\lambda_j}\right)+\exp\left(\frac{A_2}{j}|\operatorname{Im}\zeta'|-\frac{\operatorname{Im}\zeta_n}{64j}+3T_j|\zeta|\right) \tag{3.18}$$

if $P(\zeta)=0$, since by (3.1),

$$T_j\le jT_j=o(j\operatorname{dist}(\xi_j,\partial\Omega)^{4+B}) = o(j^{-3-B},o(1/(A_9j^3h(1/(8j)))))=o(1/\lambda_j). \tag{3.19}$$

iii) Let $\mu_j:=1/(8e\lambda_j)$ and $\nu_j:=386jT_j$. Then $\mu_j\to0$ and $\nu_j=o(\mu_j)$ by (3.19). Let $\zeta=\xi+zN=\xi+ze_n$ with $\xi\in\Gamma_{\mu_j}(\Theta)$, $|\xi|\ge C(j):=2C_5(j)$ and $z\in\mathbb{C}$ with $|z|\le\mu_j|\xi|$ and $\operatorname{Im}z\ge\nu_j|\xi|$. Then ξ satisfies (3.17). Indeed, if $\mu_j\le1/2$, then

$$|\zeta|\ge|\xi|/2\ge C_5(j),$$
$$|\operatorname{Im}\zeta|=|\operatorname{Im}z|\le|z|\le|\xi|/2\le|\xi+\operatorname{Re}ze_n|=|\operatorname{Re}\eta|.$$

Further, $\operatorname{Re}\zeta=\xi+\operatorname{Re}ze_n\in\overline{\Gamma_{4\mu_j}(\Theta)}\subset\Gamma_{1/\lambda_j}(\Theta)$ since $\xi\in\Gamma_{\mu_j}(\Theta)$ and $|\operatorname{Re}z|\le\mu_j|\xi|$ (e.g. by Langenbruch [30, (1.10)]. Next, ζ does not satisfy (3.18) since

$$|\operatorname{Im}\zeta|=|\operatorname{Im}z|\le\mu_j|\xi|<|\zeta|/(2e\lambda_j)$$

and since $\operatorname{Im}\zeta'=0$ and

$$\operatorname{Im}\zeta_n/(64j)=\operatorname{Im}z/(64j)\ge\nu_j|\xi|/(64j)>6T_j|\xi|\ge3T_j|\zeta|.$$

Therefore $P(\zeta)\ne0$ and the theorem is proved (for $N=e_n$).

From Theorem 3.3 we can now easily deduce the main result of this section:

THEOREM 3.4. *Let ν satisfy* (1.2) *and let*

$$\nu(\xi)=o(\operatorname{dist}(\xi,\partial\Omega)^{4+B})\quad\textit{for }\xi\to\partial\Omega,\ \xi\in\Omega, \tag{3.1}$$

with B from Theorem 2.2. Assume that for any $g\in A_\nu(\Omega)$ the equation $P(D)f=g$ has a solution $f\in A(\Omega)$. Then $P_{m,\Theta}$ is hyperbolic w.r.t. $N\in N(\partial\Omega)$ if $\Theta\in V_{P_m}$ and if N is noncharacteristic for $P_{m,\Theta}$.

Proof. We can assume that $N=e_n\in N_i(\partial\Omega)$ and then write $x\in\mathbb{C}^n$ as $x=(x',x_n)\in\mathbb{C}^{n-1}\times\mathbb{C}$.

a) With ν_j,μ_j and $C(j)$ from Theorem 3.3 we have: there are $A_1\ge1$ and $J\ge1$ such that for $j\ge J$ and any $\xi\in\Gamma_{\mu_j}(\Theta)$ with $|\xi|\ge C(j)$,

$$\widetilde{P}_{\langle e_n\rangle}(\xi,\tau)\le A_1|P(\xi+i\tau e_n)|\quad\text{for }2\nu_j|\xi|\le\tau\le4\nu_j|\xi|.$$

Indeed, let $d_Q(x)$ be the distance from $x\in\mathbb{R}^n$ to the complex roots of the polynomial Q. By Hörmander [22, Lemma 11.1.4] there is $C\ge1$ such that for any $x\in\mathbb{R}^n$ with $Q(x)\ne0$,

$$1/C\le A_Q(x)d_Q(x):=\Big(\sum_{\alpha\ne0}|Q^{(\alpha)}(x)/Q(x)|^{1/|\alpha|}\Big)d_Q(x)\le C. \tag{3.20}$$

For ξ and τ as above let $z:=\eta+i\tau$ with $\eta\in\mathbb{C}$ and $|\eta|\le\nu_j|\xi|$. Then

$$\operatorname{Im}z\ge\nu_j|\xi|\quad\text{and}\quad|z|\le5\nu_j|\xi|\le\mu_j|\xi|\quad\text{for large }j$$

since $\nu_j = o(\mu_j)$. We thus have $P(\xi + i\tau e_n + \eta e_n) \neq 0$ for $|\eta| \le \nu_j|\xi|$ and large j by (3.3) and therefore $d_Q(0) \ge \nu_j|\xi|/A$ for $Q(\eta) := P(\xi+i\tau e_n+\eta e_n)$. By (3.20) (for $n = 1$) and since $\tau \le 4\nu_j|\xi|$, for large j this implies

$$\widetilde{P}_{\langle e_n\rangle}(\xi + i\tau e_n, \tau) \le A_1|P(\xi + i\tau e_n)|.$$

Now a) follows by Taylor expansion since

$$\widetilde{P}_{\langle e_n\rangle}(\xi,\tau) \le A_2\widetilde{P}_{\langle e_n\rangle}(\xi + i\tau e_n, 2\tau) \le 2^m A_2\widetilde{P}_{\langle e_n\rangle}(\xi + i\tau e_n, \tau).$$

b) For $\xi \in \mathbb{R}^n$, $\tau > 0$ and $\zeta \in \mathbb{C}^n$ we have

$$(3.21)\qquad \widetilde{P}_{\langle e_n\rangle}(\lambda\xi,\lambda\tau)\lambda^{-m} \to (P_m)^{\sim}_{\langle e_n\rangle}(\xi,\tau) \quad\text{and}\quad P(\lambda\zeta)\lambda^{-m} \to P_m(\zeta) \quad \text{if } \lambda \to \infty.$$

Let $j \ge J$ and $x \in \mathbb{R}^n$ with $|x| < \mu_j/4$. Then $\xi := \lambda(\Theta + x) \in \Gamma_{\mu_j}(\Theta)$ for $\lambda > 0$ since

$$|\xi/|\xi| - \Theta| \le (||x+\Theta| - |\Theta|| + |x|)/|x+\Theta| < \mu_j.$$

If $\mu_j \le 1$, we also have

$$2\nu_j|\xi| \le \tau := 3\nu_j\lambda \le 4\nu_j|\xi|.$$

Thus a) and (3.21) imply

$$(3.22)\qquad A_0\nu_j^{q_\Theta} \le (P_m)^{\sim}_{\langle e_n\rangle}(\Theta + x, 3\nu_j) \le A_1|P_m(\Theta + x + 3i\nu_j e_n)|$$

if $|x| < \mu_j/4$ and $j \ge J$ since e_n is noncharacteristic for $P_{m,\Theta}$.

c) For $\xi \in \mathbb{R}^n$ let $x := \nu_j\xi$. Then $|x| < \mu_j/4$ for large j since $\nu_j = o(\mu_j)$ and (3.22) implies by (2.3) that

$$0 < A_0/A_1 \le |P_m(\Theta + (\xi + 3ie_n)\nu_j)|/\nu_j^{q_\Theta} \to |P_{m,\Theta}(\xi + 3ie_n)|$$

since $\nu_j \to 0$. This implies that $P_{m,\Theta}$ is hyperbolic w.r.t. e_n since $P_{m,\Theta}$ is homogeneous and e_n is noncharacteristic for $P_{m,\Theta}$.

REMARK 3.5. The conclusion of Theorem 3.4 holds in particular if $P(D)$ is surjective on $A(\Omega)$. We just define

$$(3.23)\qquad \nu(\xi) := \min(\operatorname{dist}(\xi, \partial\Omega)^{5+B}, \gamma)$$

for sufficiently small $\gamma > 0$ and B from Theorem 2.2. However, the statement of Theorem 3.4 is stronger: if the conclusion of Theorem 3.4 fails for P, then Theorem 3.4 provides functions g with Cauchy radii given by the polynomial bound (3.23) near the boundary such that the equation $P(D)f = g$ cannot be solved in $A(\Omega)$. The existence of this polynomial bound is due to the quantitative results on extension of analyticity from Langenbruch [30]. The same remark also holds for any of the necessary conditions for surjectivity of $P(D)$ in $A(\Omega)$ which are proved in this paper.

REMARK 3.6. The problems with solving the equation $P(D)f = g$ already arise for simple rational functions: assume that for some $\Theta \in V_{P_m}$,

$P_{m,\Theta}$ is not hyperbolic w.r.t. some $N \in N(\partial\Omega)$ which is noncharacteristic for $P_{m,\Theta}$. Let ν be defined by (3.23). Then there is an open set $\omega \subset\subset \Omega$ such that for any $k \in \mathbb{N}$ the equations

$$(3.24)\qquad P(D)f_j = |(\cdot - \xi_j, T_j)|^{1-n}, \quad \xi_j := x_0 - N/j \quad\text{and}\quad T_j := 2B_1\nu(\xi_j)$$

cannot be solved with $f_j \in \mathcal{C}^\infty(\Omega)$ such that

$$(3.25)\qquad \sup\{|f_j^{(a)}(x)|/(k^{|a|}|a|!) \mid x \in \omega,\ a \in \mathbb{N}_0^n\} < \infty \quad \text{for large } j.$$

Indeed, to show the existence of regular $\{\xi_j\}$-elementary solutions E_{ξ_j} in the proof of Theorem 1.3 we only needed the fact that for any $\omega \subset\subset \Omega$ there is $k \in \mathbb{N}$ such that (3.24) can be solved with f_j satisfying (3.25). The sequence ξ_j was used in the proof of Theorem 3.3 to show (3.3). (3.3) implies the conclusion of Theorem 3.4.

In the remaining part of this section we will prove several consequences of Theorem 3.4. The conclusion is particularly strong if there are many normal vectors:

COROLLARY 3.7. *Let $\Theta \in V_{P_m}$ and let*

$$(3.26)\qquad N(\partial\Omega) \cap V \neq \emptyset \text{ for any component } V \text{ of } S^{n-1} \setminus V_{P_{m,\Theta}}.$$

If $P(D)$ is surjective in $A(\Omega)$, then there are $\xi_{j,\Theta} \in \mathbb{R}^n$ and $c_\Theta \in \mathbb{C}$ such that

$$(3.27)\qquad P_{m,\Theta}(x) = c_\Theta \prod_{j=1}^{q_\Theta} \langle x, \xi_{j,\Theta}\rangle,$$

i.e. $P_{m,\Theta}$ is a product of real linear forms (times a complex constant).

Proof. By Theorem 3.4, $P_{m,\Theta}$ is hyperbolic w.r.t. any $N \in N(\partial\Omega)$. By (3.26) and Hörmander [22, Corollary 12.4.5] this implies that $P_{m,\Theta}$ is hyperbolic w.r.t. any $N \in S^{n-1}$ with $P_{m,\Theta}(N) \neq 0$. The claim now follows from de Cristoforis [13, Theorem 1].

REMARK 3.8. (a) (3.26) is clearly satisfied for any $\Theta \in V_{P_m}$ if Ω is bounded with $\mathcal{C}^1$-boundary since then $N(\partial\Omega) = S^{n-1}$.

(b) An example of a different kind is the following: let $\Omega := \{(x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} \mid x_n > |x'|^2\}$. Then Ω is unbounded and $N(\partial\Omega) = \{x \in \mathbb{R}^n \mid x_n \neq 0\}$ also satisfies (3.26) for any $\Theta \in V_{P_m}$.

(c) The statement of Corollary 3.7 is particularly strong if there is $0 \neq M \in \mathbb{R}^n$ such that

$$(3.28)\qquad V_{P_{m,\Theta}} \subset H_M := \{x \in \mathbb{R}^n \mid \langle x, M\rangle = 0\}.$$

If $P(D)$ is surjective on $A(\Omega)$ and if $N(\partial\Omega) \not\subset H_M$, then $(P_{m,\Theta})(x) = c\langle x, M\rangle^{q_\Theta}$ for some $c \neq 0$. Indeed, (3.26) is satisfied by (3.28) since

$N(\partial\Omega) \not\subset H_M$. Thus $P_{m,\Theta}$ has the form (3.27) by Corollary 3.7 with $\bigcup_{j\le q_\Theta} H_{\xi_j,\Theta} \cap S^{n-1} = V_{P_{m,\Theta}} \subset H_M$.

Corollary 3.9. *If $P_{m,\Theta}$ is not of the form (3.27) for some $\Theta \in V_{P_m}$, then there is a nontrivial open halfspace*

$$\Omega_N := \{x \in \mathbb{R}^n \mid \langle x, N\rangle > 0\}$$

such that $P(D)$ is not surjective on $A(\Omega_N)$.

Proof. By de Cristophoris [13, Theorem 1] there is a noncharacteristic vector N for $P_{m,\Theta}$ such that $P_{m,\Theta}$ is not hyperbolic w.r.t. N. Then $P(D)$ is not surjective on $A(\Omega_N)$ by Theorem 3.4.

The Laplace equation in two variables and the heat equation were the first examples of operators not surjective on $A(\mathbb{R}^3)$ (Piccinini [37]). In both cases P_m is independent of some variable, that is,

(3.29) $\quad \Lambda(P_m) := \{\Theta \in \mathbb{R}^n \mid P_m(x+t\Theta) = P_m(x) \text{ if } (x,t) \in \mathbb{R}^{n+1}\} \neq \{0\}.$

$\Theta \in \Lambda(P_m)$ is a root of order m for P_m (i.e. of maximal order) and $P_{m,\Theta} = P_m$. The operators satisfying (3.29) are now studied in detail:

Corollary 3.10. *Let $\Lambda(P_m) \neq \{0\}$. If $P(D)$ is surjective on $A(\Omega)$, then P_m is hyperbolic w.r.t. $N \in N(\partial\Omega)$ if N is noncharacteristic for P.*

Proof. (3.29) implies that $P_m = P_{m,\Theta}$ for some $\Theta \in V_{P_m}$. The claim now follows from Theorem 3.4.

We mention some simple examples of operators of second order:

i) If $P(x) := \sum_{j=1}^{n-1} x_j^2 + x_n$ (the Schrödinger operator) or $P(x) := \sum_{j=1}^{n-1} x_j^2 + ix_n$ (the heat operator) or $P(x) := \sum_{j=1}^{n-1} x_j^2$ (the Laplace operator in $n-1$ variables), then $P(D)$ is not surjective on $A(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is open and if there is $e_n \neq N \in N(\partial\Omega)$.

ii) If $P(x) = x_k^2 - \sum_{j=1}^{k-1} x_j^2$ with $3 \le k < n$, then $P(D)$ is not surjective on $A(\Omega)$ if there is $N \in N(\partial\Omega)$ with $P(N) < 0$ (P is not hyperbolic w.r.t. those N).

iii) If $P(x) = \sum_{j=1}^{d} x_j^2 - \sum_{j=d+1}^{k} x_j^2$ with $2 \le d < k-1$ and $k < n$, then $P(D)$ is not surjective on $A(\Omega)$ if $N(\partial\Omega)$ contains a noncharacteristic vector (P is not hyperbolic w.r.t. any vector).

Notice that the operators in ii) and iii) are surjective on $A(\mathbb{R}^n)$ by Hörmander [21, Theorem 6.6]. Hörmander [21, Theorem 6.7] has also shown that these operators are not surjective on $A(\Omega)$ for bounded convex Ω. If we combine Corollary 3.10 with the sufficient criteria known from the literature we can often characterize surjectivity. We first consider halfspaces:

Corollary 3.11. *Let $\Lambda(P_m) \neq \{0\}$ and let N be noncharacteristic for P. The following are equivalent:*

(a) *$P(D)$ is surjective on $A(\Omega)$ for some Ω with $N \in N(\partial\Omega)$.*

(b) *$P(D)$ is surjective on $A(\Omega)$ for $\Omega := \{x \in \mathbb{R}^n \mid C_1 > \langle x, N\rangle > C_2\}$, $-\infty \le C_2 < C_1 \le \infty$.*

(c) *P_m is hyperbolic w.r.t. N.*

Proof. (a)⇒(c). This follows by Corollary 3.10.

(c)⇒(b). Since P_m is hyperbolic w.r.t. N by assumption this also holds for $P_{m,\Theta}$ for any $\Theta \in V_{P_m}$. Therefore Ω satisfies the assumption of Zampieri [40, Main Theorem] (for any $\Theta \in V_{P_m}$) and (b) follows from that result.

(b)⇒(a). This is trivial.

The surjectivity problem for $P(D)$ in $A(\Omega)$ for convex Ω can be reduced to the consideration of tangent halfspaces (and their complements) defined by the inductive procedure from Andreotti–Nacinovich [3, Section 16]. In fact, we have the following results:

(3.30) $P(D)$ is surjective in $A(\Omega)$ for convex Ω if $P(D)$ is surjective on $A(\Sigma)$ and on $A(\mathbb{R}^n \setminus \overline{\Sigma})$ for any tangent halfspace Σ

(Zampieri [39, Lemma 2.2], see also Andreotti–Nacinovich [3, Section 16] for bounded convex Ω).

(3.31) If $P(D)$ is surjective in $A(\Omega)$ for convex Ω, then $P(D)$ is also surjective in $A(\Sigma)$ for any tangent halfspace Σ of Ω

(Andreotti–Nacinovich [3, Section 16]).

When extending Corollary 3.11 to convex sets Ω, we can also allow that $\partial\Omega$ has characteristic normals. The relevant notion is the following:

Definition 3.12. Let Ω be convex.

(a) The *generalized normals* $N_g(\partial\Omega)$ are defined by

$$N_g(\partial\Omega) := \{\pm N \mid \Omega \text{ has a tangent halfspace } \Sigma = x_0 + \Omega_N\}.$$

(b) Let $N(P_m, \partial\Omega)$ be the union of the closed convex hulls $\overline{\mathrm{conv}(V_j)}$ of the components V_j of $S^{n-1} \setminus V_{P_m}$ containing a vector $N \in N_g(\partial\Omega)$. Then Ω is called *P-admissible* if $N_g(\partial\Omega) \subset N(P_m, \partial\Omega)$.

The boundary of P-admissible convex sets admits sufficiently many noncharacteristic generalized normals.

Corollary 3.13. *Let $\Lambda(P_m) \neq \{0\}$ and let Ω be convex and P-admissible. The following are equivalent:*

(a) *$P(D)$ is surjective on $A(\Omega)$.*
(b) *$P(D)$ is surjective on $A(\Omega_M)$ for any $M \in N(P_m, \partial\Omega)$.*
(c) *P_m is hyperbolic w.r.t. $N \in N_g(\partial\Omega)$ if N is noncharacteristic for P.*

Proof. (a)⇒(c). Let $N \in N_g(\partial\Omega)$ and let $\Sigma = x_0 + \Omega_N$ be a tangent halfspace of Ω. Then $P(D)$ is surjective on $A(\Sigma)$ by (3.31) and (c) follows for N by Corollary 3.11 applied to Ω_N.

(c)⇒(b). For $N \in N_g(\partial\Omega)$ let V be the component of N in $S^{n-1} \setminus V_{P_m}$. Since P_m is hyperbolic w.r.t. N by assumption, V is convex and P_m is hyperbolic w.r.t. any $M \in V$. The same is then true for $P_{m,\Theta}$ for any $\Theta \in V_{P_m}$. Let V^0 be the dual cone of V. Since

$$-V^0 \subset \mathbb{R}^n \setminus \Omega_M \quad \text{for any } M \in \overline{V},$$

the halfspaces Ω_M with $M \in \overline{V}$ satisfy the assumption of Zampieri [40, Main Theorem] (for any $\Theta \in V_{P_m}$) and (b) follows from that result.

(b)⇒(a). Since $-N(P_m, \partial\Omega) = N(P_m, \partial\Omega)$, (b) means that $P(D)$ is surjective on $A(\Sigma)$ and on $A(\mathbb{R}^n \setminus \overline{\Sigma})$ for any tangent halfspace Σ of Ω since Ω is P-admissible. This implies (a) by (3.30).

The boundary of P-admissible convex sets may have large characteristic parts: let $n = 3$ and let $P(x) := x_1^2 - x_2^2$. Then the boundary of $\Omega := \{x \in \mathbb{R}^3 \mid x_1 > 0, x_2 > x_1\}$ consists of $\{0\} \times [0, \infty[\times \mathbb{R}$ and the characteristic part $\{x \in \mathbb{R}^3 \mid x_1 = x_2 > 0\}$. Hence Ω is P-admissible.

For bounded open sets we get the following characterization:

COROLLARY 3.14. *Let $\Lambda(P_m) \neq \{0\}$. The following are equivalent*:

(a) *$P(D)$ is surjective on $A(\Omega)$ for some bounded open set Ω with C^1-boundary.*
(b) *$P(D)$ is surjective on $A(\Omega)$ if Ω is convex.*
(c) *$P(D)$ is surjective on $A(\Sigma)$ for any halfspace Σ.*
(d) *P_m is the product of real linear forms (times a complex constant).*

Proof. (a)⇒(d). This follows from Corollary 3.10 and de Cristoforis [13, Theorem 1].

(d)⇒(c). $P_m(D)$ is surjective on $A(\Sigma)$ by repeated integration. Thus, $P(D)$ is surjective on these spaces by Hörmander [21].

(c)⇒(b). This holds by (3.30).

REMARK 3.15. The surjectivity of $P(D)$ in $A(\Omega)$ is very sensitive to small perturbations of the boundary: let $\Lambda(P_m) \neq \{0\}$ and let N be noncharacteristic for P. Let $\widetilde{\Omega} := \Omega_N \cup U_\varepsilon(0)$. Then $P(D)$ is surjective on $A(\Omega_N)$ if and only if P_m is hyperbolic w.r.t. N (by Corollary 3.11) while P_m is a product of real linear forms (times a complex constant) if $P(D)$ is surjective on $A(\widetilde{\Omega})$ (by Corollary 3.7 since $N(\partial\widetilde{\Omega}) = \{x \in \mathbb{R}^n \mid \langle N, x\rangle \neq 0\}$ and $P_{m,\Theta} = P_m$ for $\Theta \in \Lambda(P_m)$).

4. Local hyperbolicity. In this section we consider a situation where any surjective partial differential operator on $A(\Omega)$ must have a locally hyperbolic principal part. Local hyperbolicity was introduced by Andersson [1] and further studied by Gårding [15]. The notion is defined as follows:

DEFINITION 4.1. Let $\Theta \in V_{P_m}$.

(a) P_m is called *locally hyperbolic with respect to N at Θ* if there is $\delta > 0$ such that for $(x, z) \in \mathbb{R}^n \times \mathbb{C}$,

$$P_m(\Theta + x + zN) \neq 0 \quad \text{if } |(x,z)| < \delta \text{ and } \operatorname{Im} z \neq 0. \tag{4.1}$$

(b) P_m is called *locally hyperbolic* if for any $\Theta \in V_{P_m}$ there is $N \in \mathbb{R}^n$ such that P_m is locally hyperbolic w.r.t. N at Θ.

REMARK 4.2. Let P_m be locally hyperbolic w.r.t. N at Θ. The following are well known:

(a) $P_{m,\Theta}$ is hyperbolic w.r.t. N (e.g. Hörmander [22, Lemma 8.7.2]), but the converse is not true (Kaneko [23, Example 3.1]).

(b) P_m is locally hyperbolic w.r.t. any vector in the component of N in $\{x \in \mathbb{R}^n \mid P_{m,\Theta}(x) \neq 0\}$. This component is a convex open cone and the dual cone K_Θ is called the *local propagation cone* for P_m at Θ.

Polynomials P_m such that

$$P_m \text{ is locally hyperbolic w.r.t. } N \text{ at any } \Theta \in V_{P_m} \tag{4.2}$$

were studied by Fehrman [14]. In fact, he called P_m *hyperbolic-elliptic* w.r.t. N if there is $C > 0$ such that

$$P_m(x + itN) \neq 0 \quad \text{if } (x,t) \in \mathbb{R}^{n+1} \text{ and } 0 < |t| \leq C|x|. \tag{4.3}$$

It is easy to see that (4.2) and (4.3) are equivalent. Indeed, (4.2) is clearly necessary for (4.3), and (4.3) follows from (4.2) for any x in a neighbourhood $W \subset S^{n-1}$ of V_{P_m}. Since (4.3) trivially holds for $x \in S^{n-1} \setminus W$, (4.3) holds for $x \in S^{n-1}$ and thus for $x \in \mathbb{R}^n$ by homogeneity.

In Section 3 we generally assumed that $N \in N(\partial\Omega)$ is noncharacteristic for the localization $P_{m,\Theta}$ to deduce the hyperbolicity of $P_{m,\Theta}$. To obtain local hyperbolicity at Θ for P_m we need a stronger version of noncharacteristicity which is based on the following estimate introduced by Hörmander [20] when he studied the extension of C^∞-regularity for solutions of $P_m(D)$: assume that there is $C_0 \geq 1$ such that for any $t \geq 1$ there is $C(t) \geq 1$ such that for $(x,t) \in \mathbb{R}^{n+1}$,

$$(P_m)^\sim(x,t) \leq C_0 (P_m)^\sim_{\langle N\rangle}(x,t) \quad \text{if } |x| \geq C(t) \text{ and } t \geq 1. \tag{4.4}$$

(4.4) can be characterized by means of the localizations $Q \in L(P_m)$ of P_m at ∞ (see (4.6) below). $L(P_m)$ is defined as follows (Hörmander

[22, Definition 10.2.6]): let

$$\widetilde{P}_m(x) := \widetilde{P}_m(x,1) = \Big(\sum |P_m^{(a)}(x)|^2\Big)^{1/2}.$$

For $\Theta \in V_{P_m}$ let

$$(4.5)\quad L_\Theta(P_m) := \{Q \mid \exists x_k \in \mathbb{R}^n,\ x_k \to \infty,\ x_k/|x_k| \to \Theta,\ Q = \lim Q_k \text{ for } Q_k(x) := P_m(x_k+x)/\widetilde{P}_m(x_k)\}$$

$$L(P_m) := \bigcup_{\Theta\in V_{P_m}} L_\Theta(P_m).$$

(4.4) holds if and only if

$$(4.6)\qquad N \text{ is noncharacteristic for any } Q \in L(P_m).$$

In fact, (4.4) means that

$$\sigma_{P_m}(\langle N\rangle) := \inf_{t>1}\liminf_{\xi\to\infty}(P_m)^\sim_{\langle N\rangle}(\xi,t)/\widetilde{P}_m(\xi,t) > 0$$

in the notation of Hörmander [20]; the necessity of (4.6) follows from [20, Theorem 6.3] while the sufficiency follows from [20, Lemma 6.2].

By the homogeneity of P_m, (4.4) holds if and only if there is $C \ge 1$ such that

$$(P_m)^\sim(x,t) \le C(P_m)^\sim_{\langle N\rangle}(x,t) \quad \text{if } (x,t)\in S^{n-1}\times]0,1/C].$$

The following condition (4.7) is therefore a local version of (4.4): let $\Theta \in V_{P_m}$. Assume that there is $C \ge 1$ such that

$$(4.7)\quad (P_m)^\sim(x,t) \le C(P_m)^\sim_{\langle N\rangle}(x,t) \quad \text{if } |x-\Theta| \le 1/C \text{ and } 0 < t \le 1/C.$$

To check (4.7) for a polynomial, (4.7) must be proved only for the terms of $\widetilde{P}_m$ of lower order. In fact, if p_Θ is the order of the root $\tau = 0$ of $P_m(\Theta+\tau N)$, then (4.7) holds if and only if there is $C \ge 1$ such that

$$(4.8)\qquad \Big(\sum_{0<|\alpha|<p_\Theta} |P^{(\alpha)}(x)|^2 t^{2|\alpha|}\Big)^{1/2} \le C(P_m)^\sim_{\langle N\rangle}(x,t)$$

if $|x-\Theta| \le 1/C$ and $0 < t \le 1/C$. It is trivial that for $A \ge 1$,

$$(4.9)\qquad \widetilde{P}_m(x,t) \le \widetilde{P}_m(x,At) \le A^m \widetilde{P}_m(x,t) \quad \text{if } (x,t)\in\mathbb{R}^n\times]0,\infty[$$

(and similarly for $(P_m)^\sim_{\langle N\rangle}$). Therefore (4.7) is equivalent to the assumption (2.5) of Theorem 2.2(b) and it will be the standard assumption in this section. Similarly to (4.4) also (4.7) can be described by localizations at ∞.

LEMMA 4.3. *Let $\Theta \in V_{P_m}$ and $N \in S^{n-1}$. The following are equivalent:*

(a) *P_m satisfies (4.7).*
(b) *Any $Q \in L_\Theta(P_m)$ satisfies (4.4) for any $(x,t)\in\mathbb{R}^{n+1}$.*
(c) *N is noncharacteristic for any $Q \in L_\Theta(P_m)$.*

Proof. (a)⇒(b). Let $Q \in L_\Theta(P_m)$ and choose x_k for Q as in (4.5). Let $t > 0$ and $x\in\mathbb{R}^n$. Then

$$\begin{aligned}\widetilde{Q}(x,t) \leftarrow \widetilde{Q}_k(x,t) &= \widetilde{P}_m(x+x_k,t)/\widetilde{P}_m(x_k)\\ &= \widetilde{P}_m(x/|x_k| + x_k/|x_k|,\ t/|x_k|)|x_k|^m/\widetilde{P}_m(x_k)\\ &\le C(P_m)^\sim_{\langle N\rangle}(x/|x_k| + x_k/|x_k|,\ t/x_k)|x_k|^m/\widetilde{P}_m(x_k)\\ &= C(P_m)^\sim_{\langle N\rangle}(x+x_k,t)/\widetilde{P}_m(x_k) \to C\widetilde{Q}_{\langle N\rangle}(x,t)\end{aligned}$$

by (a) and the homogeneity of P_m since $x_k/|x_k| \to \Theta$ and $(x/|x_k|, t/|x_k|) \to 0$.

(b)⇒(c). This is trivial.

(c)⇒(a). If (a) is not true, then for any $k\in\mathbb{N}$ there are $\tau_k \in]0,1/k]$ and $\xi_k\in\mathbb{R}^n$ with $|\xi_k - \Theta| \le 1/k$ such that for $x_k := \xi_k/\tau_k$,

$$(4.10)\qquad (P_m)^\sim_{\langle N\rangle}(x_k)/\widetilde{P}_m(x_k) = (P_m)^\sim_{\langle N\rangle}(\xi_k,\tau_k)/\widetilde{P}_m(\xi_k,\tau_k) < 1/k.$$

Since $x_k\to\infty$ and $x_k/|x_k| = \xi_k/|\xi_k| \to \Theta$, we can assume that (x_k) defines $0 \ne Q \in L_\Theta(P_m)$ by (4.5). Notice that the polynomials Q_k in (4.5) are normalized such that

$$\|(Q_k^{(\alpha)}(0))\|_{l_2} = 1.$$

Then we get

$$\widetilde{Q}_{\langle N\rangle}(0) \leftarrow (Q_k)^\sim_{\langle N\rangle}(0) = (P_m)^\sim_{\langle N\rangle}(x_k)/\widetilde{P}_m(x_k) \to 0.$$

Hence N is characteristic for Q.

Lemma 4.3 and (4.6) are the motivation for the following

DEFINITION 4.4. (a) $N\in S^{n-1}$ is called *locally noncharacteristic for P_m at $\Theta\in V_{P_m}$* if P_m satisfies (4.7).

(b) $N\in S^{n-1}$ is called *locally noncharacteristic for P_m* if P_m satisfies (4.4).

Obviously, $P_{m,\Theta}/\widetilde{P}_{m,\Theta} \in L_\Theta(P_m)$ (choose $x_k = k\Theta$). Therefore, N is noncharacteristic for $P_{m,\Theta}$ if N is locally noncharacteristic at Θ for P_m. The standard assumption of this section is thus stronger than that of Section 3 (with the exception of $n \le 3$, see [31]). On the other hand, one cannot deduce local hyperbolicity w.r.t. N at Θ if N is not locally noncharacteristic for P_m at Θ:

REMARK 4.5. If P_m is locally hyperbolic w.r.t. N at Θ, then N is locally noncharacteristic for P_m at Θ. Indeed, we can assume that $N = e_n$ and write

$\xi = (\xi', \xi_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. By Zampieri [40, Lemma 1.3] there is $C \geq 1$ such that

$$P_m(\Theta + z) \neq 0 \quad \text{if } |z| \leq 1/C \text{ and } |\operatorname{Im} z_n| \geq C|\operatorname{Im} z'|.$$

This shows that $P_m(\Theta + x + z + ite_n) \neq 0$ if $x \in \mathbb{R}^n$, $z \in \mathbb{C}^n$, $t > 0$ and $|x| \leq 1/(4C)$, $t \leq 1/(4C)$ and $|z| \leq t/(4C)$. Thus $d_Q(0) \geq t/(4C)$ for $Q(\xi) := P_m(\Theta + x + \xi + ite_n)$ and (3.20) implies that for those x and t,

$$\widetilde{P}_m(\Theta + x + ite_n, t) \leq C_1|P_m(\Theta + x + ite_n)| \leq C_2(P_m)^{\sim}_{\langle e_n\rangle}(\Theta + x, t).$$

This implies (4.8) since by (4.9),

$$\widetilde{P}_m(\Theta + x, t) \leq C_3\widetilde{P}_m(\Theta + x + ite_n, 2t) \leq 2^m C_3\widetilde{P}_m(\Theta + x + ite_n, t).$$

We now continue the evaluation of the condition in Theorem 1.3 under the assumption of local noncharacteristicity. The following theorem is the main result of this section:

THEOREM 4.6. *Let $P(D)$ be surjective on $A(\Omega)$.*

(a) *P_m is locally hyperbolic w.r.t. $N \in N(\partial\Omega)$ at $\Theta \in V_{P_m}$ if N is locally noncharacteristic for P_m at Θ.*

(b) *P_m is locally hyperbolic if for any $\Theta \in V_{P_m}$ there is $N \in N(\partial\Omega)$ such that N is locally noncharacteristic for P_m at Θ.*

Proof. (a) We can assume that $N = e_n \in N_i(\partial\Omega)$. The proof of Theorem 3.3 shows that (3.3) holds for $\widehat{\Theta} \in S^{n-1}$ with $|\Theta - \widehat{\Theta}| < 1/(2C)$ with ν_j, μ_j and B independent of $\widehat{\Theta}$ by Theorem 2.2(b). By part a) of the proof of Theorem 3.4 we get $A_1 \geq 1$ and $J \geq 1$ independent of $\widehat{\Theta}$ such that for $j \geq J$ and any $\xi \in \Gamma_{\mu_j}(\widehat{\Theta})$,

$$\widetilde{P}_{\langle e_n\rangle}(\xi, \tau) \leq A_1|P(\xi + i\tau e_n)| \quad \text{for } 2\nu_j|\xi| \leq \tau \leq 4\nu_j|\xi| \text{ if } |\xi| \geq C(j).$$

We apply this to $\xi = \widehat{\Theta} =: \Theta + x$ with $|x| \leq 1/(2C)$ and get by part b) of the proof of Theorem 3.4 (see (3.22))

$$A_0\nu_j^{q_\Theta} \leq (P_m)^{\sim}_{\langle e_n\rangle}(\Theta + x, 3\nu_j) \leq A_1|P_m(\Theta + 3i\nu_j e_n)| \quad \text{if } |x| \leq 1/(2C).$$

Thus, for large j,

$$P_m(\Theta + x + 3i\nu_j e_n) \neq 0 \quad \text{if } |x| < 1/(2C). \tag{4.11}$$

Since $M := \{(x,t) \mid P_m(\Theta + x + ite_n) = 0,\ |x| < 1/(2C),\ t > 0\}$ is semialgebraic, M has only finitely many connected components (Bochnak–Coste–Roy [5, Theorem 2.4.5]). Since $\nu_j \to 0$, (4.11) therefore implies (4.1) for $\operatorname{Im} z > 0$. Since N is locally noncharacteristic for P_m also at $-\Theta$, (4.1) holds for $\operatorname{Im} z > 0$ at $-\Theta$. Thus, (4.1) at Θ also holds for $\operatorname{Im} z < 0$.

(b) This directly follows from (a).

COROLLARY 4.7. *Let $\Theta \in V_{P_m}$ and assume that in any component V of $S^{n-1} \setminus V_{P_{m,\Theta}}$ there is $N \in S^{n-1}$ which is locally noncharacteristic for*

P_m at Θ. If $P(D)$ is surjective in $A(\Omega)$ for some bounded open set Ω with C^1-boundary, then $P_{m,\Theta}$ has the form (3.27) and P_m is locally hyperbolic at Θ w.r.t. M if $\langle M, \xi_{j,\Theta}\rangle \neq 0$ for $j = 1, \ldots, q_\Theta$.

Proof. By Theorem 4.6, P_m is locally hyperbolic at Θ w.r.t. some vector of each component V of $S^{n-1} \setminus V_{P_{m,\Theta}}$. The claim thus follows from Remark 4.2 and the proof of Corollary 3.7.

COROLLARY 4.8. *Let N be locally noncharacteristic for P_m. The following are equivalent:*

(a) *$P(D)$ is surjective on $A(\Omega)$ for some Ω with $N \in N(\partial\Omega)$.*

(b) *$P(D)$ is surjective on $A(\Omega)$ for $\Omega := \{x \in \mathbb{R}^n \mid C_1 > \langle x, N\rangle > C_2\}$, $-\infty \leq C_2 < C_1 \leq \infty$*

(c) *$P_m(D)$ is hyperbolic-elliptic w.r.t. N.*

Proof. (a)⇒(c). This follows from Theorem 4.6(a) and the equivalence of (4.2) and (4.3).

(c)⇒(b). By (3.30) we have to show (b) only for the halfspaces $\Omega_{\pm N}$. Since P_m is also hyperbolic-elliptic w.r.t. $-N$, we only need to consider Ω_N. Let $\Theta \in V_{P_m}$. Then P_m is locally hyperbolic w.r.t. N at Θ by the equivalence of (4.2) and (4.3), and the local propagation cone K_Θ containing $-N$ is contained in $\mathbb{R}^n \setminus \Omega_N$. The assumption of Zampieri [40, Main Theorem] is thus satisfied for Ω_N and $P(D)$ is surjective on $A(\Omega_N)$ by that result.

To extend Corollary 4.8 to convex sets we need a local version of Definition 3.12:

DEFINITION 4.9. Let Ω be convex. For $\Theta \in V_{P_m}$ let $N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega)$ be the union of the closed convex hulls $\overline{\operatorname{conv}(V_{\Theta,j})}$ of the components $V_{\Theta,j}$ of $S^{n-1} \setminus V_{P_{m,\Theta}}$ containing a vector $N \in N_g(\partial\Omega)$ which is locally noncharacteristic at Θ for P_m. We call Ω *locally P-admissible* if

$$N_g(\partial\Omega) \subset N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega) \quad \text{for any } \Theta \in V_{P_m}. \tag{4.12}$$

COROLLARY 4.10. *Let Ω be convex and locally P-admisible. The following are equivalent:*

(a) *$P(D)$ is surjective on $A(\Omega)$.*

(b) *$P(D)$ is surjective on $A(\Omega_N)$ for each $N \in \bigcap_{\Theta \in V_{P_m}} N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega)$.*

(c) *P_m is locally hyperbolic at any $\Theta \in V_{P_m}$ w.r.t. each $N \in N_g(\partial\Omega)$ which is locally noncharacteristic at Θ for P_m.*

Proof. (a)⇒(c). Let $N \in N_g(\partial\Omega)$ and let $\Sigma = x_0 + \Omega_N$ be a tangent halfspace of Ω. Then $P(D)$ is surjective on $A(\Sigma)$ by (3.31) and (c) follows for N by Theorem 4.6(a) applied to Ω_N.

(c)⇒(b). For Θ and N as in (c) let V be the component of N in $S^{n-1} \setminus V_{P_{m,\Theta}}$. Since P_m is locally hyperbolic w.r.t. N by assumption, V is convex and the local propagation cone $K_\Theta := -V^0$ is contained in $\mathbb{R}^n \setminus \Omega_M$ for any $M \in \overline{V}$. The union of the sets $\overline{V}$ is $N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega)$ by definition. The halfspaces Ω_M with $M \in \bigcap_{\Theta\in V_{P_m}} N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega)$ therefore satisfy the assumption of Zampieri [40, Main Theorem] and (b) follows from that result.

(b)⇒(a). If N is in $\bigcap_{\Theta\in V_{P_m}} N_{\mathrm{loc}}(P_{m,\Theta}, \partial\Omega)$ then so is $-N$. Since Ω is locally P-admissible, (b) means that $P(D)$ is surjective on $A(\Sigma)$ and on $A(\mathbb{R}^n \setminus \overline{\Sigma})$ for any tangent halfspace Σ of Ω. This implies (a) by (3.30).

COROLLARY 4.11. *Assume that for any $\Theta \in V_{P_m}$ in any component V of $S^{n-1} \setminus V_{P_{m,\Theta}}$ there is N which is locally noncharacteristic for P_m at Θ. The following are equivalent:*

(a) *$P(D)$ is surjective on $A(\Omega)$ for some bounded Ω with C^1-boundary.*

(b) *$P(D)$ is surjective on $A(\Omega)$ for any convex Ω.*

(c) *$P(D)$ is surjective on $A(\Sigma)$ for any halfspace Σ.*

(d) *For any $\Theta \in V_{P_m}$, $P_{m,\Theta}$ has the form (3.27) and P_m is locally hyperbolic at Θ w.r.t. M if $\langle M, \xi_{j,\Theta}\rangle \neq 0$ for $j = 1, \ldots, q_\Theta$.*

(e) *There are $\xi_1, \ldots, \xi_k \in \mathbb{R}^n$ such that P_m is hyperbolic-elliptic w.r.t. M if*

$$\langle M, \xi_j\rangle \neq 0 \quad \text{for } j = 1, \ldots, k. \tag{4.13}$$

Proof. (a)⇒(d). This follows from Corollary 4.7.

(d)⇒(e). By a compactness argument and (d) there are $\xi_1, \ldots, \xi_k$ such that P_m is locally hyperbolic at any $\Theta \in V_{P_m}$ w.r.t. M if M satisfies (4.13). Thus P_m is hyperbolic-elliptic w.r.t. those M.

(e)⇒(c). This holds for $\Sigma = \Omega_M$ by Corollary 4.8 if M satisfies (4.13). If $\langle M, \xi_j\rangle = 0$ for some j, then for any $\Theta \in V_{P_m}$, M is contained in the closure of some component V_Θ of $S^{n-1} \setminus P_{m,\Theta}$ and the corresponding propagation cone $-K_\Theta$ is contained in $\mathbb{R}^n \setminus \Omega_M$. Thus, $P(D)$ is surjective on $A(\Omega_M)$ by Zampieri [39, Main Theorem].

(c)⇒(b). This holds by (3.30).

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An asymptotic expansion for the distribution of the supremum of a random walk

by

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Abstract. Let $\{S_n\}$ be a random walk drifting to $-\infty$. We obtain an asymptotic expansion for the distribution of the supremum of $\{S_n\}$ which takes into account the influence of the roots of the equation $1-\int_{\mathbb{R}} e^{sx}\,F(dx)=0$, F being the underlying distribution. An estimate, of considerable generality, is given for the remainder term by means of submultiplicative weight functions. A similar problem for the stationary distribution of an oscillating random walk is also considered. The proofs rely on two general theorems for Laplace transforms.

1. Introduction. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random variables with a common nonarithmetic distribution F. Define $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Suppose the random walk $\{S_n\}$ drifts to $-\infty$, i.e., with probability one $S_n \to -\infty$ as $n \to \infty$. We set $M_\infty := \sup_{n\geq 0} S_n$.

Properties of the distribution of M_∞ have been studied by many authors for various reasons. First, the problems involving M_∞ are of interest in their own right, since the supremum is one of the underlying functionals in random walk theory. Second, the distribution of M_∞ appears in some applications; for example, it coincides with the limiting distribution of the waiting time process in the theory of queues [7, Sections XII.5 and VI.9]. The existence of moments of the form $Ef(M_\infty)$ was considered for various choices of the function $f(x)$ by Kiefer and Wolfowitz [12], Tweedie [19], Janson [10], Alsmeyer [1], and Sgibnev [16]. Note that although Theorem 5 of Tweedie [19] concerns moments of the form $\int f(x)\,\pi(dx)$ for the stationary distribution π of the Markov chain $Z_{n+1} = \max(Z_n + X_{n+1}, 0)$, it is, however, well known

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