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On a theorem of Saks for abstract polynomials

by

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The purpose of this Note[1]) is to generalize a theorem due to A. Alexiewicz ([1], Theorem 1) dealing with the structure of linear operations depending on a parameter. Because of the close connection of this Note with paper [1] I use throughout the definitions adopted there.

In particular $(T,\mathfrak{C},\mu)$ denotes a measure space, on which μ is σ-additive and $\mu(T)<\infty$. $U(x,t)$ will stand for an operation from $X\times T$ to an F-space Y, with the following properties:

1. $U(x,t)$ is Bochner measurable for fixed x;
2. $U(x,t)$ is a polynomial of degree m in x for fixed t;
3. $x\to x_0$ implies $U(x,t)\overset{as}{\to}U(x_0,t)$

(unlike [1]).

For a set $R\subset Y$ we shall denote by $\Theta(x)$ the set of those elements t for which $U(x,t)\epsilon R$.

It is easily seen that Lemmas 1 and 2 of [1] still hold in the case considered now.

The following Lemma will be proved in place of Lemma 3:

LEMMA. *Let the set $R\subset Y$ be linear and measurable (B). If the set*

$$W=\underset{x}{E}\{\mu(H-\Theta(x))<\varepsilon\}$$

is of the second category, then the set

$$V=\underset{x}{E}\{\mu(H-\Theta(x))<(m+1)\varepsilon\}$$

is residual.

Proof. It is known ([2], p. 50-51) that we can represent the polynomial operation $U(x,t)$ in the canonical form

$$(1)\qquad U(x,t)=U_0(x,t)+U_1(x,t)+\ldots+U_m(x,t),$$

[1]) I am indebted to Professor A. Alexiewicz for his help in the preparation of this Note.

where $U_k(x,t)$ $(k=0,1,\ldots,m)$ is a homogeneous polynomial (power) of degree k, moreover

(2) $$U_k(x,t)=a_{k0}U(0\cdot x,t)+a_{k1}U(1\cdot x,t)+\ldots+a_{km}U(m\cdot x,t),$$

with a_{ki} independent of U, x and t.

Setting for every set $R\subset Y$

$$\Theta_k(x)=\underset{t}{E}\{U_k(x,t)\epsilon R\},\qquad \Theta(kx)=\underset{t}{E}\{U(kx,t)\epsilon R\},$$

we obtain from (1) and (2)

$$\bigcap_{k=0}^{m}\Theta_k(x)\subset\Theta(x),\qquad \bigcap_{i=0}^{m}\Theta(ix)\subset\Theta_k(x),$$

and therefore

$$H-\Theta(x)\subset\bigcup_{k=0}^{m}\big(H-\Theta(kx)\big).$$

Then for $t\epsilon\bigcap_{k=0}^{m}\Theta(kx)$ we have

(3) $$\mu\big(H-\Theta(x)\big)\leqslant\sum_{k=0}^{m}\mu\big(H-\Theta(kx)\big).$$

Since the set W is measurable (B) and is of the second category, there exists a sphere $K(0,r)$ with centre 0[2]) and radius r in which this set is residual. Therefore

$$K(0,r)-M\subset W,$$

where M denotes a set of the first category. Setting

$$N=\bigcup_{k=0}^{m}kM\,^{3}),\qquad K=K\left(0,\frac{r}{m}\right)-N$$

we see that if $x\epsilon K$ then

$$kx\epsilon K(0,r)-M \text{ for } k=0,1,\ldots,m.$$

Since we have $K\subset K(0,r)-M\subset W$, therefore if $x\epsilon K$, then $\mu\big(H-\Theta(kx)\big)<\varepsilon$ for $k=0,1,\ldots,m$. By formula (3) this gives: if $x\epsilon K, t\epsilon\bigcap_{k=0}^{m}\Theta(kx)$, then $\mu\big(H-\Theta(x)\big)<(m+1)\varepsilon$. Therefore

$$K\subset\underset{x}{E}\{\mu\big(H-\Theta(x)\big)<(m+1)\varepsilon\}.$$

[2]) In the contrary case it is sufficient to consider the polynomial operation $\bar{U}(x,t)=U(x+x_0,t)$.

[3]) kM denotes the set of elements kx where $x\epsilon M$.

Since

$$U(\lambda x,t)=\sum_{k=0}^{m}\lambda^k U_k(x,t),$$

we have, for $\lambda\neq 0$,

$$\bigcap_{k=0}^{m}\Theta_k\left(\frac{x}{\lambda}\right)\subset\Theta(x).$$

Therefore, for λ sufficiently large,

$$\mu\big(H-\Theta(x)\big)\leqslant\sum_{k=0}^{m}\mu\left(H-\Theta_k\left(\frac{x}{\lambda}\right)\right)<(m+1)\varepsilon.$$

This Lemma being proved we see that Lemma 4 of [1] is also true in this case.

Now we can prove, in the same way as in the paper of A. Alexiewicz, the following

THEOREM. *Let the set R be linear and measurable (B). Then there exists a decomposition $T=A+B$ into two measurable sets and a residual set $Z\subset X$ such that*

1. *$U(x,t)\epsilon R$ for any $x\epsilon X$ a. e.*[4]) *in A,*
2. *$U(x,t)$ non ϵR for any $x\epsilon Z$ a. e. in B.*

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[4]) a. e. = almost everywhere.