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Approximation of abstract linear integrodifferential equations

by

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Abstract. This paper is devoted to the approximation of abstract linear integrodifferential equations by finite difference equations. The result obtained here is applied to the problem of convergence of the backward Euler type discrete scheme.

1. Introduction. In this paper we discuss the problem of approximation of solutions of a linear integrodifferential equation

$$(\mathrm{IE}; u_0)\qquad \begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds & \text{for } t \ge 0, \\ u(0) = u_0 \end{cases}$$

in a general Banach space X. Here A is the infinitesimal generator of a semigroup of class (C_0) on X and $\{B(t) : t \ge 0\}$ is a family of bounded linear operators from Y to X, where Y is the Banach space $D(A)$ equipped with its graph norm.

The notion of the resolvent operator is central for the theory of linear integrodifferential equations. Recall that a family $\{R(t) : t \ge 0\}$ of bounded linear operators on X is called a *resolvent operator* if the following conditions are satisfied:

(i) $R(\cdot)x \in C([0,\infty); X)$ for $x \in X$ and $R(0) = I$ (identity);
(ii) $R(\cdot)x \in C^1([0,\infty); X) \cap C([0,\infty); Y)$ for $x \in Y$;
(iii) the following resolvent equations hold:

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t B(t-s)R(s)x\,ds$$
$$= R(t)Ax + \int_0^t R(t-s)B(s)x\,ds \quad \text{for } t \ge 0 \text{ and } x \in Y.$$

Many authors studied the problem of existence and uniqueness of solutions

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of (IE; u_0) by proving the existence of a resolvent operator (see Chen and Grimmer [1], Grimmer and Prüss [2] and Prüss [8]).

The approximation theorem which says that a solution depends continuously on A and $\{B(t) : t \ge 0\}$ may be proved by using the so-called "Kisyński method" proposed in his paper [4]. We are here interested in studying the problem of approximation for linear integrodifferential equations by finite difference equations. In the special case of $B(t) = 0$, the problem of this kind is known as the problem of approximation of semigroups and was extensively studied by many authors (see Kurtz [5], Pazy [7] and Trotter [11]). For the time-discretization of (IE; u_0) we consider the following type of approximation:

$$(1.1)\quad \begin{cases} F_{n,k+1} = T_nF_{n,k} + \displaystyle\sum_{i=0}^{k} h_n^2 B_n((k-i)h_n)F_{n,i} & \text{for } k = 0,1,2,\ldots, \\ F_{n,0} = I_n, \end{cases}$$

where T_n is a bounded linear operator on a Banach space X_n such that $(T_n - I_n)/h_n$ is a finite difference approximation to A and B_n is a finite difference approximation to B. Here $\{h_n\}$ is a positive null sequence and $\{X_n\}$ is a *sequence of Banach spaces approximating* X in the following sense: There exist bounded linear operators P_n from X to X_n such that $\lim_{n\to\infty} \|P_nx\|_n = \|x\|$ for each $x \in X$. We note that there exists $\beta > 0$ such that

$$(1.2)\qquad \|P_nx\|_n \le \beta\|x\| \quad \text{for } x \in X \text{ and } n \ge 1.$$

Now we introduce a few definitions used in the present paper. A sequence $\{x_n\}$ with $x_n \in X_n$ is said to *converge* to $x \in X$ if $\lim_{n\to\infty} \|P_nx - x_n\|_n = 0$. When there is no danger of confusion, this type of convergence will be denoted by $\lim_{n\to\infty} x_n = x$ and we then say that $\{x_n\}$ is a *convergent sequence in* $\{X_n\}$. The main result in this paper is Theorem 3.1, which asserts that for $x \in X$, $x_n \in X_n$ with $\lim_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} F_{n,[t/h_n]}x_n = R(t)x$$

holds for $t \ge 0$ and the convergence is uniform on every compact subinterval of $[0,\infty)$, where $[r]$ denotes the integer part of $r \ge 0$. This result can be applied to the problem of convergence of the backward Euler type discrete scheme for (IE; u_0).

2. Preliminaries.

In this section we show four fundamental lemmas used later.

LEMMA 2.1. *Let $\tau > 0$ and $\{C_n(t) : t \in [0,\tau]\}$ a sequence of operators with $C_n(t) \in B(X_n)$ for each $t \in [0,\tau]$. If for all convergent sequences $\{x_n\}$ in $\{X_n\}$, $\sup\{\|C_n(t)x_n\|_n : t \in [0,\tau], n \ge 1\} < \infty$, then*

$$\sup\{\|C_n(t)\|_n : t \in [0,\tau], n \ge 1\} < \infty.$$

Proof. Denote by $\widetilde{X}$ the space consisting of all sequences $\widetilde{x} = \{x_n\}$ in $\{X_n\}$ such that $\sup\{\|x_n\|_n : n \ge 1\} < \infty$. Then $\widetilde{X}$ is a Banach space equipped with norm $\|\cdot\|_{\widetilde{X}}$ defined by $\|\widetilde{x}\|_{\widetilde{X}} = \sup\{\|x_n\|_n : n \ge 1\}$ for $\widetilde{x} = \{x_n\} \in \widetilde{X}$. Let $\widetilde{X}_0$ be the space of all convergent sequences in $\{X_n\}$. It is obvious that $\widetilde{X}_0$ is a closed linear subspace of $\widetilde{X}$. For each $t \in [0,\tau]$, the operator $\widetilde{C}(t)$ from $\widetilde{X}_0$ into $\widetilde{X}$ defined by

$$\widetilde{C}(t)\widetilde{x} = \{C_n(t)x_n\} \quad \text{for } \widetilde{x} = \{x_n\} \in \widetilde{X}_0$$

is linear and everywhere defined, by assumption. Moreover, it is easily seen to be closed. It follows from the closed graph theorem that $\widetilde{C}(t)$ is bounded. By assumption again we have $\sup\{\|\widetilde{C}(t)\widetilde{x}\|_{\widetilde{X}} : t \in [0,\tau]\} < \infty$ for each $\widetilde{x} \in \widetilde{X}_0$, and so the uniform boundedness principle gives $\sup\{\|\widetilde{C}(t)\|_{\widetilde{X}_0,\widetilde{X}} : t \in [0,\tau]\} =: M < \infty$. Let $n \ge 1$ and $x \in X_n$. Considering the sequence whose nth component is x and all the other components are zero in $\widetilde{X}_0$ we have $\|C_n(t)x\|_n \le M\|x\|_n$, which implies the desired claim. ■

LEMMA 2.2. *Let $\tau > 0$. Suppose that $\{C_n(t) : t \in [0,\tau]\}$ is a sequence of operators with $C_n(t) \in B(X_n)$ for each $t \in [0,\tau]$ and $\{x_n\}$ is a sequence in $\{X_n\}$ such that $C_n(\cdot)x_n \in \mathrm{BV}([0,\tau]; X_n)$ for each $n \ge 1$ and $\sup\{\mathrm{Var}(C_n(\cdot)x_n; [0,\tau]) : n \ge 1\} < \infty$, where* Var *denotes the total variation. Let $\{s_n\}$ be a sequence of step functions with $s_n([0,\tau]) \subset [0,\tau]$ for $n \ge 1$. If the sequence $\{s_n(t)\}$ converges to t uniformly on $[0,\tau]$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \int_0^\tau \|C_n(s_n(t))x_n - C_n(t)x_n\|_n \, dt = 0.$$

Proof. For $n \ge 1$ we define a function φ_n from $\mathbb{R}$ into $[0,\infty)$ by

$$\varphi_n(t) = \begin{cases} 0 & \text{for } t \le 0, \\ \mathrm{Var}(C_n(\cdot)x_n; [0,t]) & \text{for } t \in [0,\tau], \\ \mathrm{Var}(C_n(\cdot)x_n; [0,\tau]) & \text{for } t \ge \tau. \end{cases}$$

Then for each $n \ge 1$, φ_n is nondecreasing and $\|C_n(t_1)x_n - C_n(t_2)x_n\|_n \le \varphi_n(t_2) - \varphi_n(t_1)$ for $0 \le t_1 \le t_2 \le \tau$. Now, let $\varepsilon > 0$. By assumption we choose an integer $N_\varepsilon \ge 1$ such that $n \ge N_\varepsilon$ implies $t - \varepsilon < s_n(t) < t + \varepsilon$ for all $t \in [0,\tau]$. We then have

$$\begin{aligned} \|C_n(s_n(t))x_n - C_n(t)x_n\|_n &\le \varphi_n(s_n(t) \vee t) - \varphi_n(s_n(t) \wedge t) \\ &\le \varphi_n(t+\varepsilon) - \varphi_n(t-\varepsilon) \end{aligned}$$

for $n \ge N_\varepsilon$ and $t \in [0,\tau]$. Integrating both sides of this inequality over $[0,\tau]$, and using the fact that $\varphi_n(t) = 0$ for $t \le 0$, we have

$$\int_0^\tau \|C_n(s_n(t))x_n - C_n(t)x_n\|_n\,dt \le \int_\varepsilon^{\tau+\varepsilon} \varphi_n(t)\,dt - \int_0^{\tau-\varepsilon} \varphi_n(t)\,dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} \varphi_n(t)\,dt - \int_0^\varepsilon \varphi_n(t)\,dt$$

for $n \ge N_\varepsilon$, and the last term is bounded by $2\varepsilon \sup\{\mathrm{Var}(C_n(\cdot)x_n;[0,\tau]) : n \ge 1\}$. ∎

Let $N \ge 1$ be an integer. By $\widehat{Z}$ we denote the linear space consisting of all sequences $\widehat{z} = \{z_i\}_{i=0}^N$ in Z. For convenience, we introduce two notations similar to the convolution of functions:

$$\widehat{K} * \widehat{u} = \Big\{\sum_{l=0}^{i} K_{i-l}u_l\Big\}_{i=0}^{N} \in \widehat{X} \qquad \text{for } \widehat{K} = \{K_i\} \in \widehat{B(X)},\ \widehat{u} = \{u_i\} \in \widehat{X};$$

$$\widehat{L} * \widehat{K} = \Big\{\sum_{l=0}^{i} L_{i-l}K_l\Big\}_{i=0}^{N} \in \widehat{B(X)} \quad \text{for } \widehat{L} = \{L_i\},\ \widehat{K} = \{K_i\} \in \widehat{B(X)}.$$

Moreover, we define $\widehat{K}\widehat{u} = \{K_iu_i\}_{i=0}^N$ for $\widehat{K} = \{K_i\} \in \widehat{B(X)}$ and $\widehat{u} = \{u_i\} \in \widehat{X}$.

Lemma 2.3. (i) *$\alpha(\widehat{L} * \widehat{u}) = (\alpha\widehat{L}) * \widehat{u} = \widehat{L} * (\alpha\widehat{u})$ for $\alpha \in \mathbb{R}$, $\widehat{L} \in \widehat{B(X)}$ and $\widehat{u} \in \widehat{X}$.*

(ii) *$\widehat{L} * (\widehat{K}_1 + \widehat{K}_2) = \widehat{L} * \widehat{K}_1 + \widehat{L} * \widehat{K}_2$ and $(\widehat{K}_1 + \widehat{K}_2) * \widehat{L} = \widehat{K}_1 * \widehat{L} + \widehat{K}_2 * \widehat{L}$ for $\widehat{L}, \widehat{K}_1, \widehat{K}_2 \in \widehat{B(X)}$.*

(iii) *$\widehat{L} * (\widehat{K} * \widehat{u}) = (\widehat{L} * \widehat{K}) * \widehat{u}$ for $\widehat{L}, \widehat{K} \in \widehat{B(X)}$ and $\widehat{u} \in \widehat{X}$.*

(iv) *Let $\widehat{K}, \widehat{L} \in \widehat{B(X)}$ and $x \in X$. If we set $\widehat{J} = \widehat{K} * \widehat{L}$, then*

$$\mathrm{lip}_x(\widehat{J}) \le \|\widehat{K}\|(\|\widehat{L}\| \cdot \|x\| + \mathrm{var}_x(\widehat{L})). \tag{2.1}$$

Here and subsequently $\|\widehat{Q}\|$, $\mathrm{lip}_x(\widehat{Q})$ and $\mathrm{var}_x(\widehat{Q})$ are defined by

$$\|\widehat{Q}\| = \max\{\|Q_i\| : 0 \le i \le N\},$$
$$\mathrm{lip}_x(\widehat{Q}) = \max\{\|Q_ix - Q_{i-1}x\| : 1 \le i \le N\},$$
$$\mathrm{var}_x(\widehat{Q}) = \sum_{i=1}^{N} \|Q_ix - Q_{i-1}x\|$$

for $\widehat{Q} = \{Q_i\}_{i=0}^N \in \widehat{B(X)}$ and $x \in X$.

Proof. Properties (i) and (ii) are obvious by definition. Assertion (iii) is proved by changing the order of summation. Indeed, for $0 \le i \le N$ we have

$$(\widehat{L} * (\widehat{K} * \widehat{u}))_i = \sum_{l=0}^{i} L_{i-l}\Big(\sum_{k=0}^{l} K_{l-k}u_k\Big) = \sum_{k=0}^{i}\sum_{l=k}^{i} L_{i-l}K_{l-k}u_k = \sum_{k=0}^{i}\sum_{l=0}^{i-k} L_{i-l-k}K_lu_k = \sum_{k=0}^{i} (\widehat{L} * \widehat{K})_{i-k}u_k = ((\widehat{L} * \widehat{K}) * \widehat{u})_i.$$

Here we write $(\widehat{L} * \widehat{K})_i$ for the ith component of $\widehat{L} * \widehat{K}$. Inequality (2.1) of (iv) is obtained by estimating

$$J_ix - J_{i-1}x = K_iL_0x + \sum_{l=0}^{i-1} K_l(L_{i-l}x - L_{i-1-l}x)$$

for $1 \le i \le N$. ∎

Lemma 2.4. *If $\widehat{K} \in \widehat{B(X)}$ and $h \ge 0$ satisfy $h\|\widehat{K}\| \le 1/2$, then there exists $\widehat{L} \in \widehat{B(X)}$ such that*

$$\widehat{L} = \widehat{K} - h\widehat{L} * \widehat{K} = \widehat{K} - h\widehat{K} * \widehat{L}. \tag{2.2}$$

Moreover, the following estimates hold:

(i) *$\|\widehat{L}\| \le \|\widehat{K}\|(1 + h\|\widehat{K}\|(1 - h\|\widehat{K}\|)^{-1}) \exp(2hN\|\widehat{K}\|)$.*

(ii) *$\mathrm{var}_x(\widehat{L}) \le \mathrm{var}_x(\widehat{K}) + hN\|\widehat{L}\| \cdot \|\widehat{K}\| \cdot \|x\| + hN\|\widehat{L}\|\,\mathrm{var}_x(\widehat{K})$ for $x \in X$.*

Proof. Since $h\|\widehat{K}\| \le 1/2$ there exists $(I + hK_0)^{-1} \in B(X)$. We can define $\widehat{L}, \widehat{Q} \in \widehat{B(X)}$ inductively by

$$L_0 = K_0(I + hK_0)^{-1}, \quad L_j = \Big(K_j - h\sum_{l=1}^{j} L_{j-l}K_l\Big)(I + hK_0)^{-1},$$

$$Q_0 = (I + hK_0)^{-1}K_0, \quad Q_j = (I + hK_0)^{-1}\Big(K_j - h\sum_{l=1}^{j} K_lQ_{j-l}\Big)$$

for $j = 1, \ldots, N$. Rewriting these equalities we have

$$\widehat{L} = \widehat{K} - h\widehat{L} * \widehat{K} \quad \text{and} \quad \widehat{Q} = \widehat{K} - h\widehat{K} * \widehat{Q}. \tag{2.3}$$

Since

$$\widehat{K} * \widehat{Q} - h(\widehat{L} * \widehat{K}) * \widehat{Q} = \widehat{L} * \widehat{Q} = \widehat{L} * \widehat{K} - h\widehat{L} * (\widehat{K} * \widehat{Q}),$$

we have $\widehat{K} * \widehat{Q} = \widehat{L} * \widehat{K}$, which we combine with (2.3) to obtain $\widehat{L} = \widehat{Q}$. The first part of our assertion is therefore proved. To prove (i), set $a_j = \|L_j\|$. By (2.2) we have $a_j \le \|\widehat{K}\|(1 + h\sum_{l=0}^{j} a_l)$ for $0 \le j \le N$. We denote by b_j the right-hand side. Then $a_j \le b_j$ for $0 \le j \le N$. Since $b_j - b_{j-1} = h\|\widehat{K}\|a_j \le h\|\widehat{K}\|b_j$, we have $b_j \le (1 - h\|\widehat{K}\|)^{-1}b_{j-1}$, which gives

$$a_j \le b_j \le (1 - h\|\widehat{K}\|)^{-j}b_0 = \exp(2hj\|\widehat{K}\|)\|\widehat{K}\|(1 + ha_0)$$

for $0 \le j \le N$. The desired estimate (i) is obtained by noting that $a_0 = \|L_0\| \ (= \|K_0(I+hK_0)^{-1}\|) \le \|\widehat{K}\|(1-h\|\widehat{K}\|)^{-1}$.

We now prove (ii). We use (2.2) to represent the difference between L_jx and $L_{j-1}x$ and estimate the resultant equalities. This yields

$$\|L_jx - L_{j-1}x\| \le \|K_jx - K_{j-1}x\| + h\|L_j\|\cdot\|K_0\|\cdot\|x\| + h\sum_{l=0}^{j-1}\|L_l\|\cdot\|K_{j-l}x - K_{j-1-l}x\|$$

for $1 \le j \le N$. (See also the proof of (iv) of Lemma 2.3.) The desired estimate (ii) follows readily by summing up the inequalities above from $j=1$ to $j=N$. ■

3. Main result. In this section we prove that if $x \in X$ and $x_n \in X_n$ satisfy $\lim_{n\to\infty} x_n = x$, then a sequence $\{F_{n,[t/h_n]}x_n\}$ in $\{X_n\}$ obtained by (1.1) converges uniformly to the solution $R(t)x$ of (IE; x) on each compact subinterval of $[0,\infty)$. The main result of this paper is given in the following theorem.

THEOREM 3.1. *Let $\{T_n\}$ be a sequence of operators with $T_n \in B(X_n)$, let A be a closed linear operator with dense domain $D(A)$ in X, and let $\{h_n\}$ be a positive null sequence with the following properties:*

(a_1) *There exist constants $M \ge 1$ and $\omega \ge 0$ such that*

$$\|T_n^k\|_n \le Me^{\omega k h_n} \quad \text{for } k \ge 0 \text{ and } n \ge 1.$$

(a_2) *For $x \in D(A)$ there exists a sequence $\{x_n\}$ in $\{X_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} A_nx_n = Ax$, where $A_n = h_n^{-1}(T_n - I_n)$ and I_n is the identity operator on X_n.*

(a_3) *For some $\lambda_0 > \omega$, the range $R(\lambda_0I - A)$ of $\lambda_0I - A$ is dense in X.*

Let $\{B(t) : t \ge 0\}$ be a family of bounded linear operators from Y to X and $\{B_n(t) : t \ge 0\}$ a family of operators with $B_n(t) \in B(X_n)$ for each $t \ge 0$ satisfying the two conditions below, where Y is the Banach space $D(A)$ with its graph norm.

(b_1) *For each $x \in Y$, $B(\cdot)x \in \mathrm{BV}_{\mathrm{loc}}([0,\infty); X)$.*

(b_2) *For each $x \in Y$, $x_n \in X_n$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} A_nx_n = Ax$,*

(i) $B_n(\cdot)x_n \in \mathrm{BV}_{\mathrm{loc}}([0,\infty); X_n)$,

(ii) $\sup\{\|B_n(t)x_n\|_n : t \in [0,\tau],\ n \ge 1\} < \infty$ *for each* $\tau > 0$,

(iii) $\sup\{\mathrm{Var}(B_n(\cdot)x_n; [0,\tau]) : n \ge 1\} < \infty$ *for each* $\tau > 0$,

(iv) $\lim_{n\to\infty} \int_0^\tau \|B_n(s)x_n - P_nB(s)x\|_n\, ds = 0$ *for each* $\tau > 0$.

Then there exists a unique resolvent operator $\{R(t) : t \ge 0\}$ on X such that for $x \in X$ and $x_n \in X_n$ with $\lim_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} F_{n,[t/h_n]}x_n = R(t)x \tag{3.1}$$

holds for $t \ge 0$, and the convergence is uniform on every compact subinterval of $[0,\infty)$.

We start with a few remarks important for proving this theorem.

REMARK 3.1. (i) It is well known [5] that under assumptions (a_1)–(a_3), A is the infinitesimal generator of a semigroup $\{T(t) : t \ge 0\}$ of class (C_0) on X satisfying $\|T(t)\| \le Me^{\omega t}$ for $t \ge 0$ such that for $x \in X$ and $x_n \in X_n$ with $\lim_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} T_n^{[t/h_n]}x_n = T(t)x \tag{3.2}$$

holds for $t \ge 0$, where the convergence is uniform on every compact subinterval of $[0,\infty)$. Note that $\lambda_0 \in \varrho(A)$ since $\lambda_0 > \omega$.

(ii) Assumption (b_1) and (i) of Remark 3.1 together imply that there exists a unique resolvent operator $\{R(t) : t \ge 0\}$ on X satisfying

$$R(t)x = T(t)x + \frac{d}{dt}(V * R)(t)x - V(t)x - \lambda_0(V * R)(t)x \tag{3.3}$$

for $t \ge 0$ and $x \in X$, where $\{V(t) : t \ge 0\}$ is a locally Lipschitz continuous family in $B(X)$ defined by $V(t)x = (T * L)(t)x$ for $t \ge 0$ and $x \in X$, and $L(t)$ is the resolvent kernel of $K(t) := -B(t)(\lambda_0I - A)^{-1}$ for $t \ge 0$, namely

$$L(t)x = K(t)x - (L * K)(t)x = K(t)x - (K * L)(t)x \tag{3.4}$$

for $t \ge 0$ and $x \in X$ (see [2, Corollary 4] and [6, p. 214, (3.5)]). Here we write $f * g$ for the convolution of $f, g \in L^1_{\mathrm{loc}}([0,\infty); X)$. It should be noted that by (b_1), $K(\cdot)x \in \mathrm{BV}_{\mathrm{loc}}([0,\infty); X)$ for $x \in X$, and so does $L(\cdot)x$ for $x \in X$, and that if $f \in L^1_{\mathrm{loc}}([0,\infty); X)$, then an X-valued function $V * f$ is of class C^1 and

$$\|(d/dt)(V * f)(t)\| \le \|V\|_{\mathrm{Lip}[0,\tau]} \int_0^\tau \|f(s)\|\, ds$$

for $t \in [0,\tau]$, where $\|V\|_{\mathrm{Lip}[0,\tau]}$ denotes the Lipschitz constant of V on $[0,\tau]$, due to the local Lipschitz continuity of $\{V(t) : t \ge 0\}$ and the fact that $V(0)$ is the zero operator on X. (See [3, Proof of Theorem 2.5] and [2, Lemma 2].)

Proof of Theorem 3.1. By Remark 3.1 we only have to prove (3.1). Let $\tau > 0$ and $\{x_n\}$ be an arbitrary convergent sequence in $\{X_n\}$ with $\lim_{n\to\infty} x_n = x$. We take a real number a such that $\lambda_0 > a > \omega$, and choose an integer n_0 such that $(e^{\omega h_n} - 1)/h_n \le a$ for $n \ge n_0$. Then we have $\lambda_0 \in \varrho(A_n)$ for $n \ge n_0$ and $\lim_{n\to\infty}(\lambda_0I_n - A_n)^{-1}x_n = (\lambda_0I - A)^{-1}x$; it follows that $\lim_{n\to\infty} A_n(\lambda_0 I_n - A_n)^{-1}x_n = A(\lambda_0 I - A)^{-1}x$. (See also [9, Proof of Theorem 2.1].) This together with (ii) of (b_2) implies

$$\sup\{\|B_n(t)(\lambda_0 I_n - A_n)^{-1}x_n\|_n : t \in [0,\tau] \text{ and } n \ge n_0\} < \infty.$$

We set

$$K_n(t) = -B_n(t)(\lambda_0 I_n - A_n)^{-1}$$

for $t \in [0,\tau]$ and $n \ge n_0$. By Lemma 2.1 we have

$$c_K(\tau) := \sup\{\|K_n(t)\|_n : t \in [0,\tau] \text{ and } n \ge n_0\} < \infty.$$

From (iii) and (iv) of (b_2) it follows that

$$\sup\{\mathrm{Var}(K_n(\cdot)x_n; [0,\tau]) : n \ge n_0\} < \infty,$$

$$\lim_{n\to\infty} \int_0^\tau \|K_n(s)x_n - P_n K(s)x\|_n\,ds = 0.$$

By Lemma 2.2 we have

$$\lim_{n\to\infty} \int_0^\tau \|K_n(s_n(t))z_n - P_n K(t)z\|_n\,dt = 0, \tag{3.5}$$

if $\{s_n\}$ is a sequence of step functions such that $s_n([0,\tau]) \subset [0,\tau]$ for $n \ge 1$ and that it converges to t uniformly on $[0,\tau]$ as $n \to \infty$, and if $\{z_n\}$ is a convergent sequence in $\{X_n\}$ such that $\lim_{n\to\infty} z_n = z$.

Moreover, we take an integer n_1 such that $n_1 \ge n_0$ and $c_K(\tau)h_n \le 1/2$ for all $n \ge n_1$. Let $n \ge n_1$ and set $N_n = [\tau/h_n]$. We now define $\widehat{K}_n = \{K_n(ih_n)\}_{i=0}^{N_n}$ in $B(\widehat{X_n})$. Since $\|\widehat{K}_n\|h_n \le 1/2$ we apply Lemma 2.4 to find $\widehat{L}_n = \{L_{n,i}\}_{i=0}^{N_n} \in B(\widehat{X_n})$ satisfying

$$\widehat{L}_n = \widehat{K}_n - h_n\widehat{L}_n * \widehat{K}_n = \widehat{K}_n - h_n\widehat{K}_n * \widehat{L}_n, \tag{3.6}$$

$$c_L(\tau) := \sup\{\|L_{n,i}\|_n : 0 \le i \le N_n,\ n \ge n_1\} < \infty, \tag{3.7}$$

$$\widetilde{c}_L(\tau) := \sup\Big\{\sum_{i=1}^{N_n} \|L_{n,i}x_n - L_{n,i-1}x_n\|_n : n \ge n_1\Big\} < \infty. \tag{3.8}$$

We shall prove the equality

$$\begin{aligned} F_{n,k}x_n = T_n^k x_n &+ \sum_{l=0}^{k-1}(V_{n,k-l} - V_{n,k-1-l})F_{n,l}x_n \\ &+ V_{n,0}F_{n,k}x_n - V_{n,k}x_n - \lambda_0 h_n \sum_{l=0}^{k-1} V_{n,k-1-l}F_{n,l}x_n \end{aligned} \tag{3.9}$$

for $1 \le k \le N_n$, where $V_{n,k} = h_n \sum_{i=0}^k T_n^{k-i}L_{n,i} \in B(X_n)$ for $0 \le k \le N_n$. To do so, set $\widehat{u}_n = \{F_{n,k}x_n\}_{k=0}^{N_n} \in \widehat{X}_n$ and $\widehat{v}_n = \{((F_{n,k+1} - F_{n,k})/h_n)x_n\}_{k=0}^{N_n}$ $\in \widehat{X}_n$, and define $\widehat{A}_n, \widehat{B}_n \in B(\widehat{X_n})$ by $\widehat{A}_n = \{A_n, \ldots, A_n\}$ and $\widehat{B}_n = \{B_n(ih_n)\}_{i=0}^{N_n}$. Then the first equation in (1.1) can be written as

$$\widehat{v}_n = \widehat{A}_n\widehat{u}_n + h_n\widehat{B}_n * \widehat{u}_n. \tag{3.10}$$

Using Lemma 2.3 we have, by (3.6) and (3.10),

$$\begin{aligned} -\widehat{B}_n * \widehat{u}_n &= \widehat{K}_n * (\lambda_0\widehat{I}_n - \widehat{A}_n)\widehat{u}_n \\ &= (\widehat{L}_n + h_n\widehat{L}_n * \widehat{K}_n) * (\lambda_0\widehat{I}_n - \widehat{A}_n)\widehat{u}_n \\ &= \lambda_0\widehat{L}_n * \widehat{u}_n - \widehat{L}_n * \widehat{A}_n\widehat{u}_n + \widehat{L}_n * (-h_n\widehat{B}_n * \widehat{u}_n) \\ &= \lambda_0\widehat{L}_n * \widehat{u}_n - \widehat{L}_n * \widehat{v}_n, \end{aligned}$$

where $\widehat{I}_n$ is the identity operator on $\widehat{X}_n$. Substituting this identity into (3.10) we obtain $\widehat{v}_n = \widehat{A}_n\widehat{u}_n + h_n(\widehat{L}_n * \widehat{v}_n - \lambda_0\widehat{L}_n * \widehat{u}_n)$ whose kth component leads us to the equality

$$(\widehat{u}_n)_{k+1} = T_n(\widehat{u}_n)_k + h_n^2(\widehat{L}_n * \widehat{v}_n - \lambda_0\widehat{L}_n * \widehat{u}_n)_k$$

for $0 \le k \le N_n - 1$. Solving this equation we find

$$\begin{aligned} (\widehat{u}_n)_k &= T_n^k(\widehat{u}_n)_0 + h_n^2(\widehat{T}_n * (\widehat{L}_n * \widehat{v}_n - \lambda_0\widehat{L}_n * \widehat{u}_n))_{k-1} \\ &= T_n^k x_n + (\widehat{V}_n * h_n\widehat{v}_n)_{k-1} - \lambda_0 h_n(\widehat{V}_n * \widehat{u}_n)_{k-1} \end{aligned}$$

for $1 \le k \le N_n$, where $\widehat{T}_n = \{T_n^k\}_{k=0}^{N_n} \in B(\widehat{X_n})$ and $\widehat{V}_n = \{V_{n,k}\}_{k=0}^{N_n} \in B(\widehat{X_n})$. Here we have used the equality $\widehat{V}_n = h_n\widehat{T}_n * \widehat{L}_n$ and Lemma 2.3. The equality above implies (3.9), since $F_{n,0} = I_n$ and $\sum_{l=0}^{k-1} V_{n,k-1-l}F_{n,l+1} = \sum_{l=0}^{k-1} V_{n,k-l}F_{n,l} + V_{n,0}F_{n,k} - V_{n,k}$ for $1 \le k \le N_n$.

We shall prove in passing two properties of $\{V_{n,k}\}_{k=0}^{N_n}$.

Lemma 3.1. *If $\{z_n\}$ is a convergent sequence in $\{X_n\}$ with $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} V_{n,[t/h_n]}z_n = V(t)z$ uniformly on $[0,\tau]$.*

Proof. Since $V_{n,[t/h_n]}z_n = \sum_{l=0}^{[t/h_n]} \int_{lh_n}^{(l+1)h_n} T_n^{[t/h_n]-l}L_{n,l}z_n\,ds$ we find

$$\begin{aligned} \|V_{n,[t/h_n]}z_n - P_nV(t)z\|_n &\le Me^{\omega\tau}c_L(\tau)\|z_n\|_n h_n + Me^{\omega\tau}\phi_n(\tau) \\ &\quad + \int_0^\tau \sup_{t\in[s,\tau]} \|T_n^{[t/h_n]-[s/h_n]}P_nL(s)z - P_nT(t-s)L(s)z\|_n\,ds \end{aligned}$$

for $t \in [0,\tau]$, where $\phi_n(t) = \int_0^t \|L_{n,[s/h_n]}z_n - P_nL(s)z\|_n\,ds$ for $t \in [0,\tau]$. By (3.2) the integrand of the last term on the right-hand side tends to zero as $n \to \infty$. Lebesgue's convergence theorem implies that the last term vanishes as $n \to \infty$.

It remains to show that $\lim_{n\to\infty}\phi_n(\tau)=0$. By (3.4) and (3.6) we find

$$\begin{aligned}\phi_n(t) \le \int_0^\tau \|K_n([s/h_n]h_n)z_n - P_nK(s)z\|_n\,ds \\ + h_nc_K(\tau)c_L(\tau)\tau\|z_n\|_n + c_K(\tau)\int_0^t \phi_n(s)\,ds \\ + \int_0^\tau\int_0^s \|K_n(([s/h_n]-[r/h_n])h_n)P_nL(r)z - P_nK(s-r)L(r)z\|_n\,dr\,ds\end{aligned}$$

for $t\in[0,\tau]$. By Fubini's theorem and a change of variables the last term on the right-hand side is equal to

$$\int_0^\tau\Big(\int_0^{\tau-r} \|K_n(([(s+r)/h_n]-[r/h_n])h_n)P_nL(r)z - P_nK(s)L(r)z\|_n\,ds\Big)dr.$$

By (3.5) the sequence in parentheses converges to zero as $n\to\infty$. It follows from Lebesgue's convergence theorem that the last term tends to zero as $n\to\infty$. By (3.5) again the first term vanishes as $n\to\infty$. Therefore, there exists a null sequence $\{\eta_n\}$ such that $\phi_n(t)\le\eta_n + c_K(\tau)\int_0^t\phi_n(s)\,ds$ for $t\in[0,\tau]$. By Gronwall's inequality we have $\phi_n(\tau)\le\eta_ne^{c_K(\tau)\tau}$, which tends to zero as $n\to\infty$. ■

LEMMA 3.2. $\mathrm{Lip}_V(\tau) := \sup\{\|V_{n,k}-V_{n,k-1}\|_n/h_n : 1\le k\le N_n, n\ge n_1\} < \infty$.

Proof. Since $\widehat{V}_n = h_n\widehat{T}_n * \widehat{L}_n$ we have, by (iv) of Lemma 2.3,

$$\|(V_{n,k}x_n - V_{n,k-1}x_n)/h_n\|_n \le Me^{\omega\tau}(c_L(\tau)\|x_n\|_n + \widetilde{c}_L(\tau))$$

for $1\le k\le N_n$. Here we have used (3.7) and (3.8). The desired conclusion follows by applying Lemma 2.1 to the operator $C_n(t)$ defined by $C_n(t) = (V_{n,[t/h_n]} - V_{n,[t/h_n]-1})/h_n$ for $t\in[h_n,\tau]$ and the zero operator on X_n for $t\in[0,h_n)$. ■

End of proof of Theorem 3.1. Since $\|V_{n,0}\|_n\le h_nc_L(\tau)\to0$ as $n\to\infty$, it is easily seen that $(I_n-V_{n,0})^{-1}\in B(X_n)$ exists for sufficiently large n, and $\lim_{n\to\infty}\|(I_n-V_{n,0})^{-1}-I_n\|_n=0$. By (3.3) and (3.9) it suffices to estimate the difference between $T(t)x+\frac{d}{dt}(V*R)(t)x - V(t)x - \lambda_0(V*R)(t)x$ and

$$T_n^kx_n + \sum_{l=0}^{k-1}(V_{n,k-l}-V_{n,k-1-l})F_{n,l}x_n - V_{n,k}x_n - \lambda_0h_n\sum_{l=0}^{k-1}V_{n,k-1-l}F_{n,l}x_n$$

for $k=[t/h_n]$, $t\in[0,\tau]$ and sufficiently large n. Here and subsequently, the sum of the form $\sum_{l=0}^{-1}(\ldots)$ is meant to be zero.

First we have

$$\begin{aligned}(3.11)\quad \Big\|h_n\sum_{l=0}^{[t/h_n]-1}V_{n,[t/h_n]-1-l}F_{n,l}x_n - P_n(V*R)(t)x\Big\|_n \\ \le \sum_{l=0}^{[t/h_n]-1}\int_{lh_n}^{(l+1)h_n}\|V_{n,[t/h_n]-1-l}(F_{n,l}x_n - P_nR(s)x)\|_n\,ds \\ + \sum_{l=0}^{[t/h_n]-1}\int_{lh_n}^{(l+1)h_n}\|(V_{n,[t/h_n]-1-l}-V_{n,[t/h_n]-l})P_nR(s)x\|_n\,ds \\ + \sum_{l=0}^{[t/h_n]-1}\int_{lh_n}^{(l+1)h_n}\|V_{n,[t/h_n]-l}P_nR(s)x - P_nV(t-s)R(s)x\|_n\,ds \\ + h_n\beta\sup\{\|V(t-s)R(s)x\| : 0\le s\le t\le\tau\}\end{aligned}$$

for $t\in[0,\tau]$. If $0\le t<h_n$ then the first three terms are equal to zero. We estimate them in the case where $h_n\le t\le\tau$. The first term on the right-hand side is majorized by

$$c_V(\tau)\int_0^t\|F_{n,[s/h_n]}x_n - P_nR(s)x\|_n\,ds,$$

where $c_V(\tau):=\sup\{\|V_{n,k}\|_n : 0\le k\le N_n,\ n\ge n_1\}<\infty$, which follows from condition (a$_1$) and (3.7). Lemma 3.2 implies that the second term is bounded by $\tau\,\mathrm{Lip}_V(\tau)h_n\beta\sup\{\|R(s)x\| : s\in[0,\tau]\}$. The third term is dominated by

$$\int_0^\tau\sup_{t\in[s,\tau]}\|V_{n,[t/h_n]-[s/h_n]}P_nR(s)x - P_nV(t-s)R(s)x\|_n\,ds,$$

which tends to zero as $n\to\infty$, by Lemma 3.1 and Lebesgue's convergence theorem.

Next, let $g\in C^1([0,\tau];X)$. We have, by Lemma 3.2,

$$\begin{aligned}(3.12)\quad \Big\|\sum_{l=0}^{[t/h_n]-1}(V_{n,[t/h_n]-l}-V_{n,[t/h_n]-1-l})F_{n,l}x_n - P_n\frac{d}{dt}(V*R)(t)x\Big\|_n \\ \le \mathrm{Lip}_V(\tau)\sum_{l=0}^{[t/h_n]-1}\int_{lh_n}^{(l+1)h_n}(\|F_{n,l}x_n - P_nR(s)x\|_n \\ + \|P_nR(s)x - P_ng(lh_n)\|_n)\,ds\end{aligned}$$

$$+\Big\|\sum_{l=0}^{[t/h_n]-1}(V_{n,[t/h_n]-l}-V_{n,[t/h_n]-1-l})P_n g(lh_n)-P_n\frac{d}{dt}(V*g)(t)\Big\|_n$$
$$+\Big\|P_n\frac{d}{dt}(V*(g-R(\cdot)x))(t)\Big\|_n$$

for $t\in[0,\tau]$. By (ii) of Remark 3.1 the last term is less than or equal to $\beta\|V\|_{\mathrm{Lip}[0,\tau]}\int_0^\tau\|g(s)-R(s)x\|\,ds$. We have

$$\sum_{l=0}^{[t/h_n]-1}V_{n,[t/h_n]-1-l}P_n(g((l+1)h_n)-g(lh_n))$$
$$=\int_0^{[t/h_n]h_n}V_{n,[t/h_n]-1-[s/h_n]}P_n g'(s)\,ds,$$

which converges to $\int_0^t V(t-s)g'(s)\,ds$ uniformly on $[0,\tau]$ as $n\to\infty$, by the same arguments as in the estimates of the second and third terms on the right-hand side of (3.11). Since

$$\sum_{l=0}^{[t/h_n]-1}(V_{n,[t/h_n]-l}-V_{n,[t/h_n]-1-l})P_n g(lh_n)$$
$$=V_{n,[t/h_n]}P_n g(0)-V_{n,0}P_n g([t/h_n]h_n)$$
$$+\sum_{l=0}^{[t/h_n]-1}V_{n,[t/h_n]-1-l}(P_n g((l+1)h_n)-P_n g(lh_n))$$

and $V(t)g(0)+\int_0^t V(t-s)g'(s)\,ds=(d/dt)(V*g)(t)$ it follows that the second term on the right-hand side of (3.12) tends to zero uniformly on $[0,\tau]$ as $n\to\infty$.

We combine these inequalities and use Lemma 3.1 and (3.2). This yields

$$\|F_{n,[t/h_n]}x_n-P_nR(t)x\|_n\le\varepsilon_n+c_1\int_0^\tau\|g(s)-R(s)x\|ds$$
$$+c_2\int_0^t\|F_{n,[s/h_n]}x_n-P_nR(s)x\|_n\,ds$$

for $t\in[0,\tau]$ and $g\in C^1([0,\tau];X)$, where $\{\varepsilon_n\}$ is a null sequence of positive numbers and c_i are positive constants. The proof is completed by an application of Gronwall's inequality, since $C^1([0,\tau];X)$ is dense in $L^1(0,\tau;X)$. ∎

4. Application to the backward Euler type discrete version. This section is devoted to the problem of convergence of the backward Euler type discrete scheme of (IE; u_0). This problem of parabolic type was studied by Thomée and Wahlbin [10].

We now define a sequence $\{U_{n,k}:n\ge1\text{ and }k\ge0\}$ in $B(X)$ by the recursion formula

$$\begin{cases}\dfrac{U_{n,k}-U_{n,k-1}}{h_n}=AU_{n,k}+\displaystyle\sum_{i=0}^{k-1}h_nB((k-i)h_n)U_{n,i}\quad\text{for }k=1,2,\ldots,\\ U_{n,0}=(I-h_nA)^{-1}.\end{cases}\tag{4.1}$$

Here $\{h_n\}$ is an arbitrary null sequence, A is the infinitesimal generator of a semigroup $\{T(t):t\ge0\}$ of class (C_0) on X satisfying $\|T(t)\|\le Me^{\omega t}$ for $t\ge0$ where $M\ge1$ and $\omega\ge0$, and $\{B(t):t\ge0\}$ is a family of bounded linear operators from Y to X such that $B(\cdot)x\in\mathrm{BV_{loc}}([0,\infty);X)$ for each $x\in Y$.

LEMMA 4.1. *For $\tau>0$ there exists a constant $M_\tau>0$ such that*

$$\sum_{i=1}^{k}\|B(t_i)x-B(t_{i-1})x\|\le M_\tau\|x\|_Y$$

for $0=t_0<t_1<\ldots<t_k=\tau$, $k\ge1$ and $x\in Y$.

Proof. Denote by Z the Banach space consisting of all sequences $z=\{x_i\}$ in X such that $\sum_{i=1}^\infty\|x_i\|<\infty$ equipped with the norm $\|z\|_Z=\sum_{i=1}^\infty\|x_i\|$ for $z\in Z$.

For each partition $P=\{0=t_0<t_1<\ldots<t_k=\tau\}$ of $[0,\tau]$ define the linear operator T_P from Y to Z by

$$T_Px=\{B(t_1)x-B(t_0)x,B(t_2)x-B(t_1)x,\ldots,B(t_k)x-B(t_{k-1})x,0,0,\ldots\}$$

for $x\in Y$. Each T_P is easily seen to be bounded. Our assumption on B implies $\sup_P\|T_Px\|_Z<\infty$ for each $x\in Y$. By the uniform boundedness principle there exists $M_\tau>0$ such that $\|T_P\|_{Y,Z}\le M_\tau$ for all partitions P of $[0,\tau]$, from which the desired result follows readily. ∎

THEOREM 4.1. $\lim_{n\to\infty}U_{n,[t/h_n]}x=R(t)x$ *holds for $t\ge0$ and $x\in X$, and the convergence is uniform on every compact subinterval of $[0,\infty)$.*

Proof. For each $n\ge1$ with $\omega h_n\le1/2$, we set $X_n=X$, $P_n=I$, $T_n=(I-h_nA)^{-1}$, $B_n(t)=B(t+h_n)(I-h_nA)^{-1}$ for $t\ge0$, and $F_{n,k}=(I-h_nA)U_{n,k}$ for $k\ge0$. We note here that $A_n=A(I-h_nA)^{-1}$ in this setting. Relation (1.1) is then derived from the recursion formula (4.1). All the other assumptions of Theorem 3.1 except for (iii) and (iv) of (b_2) can be easily checked. Condition (iii) of (b_2) is a direct consequence of Lemma 4.1. To check (iv) of (b_2), let $x\in Y$ and $x_n\in X$ satisfy $\lim_{n\to\infty}x_n=x$ and

$\lim_{n\to\infty} A(I-h_nA)^{-1}x_n = Ax$. For each $\tau > 0$ we have

$$\int_0^\tau \|B_n(s)x_n - B(s)x\|\,ds \le \tau \sup\{\|B(t)\|_{Y,X} : t\in[0,\tau+1]\}\|(I-h_nA)^{-1}x_n - x\|_Y + \int_0^\tau \|B(s+h_n)x - B(s)x\|\,ds$$

for $n \ge 1$. Since $B(\cdot)x \in L^1_{\mathrm{loc}}([0,\infty);X)$ the last term tends to zero as $n\to\infty$. It follows that condition (iv) of (b_2) is satisfied. Theorem 3.1 therefore asserts that $\lim_{n\to\infty} F_{n,[t/h_n]}x = R(t)x$, which implies in turn that $\lim_{n\to\infty} U_{n,[t/h_n]}x = R(t)x$ for $t\ge 0$ and $x\in X$. ■

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Localizations of partial differential operators and surjectivity on real analytic functions

by

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Abstract. Let $P(D)$ be a partial differential operator with constant coefficients which is surjective on the space $A(\Omega)$ of real analytic functions on an open set $\Omega\subset\mathbb{R}^n$. Then $P(D)$ admits shifted (generalized) elementary solutions which are real analytic on an arbitrary relatively compact open set $\omega\subset\subset\Omega$. This implies that any localization $P_{m,\Theta}$ of the principal part P_m is hyperbolic w.r.t. any normal vector N of $\partial\Omega$ which is noncharacteristic for $P_{m,\Theta}$. Under additional assumptions P_m must be locally hyperbolic.

Surjectivity criteria for partial differential operators have been obtained in most of the classical spaces of (generalized) functions in the fifties and early sixties. However, the basic question of when

$$P(D): A(\Omega)\to A(\Omega) \quad \text{is surjective,} \tag{0.1}$$

remained open. Here $P(D)$ is a partial differential operator with constant coefficients, $\Omega\subset\mathbb{R}^n$ is an open set and $A(\Omega)$ is the space of real analytic functions on Ω.

Piccinini [37] showed that the heat equation is not surjective on $A(\mathbb{R}^3)$ as was conjectured by Cattabriga–de Giorgi [12]. Then Hörmander [21] characterized (0.1) for convex sets Ω by means of a Phragmén–Lindelöf condition valid on the complex variety of P. Since then Hörmander's method has been adapted by several authors for further studies on this problem (Miwa [36], Andreotti–Nacinovich [3], Zampieri [40], Braun [9]), and on the related surjectivity problem on nonquasianalytic Gevrey classes (Zampieri [41], Braun–Meise–Vogt [10, 11]).

Specifically, (0.1) was proved to hold for operators having a locally hyperbolic principal part P_m if $\Omega=\mathbb{R}^n$ (see Andersson [2] and Hörmander [21]) or if Ω is convex and additional conditions on the local propagation cones of

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