Now for a numerical polynomial \( p_n(z) \) of degree \( n \) the following inequality of S. N. Bernstein holds:

\[
|p_n(z)| \leq \left( \frac{a' + b'}{a} \right)^n \sup_{|t| \leq a} |p_n(t)|,
\]

where \( t \) is real, \( z \) complex and \( a' \) and \( b' \) are the semiaxes of the ellipse with foci \( a \) and \( -a \), passing through the point \( z \).

We fix an \( a > 0 \) such that

\[
g \left( 1 + \frac{2}{a} \right) < 1.
\]

There are (uniquely determined) polynomials \( P_n(z), Q_n(z), \ldots \) defined on \( X + iX \) with values in \( Y + iY \) (see [1]) and with the property

\[
P_n(z) = P_1(z), \quad Q_n(z) = Q_1(z), \quad \ldots \quad \text{if} \ z \in X.
\]

We define the set \( S_\delta(U) \): the point \( z_1 + \delta z_2 (z_1, z_2, X) \) belongs to the set \( S_\delta(U) \) if \( z_1 + \delta z_2 \in U \) for every real \( t \), \( -a \leq t \leq a \). The set \( S_\delta(U) \) is open (see [1]) and the following inequality holds whenever \( z \in S_\delta(U) \):

\[
|Q_j(z)| \leq 2C g \left( 1 + \frac{2}{a} \right)^{j/2} \left( 1 + \frac{2}{a} \right)^{j/2} (j = 1, 2, \ldots).
\]

This follows easily from Bernstein's inequality if we set \( z = z_1 + \delta z_2 \) and note that \( |Q_j(z_1 + \delta z_2)| \leq 2C g \), if \( -a \leq t \leq a \), \( j = 1, 2, \ldots \).

The sum of the series

\[
P_1(z) + \frac{Q_1(z)}{z} + \frac{Q_1(z)}{z^2} + \ldots
\]

is bounded and \( G \)-differentiable in \( S_\delta(U) \). By definition this sum is an analytic operation (see [2], p. 51).

Norm. An analytic operation may be uniformly approximated by polynomials only locally. This is an easy consequence of the existence of analytic operations defined on the whole space \( X \) and unbounded in a sphere. Take for example

\[
X = E^n, \quad x = \sum_{j=1}^{N} x_j, \quad \sum_{j=1}^{N} |x_j| < \infty, \quad Z = E^1, \quad F(x) = \sum_{j=1}^{N} e^{kx^{N-k}}.
\]

This operation is analytic, as can be shown readily by extension to the complex case.

References


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