A generalization of a theorem of Khintchin

by

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The following definition is due to Khintchin and Lévy: Two one-dimensional distribution functions $F(x)$ and $G(x)$ are *of the same type* if there exist two constants $A>0$ and B such that the equality

$$G(x)=F(Ax+B)$$

holds.

Khintchin [1] has shown that if two sequences of distribution functions $F_n(x)$ and $G_n(x)$ where, for $n=1,2,\ldots$, $G_n(x)=F_n(A_n x+B_n)$, $A_n>0$ and B_n are arbitrary sequences of real constants, converge, as $n\to\infty$, to non-singular distribution functions $F(x)$ and $G(x)$ respectively, then $F(x)$ and $G(x)$ are of the same type.

This theorem of Khintchin plays an important role when the whole class of possible limiting distribution functions of some sequences of one-dimensional distribution functions is to be found.

A multidimensional generalization of this theorem is the object of this paper. The author [2] has applied this generalized theorem to the problem of finding the class of all possible limiting distributions of the multinomial distribution.

DEFINITION. We shall say that the probability functions P and G, defined in the i-dimensional space of points $(x_1,x_2,\ldots,x_i)$ are of *the same type* if there exists such a real linear transformation

$$y_m=\sum_{k=1}^{i} A_{mk}x_k+B_m \qquad (m=1,\ldots,i), \tag{1}$$

the determinant $|A_{mk}|$ $(m,k=1,2,\ldots,i)$ being different from 0, that the equality

$$G(S)=P(S') \tag{2}$$

holds, where S is an arbitrary Borel set and S' is the image of S given by (1).

The following theorem will be proved:

THEOREM. *Let the probability functions P_n and G_n for $n=1,2,\ldots$ be not singular and of the same type and let the sequences P_n and G_n converge for $n\to\infty$, to non-singular probability functions P and G respectively. Then P and G are of the same type.*

Proof. Let the assumptions of the theorem be satisfied. We shall write in (1) and (2) A_{nmk}, B_{nm} and S'_n respectively. We can choose—following the method of Cantor—such a subsequence n_a of indices that the following relations hold:

$$\begin{aligned}&\lim_{n_a\to\infty} A_{n_a mk}=A_{mk}\\ &\lim_{n_a\to\infty} B_{n_a m}=B_m\end{aligned} \qquad (k=1,2,\ldots,i;\ m=1,2,\ldots,i), \tag{3}$$

where $-\infty\leqslant A_{mk}\leqslant\infty$, $-\infty\leqslant B_m\leqslant\infty$.

For the sake of simplicity we shall assume — without restricting the generality of our considerations — that relations (3) hold for the sequence n of indices. We shall now show that for all considered m and k the inequalities

$$-\infty<A_{mk}<\infty, \qquad -\infty<B_m<\infty \tag{4}$$

hold. Indeed let us assume that for some m, say $m=1$, some A_{1k} $(k=1,\ldots,j;$ $j\leqslant i)$ are not finite. Let us now assume that there exists in the space $(x_1,x_2,\ldots,x_i)$ a hyperplane L such that for each point lying on one "side" of L, for instance "below" L, the relation

$$\lim_{n\to\infty}\Big(\sum_{k=1}^{i} A_{n1k}x_k+B_{n1}\Big)<\infty \tag{5}$$

holds, and for each point on the other "side" of L the relation

$$\lim_{n\to\infty}\Big(\sum_{k=1}^{i} A_{n1k}x_k+B_{n1}\Big)=\infty \tag{6}$$

holds. Let us consider two artbirary points $(x'_1,x'_2,\ldots,x'_j,x_{j+1},\ldots,x_i)$ and $(x''_1,x''_2,\ldots,x''_j,x_{j+1},\ldots,x_i)$ lying "below" L where $x'_k<x''_k$ if $A_{1k}=+\infty$ and $x'_k>x''_k$ if $A_{1k}=-\infty$. Then, taking into account relation (5), we have

$$\begin{aligned}&\lim_{n\to\infty}\Big(\sum_{k=1}^{j} A_{n1k}x'_k+\sum_{k=j+1}^{i} A_{n1k}x_k+B_{n1}\Big)\\ &=\lim_{n\to\infty}\sum_{k=1}^{j} A_{n1k}(x'_k-x''_k)+\lim_{n\to\infty}\Big(\sum_{k=1}^{j} A_{n1k}x''_k+\sum_{k=j+1}^{i} A_{n1k}x_k+B_{n1}\Big)=-\infty.\end{aligned} \tag{7}$$

Relation (7) implies then that for an arbitrary set S lying "below" the hyperplane L the equality

$$G(S)=0 \tag{8}$$

holds. On the other hand relation (6) implies the relation (8) for an arbitrary set S lying "above" the hyperplane L.

Relation (8) contradicts the assumption that the probability function G is non-singular.

On the other hand the assumption that there exists no hyperplane L satisfying relations (5) and (6) leads immediately to the conclusion that G is singular.

Let us now assume that a certain B_m is not finite, say $B_1=\infty$. Since the A_{1k} $(k=1,2,\ldots,i)$ are finite, this assumption implies relation (6), from which we deduce again relation (8).

Relations (4) are thus proved.

We shall now show that the rank of the matrix $[A_{mk}]$ is equal to i. Indeed let us assume that it is equal to $r<i$. We can thus suppose, for instance, that the $(r+1)$-th, $(r+2)$-th, $\ldots$, i-th row of the matrix are linear functions of the first r rows. In other words we have

$$A_{mk}=\lambda_{m1}A_{1k}+\lambda_{m2}A_{2k}+\ldots+\lambda_{mr}A_{rk}, \tag{9}$$

where $m=r+1,r+2,\ldots,i$. Thus, as $n\to\infty$, the image S' of an arbitrary set S will lie in an r-dimensional hyperplane.

Let us consider such continuity intervals S of G that for the set S'' lying "between" and "on" the images of S given by the transformations

$$\begin{aligned} y_1,\ldots,y_r,\ y_{r+1}&=\sum_{k=1}^{i}A_{(r+1)k}x_k+B_{r+1}-\delta,\ \ldots,\ y_i=\sum_{k=1}^{i}A_{ik}x_k+B_i-\delta,\\ y_1,\ldots,y_r,\ y_{r+1}&=\sum_{k=1}^{i}A_{(r+1)k}x_k+B_{r+1}+\delta,\ \ldots,\ y_i=\sum_{k=1}^{i}A_{ik}x_k+B_i+\delta, \end{aligned} \tag{10}$$

where $\delta>0$, the relation

$$\lim_{n\to\infty}P_n(S'')=P(S'') \tag{11}$$

holds. From the relations (3) follows that for sufficiently large n

$$\sum_{k=1}^{i}A_{mk}x_k+B_m-\delta\leqslant\sum_{k=1}^{i}A_{nmk}x_k+B_{nm}\leqslant\sum_{k=1}^{i}A_{mk}x_k+B_m+\delta \quad (m=r+1,\ldots,i). \tag{12}$$

and thus for such n

$$S'_n\subset S'',\quad G_n(S)=P_n(S'_n)\leqslant P_n(S''). \tag{13}$$

In virtue of relation (11) and of the fact that S is a continuity interval of G we obtain

$$G(S)\leqslant P(S''). \tag{14}$$

As $G(S)$ may for the considered intervals S take any value between 0 and 1, it follows from the inequality (14) — since δ may be arbitrarily small — that P is a singular probability function. Thus the rank of the matrix $[A_{mk}]$ is equal to i.

Now we have only to show that the equality (2) holds. Let R,S,T, where $R\subset S\subset T$, be equidimensional continuity intervals of G, the corresponding "sides" of R,S,T being parallel and the distance of the corresponding "sides" of R from those of S and those of S from those of T being equal to $\varepsilon>0$. Let R' and T' denote the image of R and T respectively given by (1). Let us choose ε so that for R' and T' the relation (11) holds. For sufficiently large n the relations

$$R'\subset S'_n\subset T',\qquad P_n(R')\leqslant P_n(S'_n)\leqslant P_n(T')$$

hold, and thus, as $n\to\infty$

$$P(R')\leqslant\varliminf_{n\to\infty}P_n(S'_n)\leqslant\varlimsup_{n\to\infty}P_n(S'_n)\leqslant P(T').$$

Since $\varepsilon>0$ may be arbitrarily small we obtain

$$\lim_{n\to\infty}G_n(S)=P(S'). \tag{15}$$

On the other hand, according to the assumptions of the theorem, the equality

$$\lim_{n\to\infty}G_n(S)=G(S) \tag{16}$$

holds. The relation (2) follows from formulae (15) and (16).

The generalization of Khintchin's theorem is thus proved.

References

[1] A. Khintchin, *Ueber Klassenkonvergenz von Verteilungsgesetzen*, Изв. НИИ Мат.-Мех. Томского Университета 1 (1937), p. 261-262.

[2] M. Fisz, *The limiting distributions of the multinomial distribution*, Studia Mathematica, this volume, p. 272-275.

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