A characterization of analytic operations in real Banach spaces

by

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This note contains a generalization of a well-known theorem of S. N. Bernstein: A real valued function $F(x)$ defined in an interval $\langle a,b\rangle$ is analytic in it if and only if there exist polynomials $P_n(x)$ and two constants C and q such that $P_n(x)$ is of degree n, $C>0, 1>q>0$, and such that the inequality $|F(x)-P_n(x)|<Cq^n$ holds for every $x\epsilon\langle a,b\rangle$ and $n=1,2,\ldots$

Let us suppose now that X is a real Banach space and $F(x)$ is an analytic operation defined in an open set $G\subset X$ with values in a real Banach space Y. For the definitions see [1].

We shall prove the following proposition:

An operation $F(x)$ is analytic in G, if and only if for every $x_0\epsilon G$ there exists a neighbourhood $U(x_0)$ of x_0, two numbers C_{x_0}, q_{x_0}, and a sequence of polynomials $P_n(x)$ such that $P_n(x)$ is of degree n, $C_{x_0}>0, 1>q_{x_0}>0$, and such that the inequality $\|F(x)-P_n(x)\|<C_{x_0}(q_{x_0})^n$ holds for every $x\epsilon U(x_0)$ and $n=1,2,\ldots$ [1]).

Proof. The condition is necessary. This follows from the definition of an analytic operation and from the fact that a power series which converges in an open set converges there locally normally (see [1], Theorem 6.2).

Now we prove that it is sufficient as well.

Let us fix an x_0; set $U(x_0)=U, C_{x_0}=C, q_{x_0}=q$ and let us write for $x\epsilon U$

$$F(x)=P_1(x)+[P_2(x)-P_1(x)]+[P_3(x)-P_2(x)]+\ldots$$
$$=P_1(x)+Q_1(x)+Q_2(x)+\ldots$$

Apparently we have

$$|Q_j(x)|<2Cq^j,\qquad j=1,2,\ldots,\qquad x\epsilon U.$$

[1]) I am indebted to Prof. W. Orlicz for calling my attention to this problem.

Now for a numerical polynomial $p_n(\zeta)$ of degree n the following inequality of S. N. Bernstein holds:

$$|p_n(\zeta)|\leqslant\left(\frac{a'+b'}{a}\right)^n\sup_{|t|\leqslant a}|p_n(t)|,$$

where t is real, ζ complex and a' and b' are the semiaxes of the ellipse with foci a and $-a$, passing through the point ζ.

We fix an $a>0$ such that

$$q\left(1+\frac{2}{a}\right)<1.$$

There are (uniquely determined) polynomials $P_1^*(z), Q_1^*(z), Q_2^*(z),\ldots$ defined on $X+iX$ with values in $Y+iY$ (see [1]) and with the property

$$P_1^*(z)=P_1(z),\quad Q_1^*(z)=Q_1(z),\quad\ldots\quad\text{if}\quad z\epsilon X.$$

We define the set $\mathfrak{Z}_a(U)$: the point $x_1+ix_2\,(x_1,x_2\epsilon X)$ belongs to the set $\mathfrak{Z}_a(U)$ if $x_1+tx_2\epsilon U$ for every real t, $-a\leqslant t\leqslant a$. The set $\mathfrak{Z}_a(U)$ is open (see [1]) and the following inequality holds whenever $z\epsilon\mathfrak{Z}_a(U)$:

$$\|Q_j(z)\|\leqslant 2C\left(1+\frac{2}{a}\right)\left[q\left(1+\frac{2}{a}\right)\right]^j\qquad(j=1,2,\ldots).$$

This follows easily from Bernstein's inequality if we set $z=x_1+ix_2$ and note that $\|Q_j(x_1+tx_2)\|\leqslant 2Cq^j$, if $-a\leqslant t\leqslant a$, $j=1,2,\ldots$

The sum of the series

$$P_1^*(z)+Q_1^*(z)+Q_2^*(z)+\ldots$$

is bounded and G-differentiable in $\mathfrak{Z}_a(U)$. By definition this sum is an analytic operation (see [2], p. 81).

NOTE. An analytic operation may be uniformly approximated by polynomials only locally. This is an easy consequence of the existence of analytic operations defined on the whole space X and unbounded in a sphere. Take for example

$$X=l^{(1)},\qquad x=\{x_j\},\qquad\sum_{j=1}^{\infty}|x_j|<\infty,\qquad Y=E_1,\qquad F(x)=\sum_{j=1}^{\infty}e^{j(x_j-1)}.$$

This operation is analytic, as can be shown readily by extension to the complex case.

References

[1] A. Alexiewicz and W. Orlicz, *Analytic operations in real Banach spaces*, this volume, p. 57-78.

[2] E. Hille, *Functional analysis and semigroups*, American Mathematical Society Colloquium Publications (1948).

(Reçu par la Rédaction le 13.5.1953)