Remarks on the Poisson stochastic process (III)

(On a property of the homogeneous Poisson process)

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The purpose of this paper[1]) is to prove an invariance property of the homogeneous Poisson process.

Usually, realizations of the Poisson process are treated as non-decreasing integral valued functions $\omega(t)$, where $\omega(t_2)-\omega(t_1)$ is equal to the number of "calls" between t_1 and t_2.

Here we treat the Poisson process in another way, namely we consider as a realization of the process the set $\boldsymbol{X}=\{\boldsymbol{x}_j\}$ of all calls. (The numeration is arbitrary, *e. g.* by magnitude: $\ldots<\boldsymbol{x}_{-2}<\boldsymbol{x}_{-1}<0\leqslant\boldsymbol{x}_1<\boldsymbol{x}_2<\ldots$).

By definition the homogeneity of the process expresses the fact that all probability distributions for $\omega(t-c)$ are the same as for the primmitive process $\omega(t)$. This transformation of the time t corresponds to the replacement of $\boldsymbol{x}_j$ by $\boldsymbol{x}_j+c$.

In this paper we consider random translations instead of the fixed translation c. The main theorem 2 says that if the random variables $\boldsymbol{x}_j$ form a homogeneous Poisson process and the random variables $\boldsymbol{y}_j$ are independent of $\boldsymbol{x}_j$ and equivalent in the sense of de Finetti, then $\boldsymbol{x}_j+\boldsymbol{y}_j$ also form a homogeneous Poisson process.

Let us recall that random variables $\boldsymbol{y}_j$ are *equivalent in the sense of de Finetti* if the distribution function of any finite system $\boldsymbol{y}_{j_1},\boldsymbol{y}_{j_2},\ldots,\boldsymbol{y}_{j_k}$ (where the indices j_n are different) depends only on k. Obviously, stochastically independent random variables with the same distribution are equivalent. On the other hand, one repeated random variable $\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\ldots$, also forms a sequence of equivalent random variables.

In these investigations the passage from the stochastic independence to the de Finetti's equivalence has been proposed by E. Marczewski.

1. In this paper we understand by a *process* a random denumerable set $\boldsymbol{X}$ of real numbers; we shall define it by a sequence $\{x_j\}$ of random

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variables. We investigate only those properties of a process which are independent of the numeration of terms of the sequence. More precisely, we shall consider only properties of new random variables

$$\boldsymbol{n}(I)=\text{the number of indices } j \text{ such that } \boldsymbol{x}_j\in I,$$

defined for all intervals I.

We suppose that

(i) $\Pr\{\boldsymbol{x}_j\neq\boldsymbol{x}_k\}=1$ for $j\neq k$,

(ii) $\Pr\{\lim\limits_j|\boldsymbol{x}_j|=\infty\}=1$,

(iii) $\Pr\{\boldsymbol{x}_j=c\}=0$ for every j and every real c.

The condition (ii) guarantees with probability one that there is only a finite number of points $\boldsymbol{x}_j$ in every finite interval.

We say that $\{\boldsymbol{x}_j\}$ *presents a Poisson process* if the random variables $\boldsymbol{n}(I)$ have the Poisson distribution, and if, for any disjoint intervals $I_1,I_2,\ldots,I_k$, the random variables $\boldsymbol{n}(I_1),\boldsymbol{n}(I_2),\ldots,\boldsymbol{n}(I_k)$ are stochastically independent.

It is known [2] that the condition of independence alone implies the Poissonian form of the distribution functions:

$$\Pr\{\boldsymbol{n}(I)=k\}=\frac{[m(I)]^k}{k!}\,e^{-m(I)};$$

here $m(E)$ is a σ-additive measure on the line, finite for intervals, and vanishing for one-point sets.

We suppose in this paper that the considered process is *homogeneous*:

$$m(I)=\lambda|I|. \tag{1}$$

For the sake of simplicity we suppose that $\lambda=1$.

2. We shall prove formula (3) characterizing the Poisson process.

The homogeneous Poisson process has the following property:

LEMMA. *Let us fix an interval I and denote by Q_k the event $\boldsymbol{n}(I)=k$. Then, under the assumption Q_k, the conditional distribution of the set of k points of $\boldsymbol{X}$ belonging to I is the same as the uniform distribution of k independent points belonging to I.*

Proof. Let

$$I=I_1+I_2+\ldots+I_l,\quad \text{where } I_j \text{ are disjoint},$$

$$k=k_1+k_2+\ldots.+k_l,\quad \text{where } k \text{ are non negative integers.}$$

Then

$$\Pr\{\boldsymbol{n}(I_1)=k_1,\ldots,\boldsymbol{n}(I_l)=k_l\,|\,Q_k\}=\Pr\{\boldsymbol{n}(I_1)=k_1,\ldots,\boldsymbol{n}(I_l)=k_l\}/\Pr(Q_k)$$
$$=\frac{k!}{k_1!\,k_2!\ldots k_l!}\left(\frac{|I_1|}{|I|}\right)^{k_1}\ldots\left(\frac{|I_l|}{|I|}\right)^{k_l}, \qquad \text{q. e. d.}$$

THEOREM 1. *If $\boldsymbol{x}_j$ form a homogeneous Poisson process and if $f(t)$ is a complex valued function of a real variable with*

$$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty,$$

then the product

(2) $$\boldsymbol{p}_f=\prod_j[1+f(\boldsymbol{x}_j)]$$

is convergent with probability 1 and presents a random variable $\boldsymbol{p}_f$ with the expected value

(3) $$E(\boldsymbol{p}_f)=\exp\int_{-\infty}^{\infty}f(t)\,dt.$$

Conversely: if the formula (3) is satisfied for every complex valued function f with $\int|f|<\infty$, then $\{\boldsymbol{x}_j\}$ forms a homogeneous Poisson process.

Proof. Let us suppose that $f(t)\neq 0$ only in a finite interval I. If the product $\boldsymbol{p}_f$ is convergent and if $E(\boldsymbol{p}_f)$ exists, then obviously

(4) $$E(\boldsymbol{p}_f)=\sum_{k=0}^{\infty}E(\boldsymbol{p}_f|Q_k)\Pr(Q_k).$$

It follows from Lemma that

$$E(\boldsymbol{p}_f|Q_k)=\overbrace{\int_I\ldots\int_I}^{k}[1+f(t_1)]\ldots[1+f(t_k)]\,d\left(\frac{t_1}{|I|}\right)\ldots d\left(\frac{t_k}{|I|}\right)=\left\{\frac{1}{|I|}\int_I[1+f(t)]\,dt\right\}^k,$$

whence, by using (3), we obtain

$$E(\boldsymbol{p}_f)=\exp\int_I f(t)\,dt=\exp\int_{-\infty}^{\infty}f(t)\,dt.$$

Without the assumption of the existence of $\boldsymbol{p}_f$ and $E(\boldsymbol{p}_f)$, we may consider $|f|$ and $\boldsymbol{p}_{|f|}$, for which the right-hand side of (4) has been checked to be finite.

By the approximation of an arbitrary f (with $\int|f|<\infty$) by functions vanishing outside intervals, we obtain formula (3) without any additional hypotheses.

In order to prove the converse implication, let us consider an arbitrary system $I_1,I_2,\ldots,I_l$ of intervals and let us denote by $\chi_1,\chi_2,\ldots,\chi_l$ their characteristic functions (in the sense of the Theory of Sets).

Putting

$$f(t)=-1+\exp i[a_1\chi_1(t)+\ldots+a_l\chi_l(t)],$$

we easily obtain the identity

$$\boldsymbol{p}_f=\exp i[a_1\boldsymbol{n}(I_1)+\ldots+a_l\boldsymbol{n}(I_l)],$$

whence, in virtue of (3),

$$E\{\exp i[a_1\boldsymbol{n}(I_1)+\ldots+a_l\boldsymbol{n}(I_l)]\}=\exp\int_{-\infty}^{\infty}f(t)\,dt.$$

The left-hand side, treated as a function of $a_1,\ldots,a_l$ is the characteristic function (in the sense of the Probability Theory) of the random variables $\boldsymbol{n}(I_1),\ldots,\boldsymbol{n}(I_l)$. It follows from the first part of Theorem 1 that this characteristic function is the same as for the homogeneous Poisson process, q. e. d.

3. THEOREM 2. *If the random variables $\boldsymbol{x}_j$ present a homogeneous Poisson process and if the random variables $\boldsymbol{y}_j$ 1° are equivalent in the sense of de Finetti and 2° form a sequence independent of the sequence $\{\boldsymbol{x}_j\}$, then the random variables $\bar{\boldsymbol{x}}_j=\boldsymbol{x}_j+\boldsymbol{y}_j$ also present a homogeneous Poisson process (with the same value of parameter λ, cf. (1)).*

Proof. On account of Theorem 1 it suffices to prove formula (3) for the variables $\bar{\boldsymbol{x}}_j$. It follows from the hypotheses and from a theorem of Dynkin [1] on equivalent random variables that we can treat the random variables $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ as functions of three real variables defined in the unit cube $0\leqslant u,v,w\leqslant 1$, with the three-dimensional Lebesgue measure as probability:

$$\boldsymbol{x}_j=x_j(u,v,w)=x_j(u), \qquad \boldsymbol{y}_j=y_j(u,v,w)=y(v,\vartheta_j(w)),$$

where ϑ_j are independent functions having a uniform distribution in the unit interval:

$$|\{w: \alpha<\vartheta_j(w)<\beta\}|=\beta-\alpha \quad \text{for} \quad 0\leqslant\alpha\leqslant\beta\leqslant 1.$$

Let f be an arbitrary complex valued function with

$$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty.$$

Denoting by $\boldsymbol{p}_f$ the product (2) for $\boldsymbol{x}_j$, we have

$$E(\bar{\boldsymbol{p}}_f)=\int_0^1\int_0^1\int_0^1\prod_j\{1+f[x_j(u)+y(v,\vartheta_j(w))]\}\,du\,dv\,dw$$
$$=\int_0^1 dv\int_0^1\prod_j\{1+\int_0^1 f[x_j(u)+y(v,w)]\,dw\}\,du$$
$$=\int_0^1 dv\int_0^1\prod_j\{1+g[x_j(u),v]\}\,du,$$

where

$$g(t,v)=\int_0^1 f[t+y(v,w)]\,dw.$$

By applying the formula (3) for the function g, we obtain

$$E(\bar{\boldsymbol{p}}_f)=\int_0^1 dv\,\exp\int_{-\infty}^{\infty} g(t,v)\,dt=\exp\int_{-\infty}^{\infty} f(t)\,dt, \qquad \text{q. e. d.}$$

A theorem analogical to Theorem 2 is valid for the homogeneous Poisson process in n-dimensional Euclidean space.

References

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[2] C. Ryll-Nardzewski, *On the non-homogeneous Poisson process (I)*, Studia Mathematica 14 (1954), p. 124-128.

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