On Leżański's determinants of linear equations in Banach spaces

by

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The subject of this paper is Leżański's [2,3] theory of determinants of linear operators in Banach spaces. The main purpose of this paper is to prove that the determinant is a multiplicative functional, *i. e.* that, roughly speaking, the determinant of the superposition of two operators is the product of the determinants of those operators. This result was obtained by Leżański [3] (p. 23) under the very restrictive hypothesis that the operators are commutative. Therefore Leżański's theorem does not contain, as a particular case, the assertion that the determinant of the product of two non-commutative finite square matrices is equal to the product of the determinants of those matrices. Similarly this theorem of Leżański does not contain, as a particular case, the earlier result of Leżański [2] (p. 274) stating that the determinant is a multiplicative functional in the case of spaces $\boldsymbol{m}$ and $\boldsymbol{l}$.

In this paper I shall prove the theorem on the multiplication of determinants under a weaker condition, which is always satisfied in case of integral operators and of matrix operators.

In order to make this paper clear for those readers who do not know Leżański's papers [2,3], I give a brief summary of Leżański's theory. This summary differs in some points from the original theory of Leżański [2,3]. The disparity will be discussed at the end of this paper (§ 7).

The main result of §§ 1-2 is a theorem on the logarithm in Banach algebras which plays an essential part in the proof of the theorem on multiplication of determinants which is given in § 5. §§ 3-4, containing a summary of Leżański's theory, can be read independently of §§ 1-2. The verification of the hypothesis of the theorem on multiplication of determinants being fulfilled in practice is given in § 6.

All Banach spaces under consideration can be real or complex. Consequently all scalars are real or complex numbers respectively. However, it is sometimes convenient to consider certain power series of λ as holomorphic functions of the complex variable λ, also in the case where the spaces under consideration are real.

§ 1. A combinatorical lemma. By a *word* we shall understand any finite sequence each term of which is one of the letters A, B, C. We shall write words W in the form of symbolic products:

$$W = A_1 A_2 \ldots A_n$$

where $A_i = A$ or B or C for $i = 1, 2, \ldots, n$ $(n > 0)$.

The number n (the length of W) will be denoted by $l(W)$.

If $W = A_1 A_2 \ldots A_n$ and $V = B_1 B_2 \ldots B_m$ are words, then WV is the word $A_1 A_2 \ldots A_n B_1 B_2 \ldots B_m$. If W is a word, then, by induction,

$$1 \cdot W = W \quad \text{and} \quad (m+1) \cdot W = (m \cdot W) W \quad \text{for } m = 1, 2, \ldots$$

The greatest positive integer m such that $W = m \cdot V$ where V is a word will be denoted by $d(W)$. Of course, $d(W)$ is a divisor of $l(W)$.

If $W = A_1 A_2 \ldots A_n$ is a word, then τW (or $\tau^1 W$) denotes the word $A_2 \ldots A_n A_1$. By induction, $\tau^{k+1} W = \tau(\tau^k W)$ for $k = 1, 2, \ldots$

Notice that if $\tau^k W = W$, then $k = mq$ where $q = l(W)/d(W)$ and m is a positive integer. Conversely, if $k = mq$ (in particular if $k = l(W)$), then $\tau^k W = W$.

If W is any word, then W^* denotes the word obtained from W by replacing all occurences of the letter C by the pair AB. For instance, $(ABCBCA)^* = ABABBABA$.

Let W and V be two words. If there is a positive integer k such that $W^* = \tau^k V^*$, we write $W \sim V$. Obviously the relation $W \sim V$ is an equivalence relation. Consequently the set of all words can be decomposed into disjoint sets ω such that two words W, V belong to the same set ω if and only if $W \sim V$. The class of all such sets ω will be denoted by Ω.

Let $\omega \in \Omega$. The greatest integer l such that there is a word $W \in \omega$ with $l = l(W)$ will be denoted by $l(\omega)$. If $l(W) = l(\omega)$ $(W \in \omega)$, then the letter C does not appear in W. If $V \in \omega$ is another word such that $l(V) = l(\omega)$, then there is an integer k such that $W = \tau^k V$. Consequently $d(W) = d(V)$. The integer $d(W)$, where $W \in \omega$ and $l(W) = l(\omega)$, will be denoted by $d(\omega)$.

Let $\omega \in \Omega$. The greatest integer s such that there is a word $W \in \omega$ which contains s times the letter C will be denoted by $s(\omega)$. Obviously

$$0 \leqslant s(\omega) \leqslant \frac{1}{2} l(\omega).$$

The equation $s(\omega) = 0$ holds if and only if ω contains exactly one element, *viz.* either $\underbrace{AA\ldots A}_{l(\omega)\text{ times}}$ or $\underbrace{BB\ldots B}_{l(\omega)\text{ times}}$ (the "power" of A or B); in this case

$$d(\omega) = l(\omega).$$

If $\omega \in \Omega$, then ω_p will denote the set of all $W \in \omega$ such that the letter C appears in W exactly p times. Obviously ω_p is non-empty if and only if $0 \leqslant p \leqslant s(\omega)$.

If $\omega \in \Omega$ and $W \in \omega$, then $\tau^k W \in \omega$ for $k=1,2,\ldots$ Similarly if $W \in \omega_p$, then $\tau^k W \in \omega_p$ for $k=1,2,\ldots$

LEMMA. *Let $\omega \in \Omega$ and $0 \leqslant p \leqslant s(\omega)$. The set ω_p contains exactly*

$$\frac{l(\omega)-p}{d(\omega)}\binom{s(\omega)}{p}$$

words.

If $s(\omega)=0$, then $p=0$, $d(\omega)=l(\omega)$ and

$$\frac{l(\omega)-p}{d(\omega)}\binom{s(\omega)}{p}=1=\overline{\overline{\omega_p}}.$$

Suppose $s(\omega)\neq 0$. Let $W \in \omega$, $l(W)=l(\omega)$. The letters A and B appear in $W=A_1A_2\ldots A_{l(\omega)}$, but the letter C does not appear in W. We can suppose that $A_1=A$ (if not, we can take a permutation $\tau^k W$ instead of W). We recall that $d(W)=d(\omega)$.

Let S be the set of all integers i such that $A_i=A$ and $A_{i+1}=B$, and let $\mathfrak{Z}$ be the class of all p-element subsets of S. If $Z \in \mathfrak{Z}$, then W_Z denotes the word obtained from W by replacing each pair $A_iA_{i+1}(=AB)$, where $i \in Z$, by the letter C. Of course, $l(W_Z)=l(\omega)-p$ and $W_Z \in \omega_p$. More generally, $\tau^k W_Z \in \omega_p$ for $k=1,2,\ldots$ Conversely, if $V \in \omega_p$, then there are a set $Z \in \mathfrak{Z}$ and a positive integer $k \leqslant l(\omega)-p$ such that $V=\tau^k W_Z$. Consequently the number $\overline{\overline{\omega_p}}$ is the number of all words $\tau^k W_Z$ where $k=1,2,\ldots,l(\omega)-p$, and $Z \in \mathfrak{Z}$.

The word $\tau^k W_Z$ is determined by the pair (k,Z). The number of all the pairs (k,Z), where $k=1,2,\ldots,l(\omega)-p$ and $Z \in \mathfrak{Z}$, is equal to

$$\big(l(\omega)-p\big)\binom{s(\omega)}{p}.$$

Hence it suffices to prove that, for a given pair (k,Z), there exist exactly $d(\omega)$ pairs (k',Z') such that $\tau^k W_Z=\tau^{k'}W_{Z'}$. Consequently, it suffices to prove that, for given $Z \in \mathfrak{Z}$, there are exactly $d(\omega)$ pairs (k',Z') (where $1 \leqslant k' \leqslant l(\omega)-p$ and $Z' \in \mathfrak{Z}$) such that $W_Z=\tau^{k'}W_{Z'}$.

Suppose $Z=(j_1,j_2,\ldots,j_p) \in \mathfrak{Z}$. Let $Z_m=(j_1+qm, j_2+qm,\ldots,j_p+qm)$ where

$$q=\frac{l(\omega)}{d(\omega)}=\frac{l(W)}{d(W)},$$

and the integers in the brackets are reduced mod $l(\omega)$ to the interval $\langle 1, l(\omega)\rangle$. Obviously $Z_m \in \mathfrak{Z}$ for $m=1,2,\ldots,d(\omega)$. Let

$$k_m=qm - (\text{the number of all the integers } j \in Z_m \text{ such that } j<qm).$$

It is easy to see that $W_Z=\tau^{k_m}W_{Z_m}$ and $1 \leqslant k_m \leqslant l(\omega)-p$ for $m=1,2,\ldots,d(\omega)$.

On the other hand, if $W_Z=\tau^{k'}W_{Z'}$ $(1 \leqslant k' \leqslant l(\omega)-p,\ Z' \in \mathfrak{Z})$ then $W=(W_Z)^*=(\tau^{k'}W_{Z'})^*=\tau^k W$, where

$k=k'+$(the number of all $j<k'$ such that the j^{th} term of $W_{Z'}$ is C).

Consequently $k=qm$ where $q=l(W)/d(W)$ and m is one of the integers $1,2,\ldots,d(\omega)$. Moreover $k'=k_m$. Since $\tau^{k'}W_{Z'}=W_Z=\tau^{k_m}W_{Z_m}$, we find that $W_{Z'}=W_{Z_m}$. Hence $Z'=Z_m$.

We have proved that the equation $W_Z=\tau^{k'}W_{Z'}$, $(1 \leqslant k' \leqslant l(\omega)-p$, $Z' \in \mathfrak{Z})$ holds if and only if there is an integer m $\big(1 \leqslant m \leqslant d(\omega)\big)$ such that $(k',Z')=(k_m,Z_m)$. Since $k_m<k_{m+1}$, the number of all such pairs (k',Z') is equal to $d(\omega)$, q. e. d.

§ 2. The logarithm in Banach algebras. Let $\mathfrak{A}$ be a Banach algebra (non-commutative, in general). Let E denote the unit of $\mathfrak{A}$ whenever it exists. If $\mathfrak{A}$ has no unit, let E be the abstract unit which can be added to $\mathfrak{A}$ in the well known way.

We define the *logarithm* $\log(E+A)$ for $A \in \mathfrak{A}$ by the power series

$$\log(E+A)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}A^n. \tag{1}$$

The mapping $\log(E+A)$ of $\mathfrak{A}$ into itself is defined by (1) only for such $A \in \mathfrak{A}$ that the series on the right side converges. In particular, $\log(E+A)$ is defined whenever $\|A\|<1$.

Let Φ be a linear[1]) functional on $\mathfrak{A}$, and let $A, B \in \mathfrak{A}$. We shall write

$$A \sim B \bmod \Phi$$

if

$$\Phi(A_1A_2\ldots A_n)=\Phi(A_2\ldots A_nA_1)$$

for each finite sequence $A_i=$ either A or B, *i. e.* if $\Phi(A'B')=\Phi(B'A')$ for arbitrary elements A', B' belonging to the least subalgebra generated by A and B. In particular, if $AB=BA$, then $A \sim B \bmod \Phi$.

THEOREM 1. *If $A, B \in \mathfrak{A}$, $A \sim B \bmod \Phi$, and if*

$$\|A\|+\|B\|+\|AB\|<1 \tag{2}$$

(e. g. if $\|A\| \leqslant 1/3$ and $\|B\| \leqslant 1/3$), then

$$\Phi\big(\log(E+A+B+AB)\big)=\Phi\big(\log(E+A)\big)+\Phi\big(\log(E+B)\big). \tag{3}$$

Obviously the left side of (3) can be written as $\Phi\Big(\log\big((E+A)(E+B)\big)\Big)$.

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[1]) The word "linear" always means "additive and continuous".

Let $C \in \mathfrak{A}$ be such that $\|A\|+\|B\|+\|C\|<1$. We have

$$\sum_{n=1}^{\infty} \frac{1}{n} (\|A\|+\|B\|+\|C\|)^n < \infty, \tag{4}$$

and

$$\log(E+A+B+C) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A+B+C)^n. \tag{5}$$

By (4) we can express the powers $(A+B+C)^n$ in the series (5) as the sums of some "words" $A_1 A_2 \ldots A_n$ where $A_i = A$ or B or C, and we can add these words (multiplied by real coefficients) in an arbitrary way. Notice that each word W appears in the development of (5) exactly once. More precisely, if $W \in \omega_p$ $(\omega \in \Omega)$, then W appears once in the development of $(A+B+C)^{l(\omega)-p}$ since $l(W) = l(\omega) - p$. Therefore the real coefficient of the word $W \in \omega_p$ is equal to

$$\frac{(-1)^{l(\omega)-p-1}}{l(\omega)-p}.$$

Consequently

$$\log(E+A+B+C) = \sum_{\omega \in \Omega} \sum_{p=0}^{s(\omega)} \sum_{W \in \omega_p} \frac{(-1)^{l(\omega)-p-1}}{l(\omega)-p} W, \tag{6}$$

and

$$\Phi(\log(E+A+B+C)) = \sum_{\omega \in \Omega} \sum_{p=0}^{s(\omega)} \sum_{W \in \omega_p} \frac{(-1)^{l(\omega)-p-1}}{l(\omega)-p} \Phi(W). \tag{7}$$

Now let $C = AB$. Since $A \sim B \bmod \Phi$, we have $\Phi(W) = \Phi(V)$ for $W, V \in \omega$. Let $\Phi(\omega)$ be the common value of all $\Phi(W)$, $W \in \omega$. It follows from the Lemma proved in § 1 that

$$\begin{aligned} \Phi(\log(E+A+B+AB)) &= \sum_{\omega \in \Omega} (-1)^{l(\omega)-1} \sum_{p=0}^{s(\omega)} \frac{(-1)^p}{l(\omega)-p} \sum_{W \in \omega_p} \Phi(\omega) \\ &= \sum_{\omega \in \Omega} (-1)^{l(\omega)-1} \sum_{p=0}^{s(\omega)} \frac{(-1)^p}{l(\omega)-p} \cdot \frac{l(\omega)-p}{d(\omega)} \binom{s(\omega)}{p} \Phi(\omega) \\ &= \sum_{\omega \in \Omega} \frac{(-1)^{l(\omega)-1}}{d(\omega)} \Phi(\omega) \sum_{p=0}^{s(\omega)} (-1)^p \binom{s(\omega)}{p}. \end{aligned} \tag{8}$$

We have

$$\sum_{p=0}^{s(\omega)} (-1)^p \binom{s(\omega)}{p} = 0 \quad \text{if} \quad s(\omega) > 0.$$

Therefore we can omit in the last sum $\sum_{\omega \in \Omega}$ all the ω with $s(\omega) > 0$. If $s(\omega) = 0$, then either $\omega = (A^n)$ or $\omega = (B^n)$, $n = 1, 2, \ldots$ We then have $l(\omega) = d(\omega) = n$ and

$$\sum_{p=0}^{0} (-1)^p \binom{0}{p} = 1.$$

Consequently

$$\begin{aligned} \Phi(\log(E+A+B+AB)) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Phi(A^n) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Phi(B^n) \\ &= \Phi(\log(E+A)) + \Phi(\log(E+B)), \end{aligned} \tag{9}$$

q. e. d.

If $\log(E+A)$ exists, we have

$$\Phi(\log(E+A)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Phi(A^n). \tag{10}$$

It is possible that the series (1) diverges, but the series (10) converges. Therefore we shall assume that the function $\Phi(\log(E+A))$ is defined for all $A \in \mathfrak{A}$ such that (10) converges.

With this convention, Theorem I can be generalized as follows: *If $A, B \in \mathfrak{A}$, $A \sim B \bmod \Phi$, and if*

$$\sum_{\omega \in \Omega} \frac{2^{s(\omega)}}{d(\omega)} |\Phi(\omega)| < \infty,$$

then the equation (3) *holds.*

The proof is the same. Instead of (5) we should now develop the series (10) where A is replaced by $A+B+C$.

§ 3. Leżański's determinant of an element of a Banach algebra.

To make clear the sense of the fundamental notions of Leżański's theory we recall the fundamental notions of the Fredholm theory of integral equations.

Let $T(s,t)$ be, for instance, a bounded measurable function in the unit square, and let X be the space of all bounded measurable functions on the unit interval. Consider the integral equation[2])

$$x(s) + \lambda \int T(s,t) x(t) dt = x_0(s), \tag{11}$$

where $x_0 \in X$ is given, and $x \in X$ is unknown. The following notions play an essential part in the Fredholm theory:

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[2]) In this section we write $\int$ instead of $\int_0^1$.

(12) $T^1(s,t)=T(s,t), \quad T^m(s,t)=\int T(s,r)T^{m-1}(r,t)\,dr$ for $n=2,3,\ldots,$

(13)[3] $$\sigma_1=\int T(s,s)\,ds,$$

(14) $$\sigma_n=\iint T(s,t)T^{n-1}(t,s)\,dt\,ds \quad \text{for} \quad n=2,3,\ldots,$$

(15) $$a_0=1, \quad a_n=\frac{1}{n}\int A_{n-1}(s,s)\,ds \quad \text{for} \quad n=1,2,\ldots,$$

(16) $$A_0(s,t)=T(s,t), \quad A_n(s,t)=a_nT(s,t)-\int T(s,r)A_{n-1}(r,t)\,dr$$ for $n=1,2,\ldots,$

(17) $$D(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n,$$

(18) $$A(s,t;\lambda)=\sum_{n=0}^{\infty}\lambda^nA_n(s,t).$$

$D(\lambda)$ is Fredholm's determinant of the equation (11). If $D(\lambda)\neq 0$, then

(19) $$x(s)=x_0(s)-\frac{\lambda}{D(\lambda)}\int A(s,t;\lambda)x_0(t)\,dt$$

is the solution of (11). On the other hand, if λ is sufficiently small, then

(20) $$x(s)=x_0(s)+\sum_{n=1}^{\infty}(-\lambda)^n\int T^m(s,t)x_0(t)\,dt$$

is the solution of (11). Further we have Plemelj's formulae

(21) $$a_n=\frac{1}{n!}\begin{vmatrix} \sigma_1 & n-1 & 0 & 0 & \ldots & 0 & 0 \\ \sigma_2 & \sigma_2 & n-2 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n-1} & \sigma_{n-2} & \cdot & \cdot & \cdot & \sigma_1 & 1 \\ \sigma_n & \sigma_{n-1} & \cdot & \cdot & \cdot & \sigma_2 & \sigma_1 \end{vmatrix}$$

and

(22) $$A_n(s,t)=\frac{1}{n!}\begin{vmatrix} T(s,t) & n & 0 & 0 & 0 & \ldots & 0 & 0 \\ T^2(s,t) & \sigma_1 & n-1 & 0 & 0 & \ldots & 0 & 0 \\ T^3(s,t) & \sigma_2 & \sigma_1 & n-2 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T^m(s,t) & \sigma_{n-1} & \sigma_{n-2} & \cdot & \cdot & \cdot & \sigma_1 & 1 \\ T^{m+1}(s,t) & \sigma_n & \sigma_{n-1} & \cdot & \cdot & \cdot & \sigma_2 & \sigma_1 \end{vmatrix},$$

[3]) In order that σ_1 be defined we must additionally suppose that $T(s,s)$ is integrable.

and Fredholm's formulae

(23) $$a_n=\frac{1}{n!}\int\ldots\int C\begin{pmatrix}s_1,\ldots,s_n\\ s_1,\ldots,s_n\end{pmatrix}ds_1\ldots ds_n,$$

and

(24) $$A_n(s,t)=\frac{1}{n!}\int\ldots\int C\begin{pmatrix}s, s_1,\ldots,s_n\\ t, s_1,\ldots,s_n\end{pmatrix}ds_1\ldots ds_n,$$

where

$$C\begin{pmatrix}s_1,\ldots,s_n\\ r_1,\ldots,r_n\end{pmatrix}=\begin{vmatrix} T(s_1,r_1) & \ldots & T(s_1,r_n) \\ \cdot & \cdot & \cdot \\ T(s_n,r_1) & \ldots & T(s_n,r_n)\end{vmatrix}.$$

From the point of view of Functional Analysis we can interpret each function $K(s,t)$ as a linear operator (in the space X) denoted by the same letter K. Instead of

$$y(s)=\int K(s,t)x(t)\,dt$$

we shall write $y=Kx$. Hence the equation (11) should be written

(25) $$(I+\lambda T)x=x_0,$$

where I is the identical mapping of X onto X. Instead of

$$K(s,t)=\int K_1(s,r)K_2(r,t)\,dr$$

we shall write $K=K_1K_2$, *i.e.* K is the superposition of K_1 and K_2. Consequently T^m (see (12)) is the superposition of n replicas of the operation T determined by the function $T(s,t)$. However, the function $T(s,t)$ plays in (14) a different part from that played by it in (12). It is not an operator here. Leżański [2] has remarked that the function $T(s,t)$ should be interpreted here as a linear functional F determined on a class $\mathfrak{K}$ of operators, such that for operators K of the integral type

(26) $$F(K)=\iint T(s,t)K(t,s)\,dt\,ds.$$

We can now write (14) in the form

(27) $$\sigma_n=F(T^{n-1}) \quad \text{for} \quad n=2,3,\ldots$$

Notice that σ_1 and a_1 are uniquely determined by the function $T(s,t)$, but not by the operator T. In fact, if we modify the function $T(s,t)$ on the diagonal of the unit square, then the operator T remains unchanged, but the number

$$\sigma_1=a_1=\int T(s,s)\,ds$$

can be arbitrarily changed. Consequently, the determinant $D(\lambda)$ is uniquely determined by the function $T(s,t)$, but not by the operator T.

After this introduction we can pass to the definition of the determinant of a linear equation

$$(25)\qquad (I+\lambda T)x=x_0$$

in an arbitrary Banach space X, where $x,x_0\epsilon X$ and T is a linear operator of X into X. The letter I always denotes the identical mapping.

It is obvious from the above considerations that besides the operator T we must also introduce a linear functional F defined on a linear class $\mathfrak{R}$ of operators. Obviously we wish to define σ_n by equation (27) for $n=2,3,\ldots$, and, by analogy,

$$(28)\qquad \sigma_1=F(T^0),\quad \text{where}\quad T^0=I.$$

Therefore we should assume that $I\epsilon\mathfrak{R}$ and that the superposition $K_1K_2\epsilon\mathfrak{R}$ whenever $K_1,K_2\epsilon\mathfrak{R}$, *i. e.* that $\mathfrak{R}$ is a Banach algebra of operators with the unit I. The problem of solving (25) for each x_0 is, roughly speaking, the problem of finding the inverse element $(I+\lambda T)^{-1}$ in the Banach algebra $\mathfrak{R}$. In this formulation of our problem the assumption that elements of $\mathfrak{R}$ are operators is not essential. Consequently we can generalize a part of Fredholm's theory of integral equations as follows (see Leżański [3], p. 14-18):

Let $\mathfrak{R}$ be a Banach algebra with the unit I, and let F be a linear functional on $\mathfrak{R}$. Let $T\epsilon\mathfrak{R}$. Set by induction

$$(29)\qquad a_0=1,\quad a_n=\frac{1}{n}F(B_{n-1})\quad \text{for}\quad n=1,2,\ldots,$$

$$(30)\qquad B_0=I,\quad B_n=a_nI-TB_{n-1}\quad \text{for}\quad n=1,2,\ldots,$$

$$(31)\qquad A_n=B_nT\quad \text{for}\quad n=0,1,2,\ldots$$

We have

$$(32)\qquad A_0=T,\quad A_n=a_nT-TA_{n-1}\quad \text{for}\quad n=1,2,\ldots,$$

which shows that formulae (29) and (32) are an abstract formulation of (15) and (16) respectively.

B_n is a polynomial of the variable T of a degree $\leqslant n$. A_n is a polynomial of the variable T of a degree $\leqslant n+1$. Therefore

$$(33)\qquad A_nT=TA_n\quad \text{and}\quad B_nT=TB_n\quad \text{for}\quad n=0,1,2,\ldots$$

The expression B_n has no analogue in the Fredholm theory since (in the case of $\mathfrak{R}=$ a class of operators) B_n is not an operator of the integral type.

We define $D(\lambda)$ by the formula (17). Analogously to (18) we set

$$(34)\qquad B(\lambda)=\sum_{n=0}^{\infty}\lambda^nB_n,$$

$$(35)\qquad A(\lambda)=\sum_{n=0}^{\infty}\lambda^nA_n.$$

Let $\varrho=\max(\|F\|,\|T\|)$. We find by an easy induction that

$$|a_n|\leqslant\varrho^n\quad \text{and}\quad B_n\leqslant(n+1)\varrho^n\quad \text{for}\quad n=0,1,2,\ldots$$

Hence, if $|\lambda|<1/\varrho$, then

$$(36)\qquad \sum_{n=0}^{\infty}|a_n\lambda^n|<\infty\quad \text{and}\quad \sum_{n=0}^{\infty}|\lambda|^n\|B_n\|<\infty,$$

and consequently

$$\sum_{n=0}^{\infty}|\lambda|^n\|A_n\|<\infty.$$

If λ is such that (36) holds, and if $D(\lambda)\neq0$, then by (30)

$$(37)\qquad (I+\lambda T)\frac{B(\lambda)}{D(\lambda)}=I,$$

i. e. (see (33))

$$(38)\qquad \frac{B(\lambda)}{D(\lambda)}=(I+\lambda T)^{-1}.$$

Consequently also

$$(39)\qquad (I+\lambda T)^{-1}=I-\lambda\frac{A(\lambda)}{D(\lambda)}.$$

This formula is an abstract formulation of (19). Notice that the equation $a_n=F(B_n)/n$ (for $n>0$) does not enter into the proof of (38) and (39).

On the other hand, if $|\lambda|<\|T\|^{-1}$, then

$$(40)\qquad \sum_{n=0}^{\infty}|\lambda^n|\|T^n\|<\infty,$$

and consequently (see (20))

$$(41)\qquad (I+\lambda T)^{-1}=\sum_{n=0}^{\infty}(-\lambda)^nT^n.$$

If λ is such that (40) and (36) hold, then by (38) and (41)

$$(42)\qquad \frac{B(\lambda)}{D(\lambda)}=\sum_{n=0}^{\infty}(-\lambda)^nT^n.$$

Consequently

$$F(B(\lambda))=D(\lambda)\cdot\sum_{n=0}^{\infty}(-\lambda)^n\sigma_{n+1}, \tag{43}$$

where, analogously to (27) and (28),

$$\sigma_n=F(T^{n-1}) \quad \text{for} \quad n=1,2,\ldots \tag{44}$$

On the other hand, it follows from (34), (29) and (17) that

$$F(B(\lambda))=\sum_{n=0}^{\infty}\lambda^n F(B_n)=\sum_{n=0}^{\infty}\lambda^n(n+1)a_{n+1}=\frac{d}{d\lambda}D(\lambda). \tag{45}$$

It follows from (43) and (45) that

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\lambda^n=\sum_{n=0}^{\infty}a_n\lambda^n\sum_{n=0}^{\infty}(-\lambda)^n\sigma_{n+1},$$

which makes it possible to calculate a_n as a function of σ_n. We obtain again Plemelj's [4] formulae (21), and the formulae analogous to (22):

$$B_n=\frac{1}{n!}\begin{vmatrix} I & n & 0 & 0 & \ldots & 0 \\ T & \sigma_1 & n-1 & 0 & \ldots & 0 \\ T^2 & \sigma_2 & \sigma_1 & n-2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T^n & \sigma_n & \sigma_{n-1} & & \ldots & \sigma_1 \end{vmatrix}, \quad A_n=\frac{1}{n!}\begin{vmatrix} T & n & 0 & 0 & \ldots & 0 \\ T^2 & \sigma_1 & n-1 & 0 & \ldots & 0 \\ T^3 & \sigma_2 & \sigma_1 & n-2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T^{n+1} & \sigma_n & \sigma_{n-1} & & \ldots & \sigma_1 \end{vmatrix}. \tag{46}$$

It follows from (43) and (45) that

$$\frac{d}{d\lambda}\log D(\lambda)=\sum_{n=0}^{\infty}(-\lambda)^n\sigma_{n+1}.$$

Hence (see Plemelj [4], p. 121, and Leżański [3], p. 16)

$$D(\lambda)=\exp\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sigma_n\lambda^n\Big) \tag{47}$$

$$=\exp\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\lambda^n F(T^{n-1})\Big)=\exp\Big(F\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\lambda^n T^{n-1}\Big)\Big),$$

for λ sufficiently small.

Obviously the condition (36) holds for each λ only for a special kind of elements T; the functional F cannot then be arbitrary, but must be closely related to the element $T\epsilon\mathfrak{R}$. In the next section (see also the beginning of § 5) we shall formulate certain conditions which imply that (36) holds for each λ, and that $(I+\lambda T)^{-1}$ exists if and only if $D(\lambda)\neq 0$.

§ 4. Leżański's theory of linear equations in Banach spaces. Let Ξ and X be two Banach spaces. Elements of Ξ will be denoted by ξ,η,ζ, and elements of X — by x,y,z with indices.

We suppose that Ξ and X are paired, *i.e.* with each pair $(\xi,x)\epsilon\Xi\times X$ there is associated a scalar denoted by ξx in such a way that ξx is a bilinear functional on $\Xi\times X$. We suppose also that

$$\|x\|=\sup_{\|\xi\|\leqslant 1}|\xi x|, \qquad \|\xi\|=\sup_{\|x\|\leqslant 1}|\xi x| \tag{48}$$

for arbitrary $x\epsilon X$ and $\xi\epsilon\Xi$. Consequently each $x\epsilon X$ can be considered as a linear functional on Ξ, and conversely each $\xi\epsilon\Xi$ can be considered as a linear functional on X. Thus we have isometrically[4])

$$X\subset\Xi^* \quad \text{and} \quad \Xi\subset X^*. \tag{49}$$

The value of a bilinear functional K (defined on $\Xi\times X$) at the point $(\xi,x)\epsilon\Xi\times X$ will be denoted by ξKx.

Each bilinear functional K on $\Xi\times X$ can be interpreted as a linear transformation of X into Ξ^*, denoted by the same letter K and transforming an element $x\epsilon X$ into an element $y=Kx\epsilon\Xi^*$ such that $\xi y=\xi Kx$ for each $\xi\epsilon\Xi$ (where ξy is the value of the functional $y\epsilon\Xi^*$ at the point $\xi\epsilon\Xi$).

Analogously each bilinear functional K on $\Xi\times X$ can be interpreted as a linear transformation of Ξ into X^* denoted by the same letter K and transforming an element $\xi\epsilon\Xi$ into an element $\eta=\xi K\epsilon X^*$ such that $\eta x=\xi Kx$ for each $x\epsilon X$ (where ηx is the value of the functional $\eta\epsilon X^*$ at the point $x\epsilon X$)[5]).

Conversely, each linear transformation K of X into Ξ^* (or: of Ξ intd X^*) can be interpreted as a bilinear functional ξKx on $\Xi\times X$ defineo by the formula

$$\xi Kx=\xi(Kx) \qquad (\text{or: } =(\xi K)x).$$

Notice that the norm of K is the same in all the three possible interpretations of K (see (48)). Therefore we shall identify the three notions: bilinear functionals on $\Xi\times X$, linear transformations of Ξ into X^*, and linear transformations of X into Ξ^*.

Suppose that K_1 and K_2 are two bilinear functionals on $\Xi\times X$ such that $K_2x\epsilon X$ and $\xi K_1\epsilon\Xi$ for $x\epsilon X$, $\xi\epsilon\Xi$. Then K_1K_2 is the bilinear functional defined by the equation

$$\xi K_1K_2x=(\xi K_1)K_2x=\xi K_1(K_2x)=(\xi K_1)(K_2x),$$

[4]) If Z is a Banach space, then Z^* is the space of all linear functionals on Z.

[5]) The above notations will be systematically used in the sequel: if K is a linear transformation of Ξ into X^* (of X into Ξ^*), then $\xi K\epsilon X^*$ ($Kx\epsilon\Xi^*$) denotes the image of the element $\xi\epsilon\Xi$ ($x\epsilon X$).

i. e. the bilinear functional determined by the linear transformation $y=K_1(K_2x)$ of X into Ξ^* (or: by the linear transformation $\eta=(\xi K_1)K_2$ of Ξ into X^*). Thus, if we interpret K_1 and K_2 as linear operations, then K_1K_2 is the superposition of K_1 and K_2.

The linear transformations determined by the bilinear functional ξx are the identical mappings I of X onto X and of Ξ onto Ξ respectively. Therefore we may write ξIx instead of ξx.

Let $\mathfrak{K}$ be a linear subset of the space of all bilinear functionals on $\Xi\times X$ such that (see Leżański [2], p. 245):

(K) $\mathfrak{K}$ is closed and $I\in\mathfrak{K}$; if $K\in\mathfrak{K}$, then $Kx\in X$ and $\xi K\in\Xi$ for $x\in X$, $\xi\in\Xi$; if $K_1,K_2\in\mathfrak{K}$, then $K_1K_2\in\mathfrak{K}$; if $x\in X$ and $\xi\in\Xi$ are fixed, then the bilinear functional K defined by the equation

$$\eta Ky=\xi y\cdot\eta x \quad \text{for} \quad \eta\in\Xi, \quad y\in X, \tag{50}$$

belongs to $\mathfrak{K}$.

If K is defined by (50), then

$$Ky=\xi y\cdot x \qquad \text{and} \qquad \eta K=\eta x\cdot\xi,$$

i.e. K is a one-dimensional operation (transforming X into X or Ξ into Ξ respectively). The last of the conditions (K) means that each one-dimensional linear operation belongs to $\mathfrak{K}$.

Let F be a linear functional on $\mathfrak{K}$. Following Leżański [2], p. 247, instead of $F(K)$ (where $K\in\mathfrak{K}$) we shall also write $F_{\eta y}\{\eta Ky\}$ where η and y are the "bound variables" and can be replaced by other letters. For instance, if K is defined by (50), we write

$$F_{\eta y}\{\xi y\cdot\eta x\} \tag{51}$$

instead of $F(K)$.

The expression (51), depending on ξ and x, is a bilinear functional on $\Xi\times X$. Following Leżański [3], p. 19, we shall denote this bilinear functional by T_F. By definition

$$\xi T_F x=F_{\eta y}\{\xi y\cdot\eta x\}. \tag{52}$$

Notice that (see Leżański [3], p. 20)

$$\|T_F\|\leqslant\|F\|. \tag{53}$$

In this section we shall consider only one functional F on $\mathfrak{K}$ such that

$$T_F\in\mathfrak{K}. \tag{54}$$

Therefore in this section we shall write, for brevity, T instead of T_F.

It follows immediately from (52) that for $K_1,K_2\in\mathfrak{K}$[6])

$$F_{\eta y}\{\xi K_1y\cdot\eta K_2x\}=(\xi K_1)T(K_2x)=\xi K_1TK_2x. \tag{55}$$

Consequently, if $0<r\leqslant n$, if $p_1,p_2,\ldots,p_n$ is a permutation of the numbers $1,2,\ldots,n$, if $\xi_{r+1},\ldots,\xi_n\in\Xi$ and $x_{r+1},\ldots,x_n\in X$, then the expression

$$F_{\xi_1x_1}\{F_{\xi_2x_2}\{\ldots\{F_{\xi_rx_r}\{\xi_1x_{p_1}\cdot\xi_2x_{p_2}\cdot\ldots\cdot\xi_nx_{p_n}\}\}\ldots\}\} \tag{56}$$

is well defined, does not depend on the order of the signs $F_{\xi_1x_1},\ldots,F_{\xi_rx_r}$, and, for fixed

$$\xi_{r+1},\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n\in\Xi, \qquad x_{r+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n\in X,$$

it is a bilinear functional (of variables ξ_i,x_j) belonging to $\mathfrak{K}$. For the exact proof of this fact see Leżański [2], p. 248.

We shall now examine the linear equations

$$(I+\lambda T)x=x_0, \tag{57}$$

$$\xi(I+\lambda T)=\xi_0, \tag{58}$$

where $x_0\in X$, $\xi_0\in\Xi$ are fixed. We shall also examine the homogeneous equations

$$(I+\lambda T)x=0, \tag{59}$$

$$\xi(I+\lambda T)=0. \tag{60}$$

The solutions x and ξ should belong to X and Ξ respectively.

Following Leżański [2], let us set

$$C_{00}=1,$$

$$C_{r0}\begin{pmatrix}\xi_1,\ldots,\xi_r\\ x_1,\ldots,x_r\end{pmatrix}=\begin{vmatrix}\xi_1x_1 & \xi_1x_2 & \ldots & \xi_1x_r\\ \xi_2x_1 & \xi_2x_2 & \ldots & \xi_2x_r\\ \cdot & \cdot & \cdot & \cdot\\ \xi_rx_1 & \xi_rx_2 & \ldots & \xi_rx_r\end{vmatrix} \quad \text{for} \quad r=1,2,\ldots,$$

$$C_{rk}\begin{pmatrix}\xi_1,\ldots,\xi_p\\ x_1,\ldots,x_p\end{pmatrix}=\frac{1}{k!}F_{\eta_1y_1}\Big\{\ldots\Big\{F_{\eta_ky_k}\Big\{C_{r+k,0}\begin{pmatrix}\xi_1,\ldots,\xi_r,\eta_1,\ldots,\eta_k\\ x_1,\ldots,x_r,y_1,\ldots,y_k\end{pmatrix}\Big\}\Big\}\ldots\Big\}$$

for $r=0,1,2,\ldots,$ $k=1,2,\ldots$

Obviously C_{0k} is a scalar. C_{rk} (for $r>0$) is linear in each of the variables $\xi_1,\ldots,\xi_r$, $x_1,\ldots,x_r$. More precisely (see (56)), C_{rk} considered as a function of ξ_i and x_j only ($1\leqslant i,j\leqslant r$) belongs to $\mathfrak{K}$.

[6]) The last expression is well defined since $\xi(K_1TK)x=\xi((K_1T)K_2)x=\xi(K_1(TK_2))x=(\xi K_1)T(K_2x)=(\xi K_1T)K_2x=\ldots$ etc.

Developing the determinants $C_{r+k,0}$ after the j^{th} line ($1\leqslant j\leqslant r$) we obtain (see Leżański [2], p. 255)

$$(61)\quad C_{rk}\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}=\sum_{i=1}^{r}(-1)^{j+i}\,\xi_j x_i\cdot C_{r-1,k}\begin{pmatrix}\xi_1,\dots,\xi_{j-1},\xi_{j+1},\dots,\xi_r\\ x_1,\dots,x_{i-1},x_{i+1},\dots,x_r\end{pmatrix}-F_{\eta y}\left\{\xi_j y\cdot C_{r,k-1}\begin{pmatrix}\xi_1,\dots,\xi_{j-1},\eta,\xi_{j+1},\dots,\xi_r\\ x_1,\dots\dots\dots\dots,x_r\end{pmatrix}\right\}.$$

Analogously, developing the determinant $C_{r+k,0}$ after the j^{th} column ($1\leqslant j\leqslant r$), we obtain[7])

$$(62)\quad C_{rk}\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}=\sum_{i=1}^{r}(-1)^{i+j}\,\xi_i x_j\cdot C_{r-1,k}\begin{pmatrix}\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_r\\ x_1,\dots,x_{j-1},x_{j+1},\dots,x_r\end{pmatrix}-F_{\eta y}\left\{C_{r,k-1}\begin{pmatrix}\xi_1,\dots\dots\dots\dots,\xi_r\\ x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_r\end{pmatrix}\cdot\eta x_j\right\}.$$

Following Leżański [2], p. 253, let us set

$$(63)\quad D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}=\sum_{k=0}^{\infty}\lambda^k C_{rk}\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}\quad\text{for}\quad r=0,1,2,\dots$$

Clearly D_0 depends only on λ. D_r (for $r>0$) are linear in each of the variables ξ_i and x_j.

The series (63) converges for each λ since (see Leżański [2], p. 253)

$$(64)^{8)}\quad \left|C_{rk}\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}\right|\leqslant\frac{(k+r)^{\frac{k+r}{2}}}{k!}\|F\|^k\cdot\|x_1\|\cdot\ldots\cdot\|x_r\|\cdot\|\xi_1\|\cdot\ldots\cdot\|\xi_r\|.$$

It follows from (61) and (62) that for $r>0$

$$(65)\quad D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}=\sum_{i=1}^{r}(-1)^{i+j}\,\xi_j x_i\cdot D_{r-1}\begin{pmatrix}\xi_1,\dots,\xi_{j-1},\xi_{i+1},\dots,\xi_r\\ x_1,\dots,x_{i-1},x_{i+1},\dots,x_r\end{pmatrix}-\lambda F_{\eta y}\left\{\xi_j y\cdot D_r\begin{pmatrix}\xi_1,\dots,\xi_{i-1},\eta,\xi_{i+1},\dots,\xi_r\\ x_1,\dots\dots\dots\dots,x_r\end{pmatrix}\right\}$$

and

$$(66)\quad D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}=\sum_{i=1}^{r}(-1)^{i+j}\,\xi_i x_j\cdot D_{r-1}\begin{pmatrix}\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_r\\ x_1,\dots,x_{j-1},x_{j+1},\dots,x_r\end{pmatrix}-\lambda F_{\eta y}\left\{D_r\begin{pmatrix}\xi_2,\dots\dots\dots\dots,\xi_r\\ x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_r\end{pmatrix}\cdot\eta x_j\right\}.$$

[7]) If $k=0$, we omit the expression $F_{\eta y}\{\dots\}$ on the right side.

[8]) This inequality follows from Hadamard's inequality for determinants.

Replace in (65) and (66) the number r by $r+1$ adding a column $\begin{matrix}\xi\\ x\end{matrix}$, and set $j=r+1$. Then

$$(67)\quad D_{r+1}\begin{pmatrix}\xi_1,\dots,\xi_r,\xi\\ x_1,\dots,x_r,x\end{pmatrix}=-\sum_{i=1}^{r}\xi x_i\cdot D_r\begin{pmatrix}\xi_1,\dots\dots\dots\dots,\xi_r\\ x_1,\dots,x_{i-1},x,x_{i+1},x_r\end{pmatrix}+\xi x\cdot D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}-\lambda F_{\eta y}\left\{\xi y\cdot D_{r+1}\begin{pmatrix}\xi_1,\dots,\xi_r,\eta\\ x_1,\dots,x_r,x\end{pmatrix}\right\},$$

$$(68)\quad D_{r+1}\begin{pmatrix}\xi_1,\dots,\xi_r,\xi\\ x_1,\dots,x_r,x\end{pmatrix}=-\sum_{i=1}^{r}\xi_i x\cdot D_r\begin{pmatrix}\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_r\\ x_1,\dots\dots\dots\dots,x_r\end{pmatrix}+\xi x\cdot D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}-\lambda F_{\eta y}\left\{D_{r+1}\begin{pmatrix}\xi_1,\dots,\xi_r,\xi\\ x_1,\dots,x_r,y\end{pmatrix}\cdot\eta x\right\}.$$

If $r=0$, we omit the sum Σ on the right side of (67) and (68).

It follows from the definition of C_{rk} that (see Leżański [2], p. 253)

$$(69)\quad \frac{d^r}{d\lambda^r}D_0=F_{\xi_1x_1}\left\{F_{\xi_2x_2}\left\{\dots\left\{F_{\xi_rx_r}\left\{D_k\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}\right\}\right\}\dots\right\}\right\}.$$

Fix a value for λ.

Since D_0 is a holomorphic function of λ, there is an integer $r\geqslant 0$ such that

$$\frac{d^r}{d\lambda^r}D_0\neq 0$$

at this value of λ. Hence the functional D_r is not identically equal to zero.

Let $r\geqslant 0$ be the least integer such that $D_r\neq 0$ for this value of λ. In the sequel $\xi_1,\dots,\xi_r\epsilon\Xi$, $x_1,\dots,x_r\epsilon X$ will denote fixed elements such that

$$(70)\quad \delta=D_r\begin{pmatrix}\xi_1,\dots,\xi_r\\ x_1,\dots,x_r\end{pmatrix}\neq 0.$$

Let $\zeta_j\epsilon\Xi$ and $z_j\epsilon X$ ($j=1,2,\dots,r$) be such elements that

$$(71)\quad \zeta_j x=D_r\begin{pmatrix}\xi_1,\dots\dots\dots\dots,\xi_r\\ x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_r\end{pmatrix}\quad\text{for each}\quad x\epsilon X,$$

$$(72)\quad \xi z_j=D_r\begin{pmatrix}\xi_1,\dots,\xi_{j-1},\xi,\xi_{j+1},\dots,\xi_r\\ x_1,\dots\dots\dots\dots,x_r\end{pmatrix}\quad\text{for each}\quad \xi\epsilon\Xi.$$

These elements exist by (K) and (56) and are uniquely determined on account of (48). If $r=0$, then ξ_j, x_j, ζ_j, z_j are not defined.

It follows from (65), where ξ_j is replaced by ξ, that

$$\xi z_j + \lambda F_{\eta y}\{\xi y\cdot\eta z_j\} = 0,$$

i. e., by (52)

$$\xi(I+\lambda T)z_j = 0 \quad \text{for each} \quad \xi\epsilon\Xi.$$

This means that z_j is a solution of the homogeneous equation (59) for $j=1,2,\ldots,r$.

Analogously, we infer from (66) and (52) that ζ_j is a solution of the homogeneous equation (60) for $j=1,2,\ldots,r$.

Those solutions ζ_j, z_j are linearly independent since by (70) and by the skew symmetry of D_r

$$\xi_i z_j = \delta\cdot\delta_{ij} = \zeta_j x_i,$$

where δ_{ij} is the Kronecker symbol.

Set

$$\xi C x = D_{r+1}\begin{pmatrix}\xi_1,\ldots,\xi_r,\xi\\ x_1,\ldots,x_r,x\end{pmatrix} \quad \text{for} \quad \xi\epsilon\Xi,\ x\epsilon X.$$

It follows from (K) and (56) that $C\epsilon\mathfrak{K}$ (see the remark before (61)). The formulae (67) and (68) can now be written in the form (see (55))

$$(73)\qquad \xi C x = -\sum_{i=1}^{r}\xi x_i\cdot\zeta_i x + \xi x\cdot\delta - \lambda\cdot\xi TCx,$$

$$(74)\qquad \xi C x = -\sum_{i=1}^{r}\xi_i x\cdot\xi z_i + \xi x\cdot\delta - \lambda\cdot\xi CTx.$$

The elements $z_1, z_2, \ldots, z_r$ and $\zeta_1, \zeta_2, \ldots, \zeta_r$ form a basis of the space of all solutions of (59) and (60) respectively. In fact, if ξ satisfies (60), then, multiplying (60) by Cx, we find from (73) that

$$\xi x = \frac{1}{\delta}\sum_{i=1}^{r}\xi x_i\cdot\zeta_i x \quad \text{for each} \quad x\epsilon X,$$

i. e.

$$\xi = \frac{1}{\delta}\sum_{i=1}^{r}\xi x_i\cdot\zeta_i.$$

Analogously, if x satisfies (59), then multiplying (59) by ξC we find from (74) that

$$\xi x = \frac{1}{\delta}\sum_{i=1}^{r}\xi_i x\cdot\xi z_i \quad \text{for each} \quad \xi\epsilon\Xi,$$

i. e.

$$x = \frac{1}{\delta}\sum_{i=1}^{r}\xi_i x\cdot z_i.$$

If there is a solution x of the equation (57), then multiplying this equation by ζ_j we find that

$$(75)\qquad \zeta_j x_0 = 0 \quad \text{for} \quad j=1,2,\ldots,r.$$

Conversely, if (75) holds, then, by (73) where x is replaced by x_0, the element $\bar{x} = \frac{1}{\delta}Cx_0$ is the solution of (57).

Analogously, if there is a solution of the equation (58), then multiplying (58) by z_j we find that

$$(76)\qquad \xi_0 z_j = 0 \quad \text{for} \quad j=1,2,\ldots,r.$$

Conversely, if (76) holds, then, by (74), the element $\bar{\xi} = \frac{1}{\delta}\xi_0 C$ is the solution of (58).

If $r=0$, then (75) and (76) give no restriction. The equation (57) and (58) are always uniquely solvable.

The analogy between Leżański's theory and Fredholm's theory is complete.

§ 5. The theorem on multiplication of determinants. Suppose now that X, Ξ and $\mathfrak{K}$ satisfy the conditions mentioned in § 4. Interpret $\mathfrak{K}$ as a Banach algebra of linear operators in the space X (see p. 35) and apply to this algebra the theory developed in § 3.

It follows from the definition of C_{0k} and C_{1k} and from (55) and (61) that C_{0k} and C_{1k} satisfy the induction formulae (29) and (30) (the last should be multiplied by $\xi\epsilon\Xi$). Consequently

$$(77)\qquad a_k = C_{0k} = \frac{1}{k!}F_{\xi_1 x_1}\Big\{\ldots\Big\{F_{\xi_k x_k}\Big\{C_{k0}\begin{pmatrix}\xi_1,\ldots,\xi_k\\ x_1,\ldots,x_k\end{pmatrix}\Big\}\Big\}\ldots\Big\}$$

and

$$\xi B_k x = \xi C_{1k}x = \frac{1}{k!}F_{\xi_1 x_1}\Big\{\ldots\Big\{F_{\xi_k x_k}\Big\{C_{k+1,0}\begin{pmatrix}\xi,\xi_1,\ldots,\xi_k\\ x,x_1,\ldots,x_k\end{pmatrix}\Big\}\Big\}\ldots\Big\}$$

for $k=1,2,\ldots$ These formulae are the analogues of the Fredholm formulae (23) and (24).

Consequently $D(\lambda)=D_0$ and $B(\lambda)=D_1$, where the bilinear functional $D_1\begin{pmatrix}\xi\\ x\end{pmatrix}$ is interpreted as an operation of X into X.

Notice that, by (64) and (77),

$$(78)\qquad |a_k| \leqslant \frac{k^{k/2}}{k!}\|F\|^k.$$

In § 4 we considered a fixed functional F satisfying the condition (54). Now we shall consider the class $\mathfrak{M}$ of all linear continuous functionals F on $\mathfrak{K}$ satisfying the condition (54). Obviously $\mathfrak{M}$ is a Banach space (see (53)). Following Leżański [3] we make $\mathfrak{M}$ into a Banach algebra defining the product $H=F\cdot G\in\mathfrak{M}$ of two functionals $F,G\in\mathfrak{M}$ by the equation

(79) $$H(K)=F\cdot G(K)=G(KT_F)\quad\text{for}\quad K\in\mathfrak{K}.$$

The transformation $F\to T_F$ defined in § 4 is a ring homomorphism, *i. e.* (see Leżański [3], p. 20)

(80) $$T_{\lambda F}=\lambda T_F,\qquad T_{F+G}=T_F+T_G,\qquad T_{F\cdot G}=T_FT_G.$$

Consequently, by an easy induction on n,

(81) $$F^n(I)=F(T_F^{n-1})\quad\text{for}\quad F\in\mathfrak{M},$$

where F^n is the n^{th} power of $F\in\mathfrak{M}$ in the Banach algebra $\mathfrak{M}$.

It is convenient from a purely formal reason to add the abstract unit E to the algebra $\mathfrak{M}$. Obviously the transformation $\nu E+F\to\nu I+T_F$ is a ring homomorphism of this extended algebra into $\mathfrak{K}$.

We shall now examine the determinant $D(\lambda)$ (of the linear operation $I+\lambda T_F$) as a function of $F\in\mathfrak{M}$. Therefore we shall write $a_k(F)$ instead of a_k to emphasize that a_k is uniquely determined by $F\in\mathfrak{M}$. More precisely, $a_k(F)$ is defined by the right side of (77) for $k>0$, and $a_0(F)=1$ $(F\in\mathfrak{M})$.

Let us set

(82) $$D(E+F)=\sum_{k=0}^{\infty}a_k(F)\quad\text{for}\quad F\in\mathfrak{M}.$$

Since by (77)

$$a_k(\lambda F)=\lambda^k a_k(F)\quad\text{for}\quad k=0,1,2,\ldots,$$

we obtain

(83) $$D(E+\lambda F)=\sum_{k=0}^{\infty}\lambda^k a_k(F),$$

i. e. $D(E+\lambda F)$ coincides with the determinant denoted hitherto by $D(\lambda)$.

$D(E+F)$ is the determinant of the linear operation $I+T_F$. More generally, $D(E+\lambda F)$ is the determinant of the linear operation $I+T_{\lambda F}=I+\lambda T_F$. It follows from the last equality that it suffices to examine only the determinant $D(E+F)$ of the operation $I+T_F$.

Notice that, in general, the mapping $F\to T_F$ of $\mathfrak{M}$ into $\mathfrak{K}$ is not one-to-one (see p. 31). Consequently the determinant $D(E+F)$ of the operation $I+T_F$ is uniquely determined by $F\in\mathfrak{M}$, but, in general, it is not uniquely determined by the operation $I+T_F$.

Set

(84) $$\Phi(F)=F(I)\quad\text{for}\quad F\in\mathfrak{M}.$$

Φ is a linear functional on $\mathfrak{M}$. By (81) and (84)

(85) $$F(T_F^{n-1})=\Phi(F^n).$$

Hence for each $F\in\mathfrak{M}$ (see Leżański [3], p. 16)

(86) $$D(E+\lambda F)=\exp\Big(\Phi\big(\log(E+\lambda F)\big)\Big)=\exp\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\Phi(F^n)\lambda^n\Big)$$

whenever the expression on the right side is defined (see (10)), *e. g.* if $|\lambda|<\|T\|^{-1}$.

In fact, the equation (86) holds for λ sufficiently small by (47). Since $D(E+\lambda F)$ is an integer function of λ and

(87) $$\Phi\big(\log(E+\lambda F)\big)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\Phi(F^n)\lambda^n$$

is a holomorphic function of λ, the equation (86) holds for $|\lambda|<r$ where r is the radius of convergence of the series (87). By Abel's theorem, the equation (86) holds also for $|\lambda|=r$ whenever the series (87) converges.

THEOREM 2. *Let* $F,G\in\mathfrak{M}$ *be such that*

(88) $$F_1\cdot F_2\cdot\ldots\cdot F_n(I)=F_2\cdot\ldots\cdot F_n\cdot F_1(I)$$

for each finite sequence $F_i=F$ *or* G $(i=1,2\ldots,n)$. *Then*

(89) $$D\big((E+F)(E+G)\big)=D(E+F)\cdot D(E+G).$$

Obviously $F_1\cdot F_2\cdot\ldots\cdot F_n\in\mathfrak{M}$ $(F_2\cdot\ldots\cdot F_n\cdot F_1\in\mathfrak{M})$ denotes the product of $F_1,F_2,\ldots,F_n$ $(F_2,\ldots,F_n,F_1)$ in the algebra $\mathfrak{M}$ (non-commutative, in general), and $F_1\cdot F_2\cdot\ldots\cdot F_n(I)$ $\big(F_2\cdot\ldots\cdot F_n\cdot F_1(I)\big)$ denotes the value of this functional at the point $I\in\mathfrak{K}$.

Theorem 2 states that the determinant

$$D\big((E+F)(E+G)\big)=D(E+F+G+F\cdot G)$$

of the superposition

$$(I+T_F)(I+T_G)=I+T_F+T_G+T_FT_G=I+T_{F+G+F\cdot G}$$

is the product of the determinants $D(E+F)$, $D(E+G)$ of the operations $I+T_F$ and $I+T_G$ respectively.

To prove Theorem 2 let us notice that the expression

$$D\big((E+\lambda F)(E+\lambda G)\big)=D\big(E+(\lambda F+\lambda G+\lambda^2F\cdot G)\big)$$

is an integer function of λ.

In fact,

$$D\big((E+\lambda F)(E+\lambda G)\big)=\sum_{k=0}^{\infty}a_k(\lambda F+\lambda G+\lambda^2F\cdot G) \tag{90}$$

where $a_k(\lambda F+\lambda G+\lambda^2F\cdot G)$ is defined by the formula (77) where F is replaced by $\lambda F+\lambda G+\lambda^2F\cdot G$. Consequently a_k is a polynomial of λ of a degree $\leqslant 2k$. The series (90) of polynomials converges uniformly for $|\lambda|\leqslant r$, r being arbitrary, since by (78)

$$|a_k(\lambda F+\lambda G+\lambda^2F\cdot G)|\leqslant\frac{k^{k/2}}{k!}\,\|\lambda F+\lambda G+\lambda^2F\cdot G\|^k$$

$$\leqslant\frac{k^{k/2}}{k!}r^k(\|F\|+\|G\|+r\|F\cdot G\|)^k.$$

We have $\lambda F\sim\lambda G \bmod \Phi$ (see p. 27). Consequently, by Theorem 1 (where $\mathfrak{A}=\mathfrak{M}$, $A=\lambda F$, $B=\lambda G$) and (86),

$$D\big((E+\lambda F)(E+\lambda G)\big)=D(E+\lambda F)\cdot D(E+\lambda G)$$

for

$$|\lambda|<\frac{1}{3}\min\big(\|F\|^{-1},\|G\|^{-1}\big).$$

Since $D\big((E+\lambda F)(E+\lambda G)\big)$, $D(E+\lambda F)$, $D(E+\lambda G)$ are integer functions of λ, this equation holds for each λ, in particular for $\lambda=1$.

§ 6. Applications. Let Γ be a set with a measure μ defined on a σ-field of subsets of Γ. Let L_Γ^p be the Banach space of all measurable functions x on Γ such that

$$\|x\|=\big(\int|x(t)|^p dt\big)^{1/p}<\infty$$

where $1\leqslant p<\infty$. The integral is always taken over the whole space Γ.

L_Γ^∞ is the Banach space of all bounded measurable functions x on Γ with the norm

$$\|x\|=\operatorname*{sup\,ess}_{t\in\Gamma}|x(t)|.$$

If Γ is a metric space and μ is a Borel measure such that

(m) for each point $t\in\Gamma$ and for each open set U $(t\in U\subset\Gamma)$ there is an open set U_0 such that $t\in U_0\subset U$ and $0<\mu(U_0)<\infty$,

then let C_Γ be the Banach space of all bounded continuous functions x on Γ with the norm

$$\|x\|=\sup_{t\in\Gamma}|x(t)|.$$

Let X and Ξ be one of the following pairs of Banach spaces

$$L_\Gamma^p,\ L_\Gamma^q\qquad\Big(1\leqslant p,\ q\leqslant\infty,\frac{1}{p}+\frac{1}{q}=1\Big),$$

$$L_\Gamma^1,\ C_\Gamma,$$

$$C_\Gamma,\ L_\Gamma^1,$$

with the bilinear functional

$$\xi x=\int\xi(t)x(t)\,dt\quad\text{ for }\quad \xi\in\Xi,\ x\in X.$$

Let $\mathfrak{K}_0$ be the class of all bilinear functionals K on $\Xi\times X$ of the integral type, *i. e.* of the form

$$\xi Kx=\iint K(s,t)\,\xi(s)x(t)\,ds\,dt\quad\text{ for }\quad\xi\in\Xi,\ x\in X, \tag{91}$$

where $K(s,t)$ is a function on the space $\Gamma\times\Gamma$, measurable with respect to the product meassure $\mu\times\mu$, and such that

$$\iint|K(s,t)\,\xi(s)x(t)|\,ds\,dt<\infty\quad\text{ for }\quad\xi\in\Xi,\ x\in X. \tag{92}$$

All double integrals are obviously taken with respect to the product measure $\mu\times\mu$.

Let $\mathfrak{K}$ be the least closed linear subspace of the space of all bilinear functionals on $\Xi\times X$, such that $I\in\mathfrak{K}$ and $\mathfrak{K}_0\subset\mathfrak{K}$.

Clearly X,Ξ and $\mathfrak{K}$ satisfy the conditions (48) and (K). Now let $\mathfrak{M}_0$ be the class of all linear functionals on $\mathfrak{K}$ of the integral type, *i.e.* such that

$$F(K)=\iint T(s,t)K(t,s)\,dt\,ds\quad\text{ for }\quad K\in\mathfrak{K}_0, \tag{93}$$

where $T(s,t)$ is a measurable function on $\Gamma\times\Gamma$ such that

$$\iint|T(s,t)K(t,s)|\,dt\,ds<\infty\quad\text{ for }\quad K\in\mathfrak{K}_0.$$

Suppose that $F\in\mathfrak{M}_0$ is defined by (93). Then T_F (see p. 36) is the bilinear functional determined by the same function $T(s,t)$, *i. e.*

$$\xi T_Fx=\iint T(s,t)\,\xi(s)x(t)\,ds\,dt\quad\text{ for }\quad\xi\in\Xi,\ x\in X. \tag{94}$$

Consequently $T=T_F\in\mathfrak{K}_0$ and $\mathfrak{M}_0\subset\mathfrak{M}$.

We shall now prove that

$$F_1\cdot F_2\cdot\ldots\cdot F_n(I)=F_2\cdot\ldots\cdot F_n\cdot F_1(I) \tag{95}$$

for an arbitrary sequence $F_1,F_2,\ldots,F_n\in\mathfrak{M}_0$.

In fact, if

$$F_i(K)=\iint T_i(s,t)K(t,s)\,ds\,dt\quad\text{ for }\quad K\in\mathfrak{K}_0,$$

then

$$F_1\cdot F_2\cdot\ldots\cdot F_n(I)=F_n(T_{F_1\cdot\ldots\cdot F_{n-1}})=F_n(T_{F_1}\ldots T_{F_{n-1}})$$
$$=\int\ldots\int T_1(s_n,s_1)T_2(s_1,s_2)\cdot\ldots\cdot T_n(s_{n-1},s_n)\,ds_1\ldots ds_n$$
$$=F_1(T_{F_2}\ldots T_{F_n})=F_2\cdot\ldots\cdot F_n\cdot F_1(I).$$

The order of the integration is of no consequence since the integral remains finite if we replace $T_i(s,t)$ by $|T_i(s,t)|$.

Let $\mathfrak{M}_1$ be the least closed linear space such that $\mathfrak{M}_0\subset\mathfrak{M}_1\subset\mathfrak{M}$. It follows from (95) and Theorem 2 that

THEOREM 3. *If $F,G\epsilon\mathfrak{M}_1$, then*

$$D\big((E+F)(E+G)\big)=D(E+F)\cdot D(E+G).$$

The classes $\mathfrak{K}_0$ and $\mathfrak{M}_0$ (and, consequently, $\mathfrak{K}$ and $\mathfrak{M}_1$) can also be specialized in another way (see *e. g.* Leżański [2], Part II). We can define $\mathfrak{K}_0$ as a linear set of bilinear functionals of type (91) satisfying certain additional conditions. $\mathfrak{M}_0$ is then a class of functionals F such that (93) holds. It is obvious that (95) is true under certain hypotheses about the absolute integrability. Consequently Theorem 3 remains true.

Consider, for instance, the case where $\Xi=X=L^2_\Gamma$. Let $\mathfrak{K}_0$ be the class of all bilinear functionals of type (91) such that

$$\int\int|K(s,t)|^2\,ds\,dt<\infty, \tag{96}$$

and let $\mathfrak{K}$ be the least closed linear space such that $I\epsilon\mathfrak{K}$ and $\mathfrak{K}_0\subset\mathfrak{K}$. Let $\mathfrak{M}_0$ be the class of all linear functionals on $\mathfrak{K}$ such that (93) holds, where $T(s,t)$ is a measurable function on $\Gamma\times\Gamma$ and

$$\int\int|T(s,t)|^2\,ds\,dt<\infty. \tag{97}$$

If $F\epsilon\mathfrak{M}_0$, then $T_F\epsilon\mathfrak{K}_0$ is defined by (94), and equation (57) is the integral equation

$$x(s)+\lambda\int T(s,t)x(t)\,dt=x_0(s)$$

of Carleman's [1] type. We can apply the theory developed in § 4. In particular we obtain Carleman's [1] theorem stating that series (17) and (18) are integer functions of λ. Theorem 3 is also true.

Let us return to the general case.

The functional $F\epsilon\mathfrak{M}_0$ (see (93)) is uniquely determined by the function $T(s,t)$ if and only if I belongs to the closure of $\mathfrak{K}_0$. If I belongs to the closure of $\mathfrak{K}_0$, then $D(E+\lambda F)$ is uniquely determined by the operation $I+\lambda T$. If I does not belong to the closure of $\mathfrak{K}_0$, then σ_n is determined by $I+\lambda T$ for $n\geqslant 2$, since for $n>1$,

$$\sigma_n=\int\ldots\int T(s_1,s_2)T(s_2,s_3)\ldots T(s_n,s_1)\,ds_1\ldots ds_n.$$

However, $\sigma_1=F(I)$ is then completely arbitrary. Therefore the determinant $D(E+\lambda F)$ of the operation $I+\lambda T_F$ is not uniquely determined by $I+\lambda T_F$ (see (47)). This phenomenon is not unexpected since it appears in Fredholm's theory of integral equations (see the remark on p. 31).

Notice that if Γ is the unit integral with the Lebesgue measure, then $L^p_\Gamma=\boldsymbol{L}^p$ and $C_\Gamma=\boldsymbol{C}$. If Γ is the set N of all positive integers and $\mu(Z)=(\overline{\overline{Z}})$ (the cardinal of Z) for $Z\subset N$, then $L^p=\boldsymbol{l}^p$. If Γ is the set N_∞ composed of all positive integers and of the number ∞, and if $\mu(Z)=\overline{\overline{Z}}$ for $Z\subset N_\infty$, then C_Γ is the space $\boldsymbol{c}$ of all convergent sequences and $L^1=\boldsymbol{l}$ (the topology in N_∞ is the usual one).

Obviously, if $\Gamma=N$ or N_∞, then the integrals can be replaced by the infinite series, and the functions $K(s,t)$, $T(s,t)$ — by infinite double matrices.

If Γ is finite and if $\mu(Z)=\overline{\overline{Z}}$ for $Z\subset\Gamma$, we obtain the theory of linear equations in Euclidean spaces.

§ 7. Final remarks. Leżański's [2,3] original theory differs from the theory developed in § 4. He supposed only that Ξ is a closed subspace of the space X^*. Consequently he could identify bilinear functionals on $\Xi\times X$ with linear transformations of Ξ into X^*, but not with linear transformations of X into Ξ^* (see p. 35). Consequently instead of (55) he had only the equation

$$F_{\eta y}\{\xi Ky\cdot\eta x\}=\xi KTx.$$

The equation

$$F_{\eta y}\{\xi y\cdot\eta Kx\}=\xi TKx \tag{F}$$

was not a consequence of his hypothesis, therefore he had to assume additionally that each operation $K\epsilon\mathfrak{K}$ satisfies the condition (F). Notice that the condition (F) can be briefly written as follows[9]):

$$T_{F_K}=T_FK \quad\text{for each}\quad K\epsilon\mathfrak{K}, \tag{F'}$$

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[9]) Leżański wrote $K\xi x$ instead of ξKx, and $K\xi$ instead of ξK. Therefore the order of all superpositions in my paper is inverse as compared with Leżański's papers. In the original notations of Leżański we should write

$$T_{F_K}=KT_F \quad\text{for each}\quad K\epsilon\mathfrak{K}, \tag{F'}$$

where

$$F_K(M)=F(MK) \quad\text{for each}\quad M\epsilon\mathfrak{K}.$$

where F_K is the functional

$$F_K(M)=F(KM) \quad \text{for} \quad M\in\mathfrak{R}.$$

Since Leżański's hypotheses about Ξ and X were not symmetrical, his results are more complicated than those in § 4. He examined only the equations (58) and (60). Instead of the equations (57) and (59) he examined the equations conjugate to (58) and (60) in the space Ξ^*. Besides the equation (58), he examined, more generally, the equation[10])

$$\xi(I+\lambda TK)=\xi_0.$$

However, this generalization is not essential since Leżański's [2] (p. 252) determinant of this equation coincides with the determinant $D(E+\lambda F_K)$ of the operation $I+\lambda T_{F_K}=I+\lambda T_F K$.

Notice that Theorem 2 remains true if we admit the original hypothesis of Leżański[11]).

The connexion between Leżański's [2,3] theory and Ruston's [5,6] theory should be discussed separately. We notice here only that Leżański's formalism is more general than that of Ruston (the question whether they are equivalent remains open). Therefore the theorem on multiplication of determinants holds also in Ruston's theory.

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[10]) $(I+\lambda KT)\xi=\xi_0$ in the original notation of Leżański.

[11]) See Sikorski [7].

On the two-norm convergence

by

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G. Fichtenholz [4] has introduced in some concrete Banach spaces a kind of convergence weaker than that generated by norm. In a previous paper [2] I introduced a general convergence in linear spaces which I called *two-norm convergence*, containing as particular cases the convergences of Fichtenholz. In this paper[1]) I shall complete the results obtained in [2].

1. Let X be an F-space (Banach [3], p. 35) and denote by $\|x\|$ the norm[2]) in X. Suppose that in X a second norm $\|x\|^*$ is defined, not stronger than $\|x\|$, i. e. such that

(i) $$\|x_n\|\to 0 \quad \text{implies} \quad \|x_n\|^*\to 0.$$

A sequence $\{x_n\}$ of elements of X will be called *γ-convergent to x_0* if it is bounded with respect to the norm $\|x\|$[3]) and if $\|x_n-x_0\|^*\to 0$; we shall then write

$$\gamma\text{-}\lim_n x_n=x_0 \quad \text{or} \quad x_n\xrightarrow{\gamma}x_0.$$

Convergence γ will be termed the *two-norm convergence*. The space X supplied with this convergence will be denoted by X_γ — it is evidently an L^*-space (Kuratowski [5], p. 84), moreover, addition of elements and multiplication by scalars are continuous.

A convergence generated by norm will be termed the *norm-convergence*. The convergence γ is in general not equivalent[4]) to a norm-con-

[1]) The results of which were presented on May 23th 1947 to the Polish Mathematical Society, Section of Poznań. Since that time Orlicz [7] has developed a theory of Saks spaces which are closely related to the notion of the two-norm convergence.

[2]) Here by a *norm* is meant an *F-norm*; it is a non-negative functional $\|x\|$, satisfying the postulates: (a) $\|x\|=0$ if and only if $x=0$; (b) $\|x+y\|\leqslant\|x\|+\|y\|$; (c) $a_n\to a_0$, $\|x_n-x_0\|\to 0$ implies $\|a_nx_n-a_0x_0\|\to 0$.

[3]) The sequence is *bounded with respect to* (or *under*) *the norm* $\|x\|$ if $t_n\to 0$ implies $\|t_nx_n\|\to 0$. This notion goes back to Banach.

[4]) Two convergences α and β in L^*-space are said to be *equivalent* if the classes of convergent sequences in both convergencies coincide and the limits under both are equal.