On a class of operations over the space of integrable functions

by

W. ORLICZ (Poznań)

In this paper $M[u]$ will denote a convex non decreasing continuous function in $\langle 0,\infty)$, vanishing only for $u=0$ and such that $M[u]/u\to 0$ as $u\to 0$ and $M[u]/u\to\infty$ as $u\to\infty$. L^M will stand for the set of all measurable functions $x(t)$ in $\langle a,b\rangle$ for which there exists the integral

$$\int_a^b M[k|x(t)|]\,dt, \tag{1}$$

k being a constant (depending on $x(t)$) such that $0\leqslant k<1$.

In L^M a homogeneous norm $\|x\|$ may be defined as the infimum of the numbers l satisfying the condition

$$\int_a^b M[l^{-1}|x(t)|]\,dt\leqslant 1.$$

It may be shown[1]) that under the usual definitions of addition of elements and multiplication by scalars, with this norm L^M is a Banach space. The space L^M is separable if and only if the function $M[u]$ satisfies also the following condition:

$$M[2u]\leqslant cM[u] \quad \text{as} \quad u\geqslant u_0. \tag{Δ_2}$$

[1]) Z. W. Birnbaum and W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Mathematica 3 (1931), p. 1-67; W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Acad. Polonaise (1932), p. 207-220; W. Orlicz, *Über Räume L^M*, ibidem (1936), p. 93-107.

In the second and the third of the above papers another norm is defined, namely

$$\|x\|_M=\sup\left|\int_a^b x(t)\,y(t)\,dt\right|,$$

$y(t)$ denoting a measurable function such that $\int_a^b M'[|y(t)|]\,dt\leqslant 1$ and $M'(v)$ being a function called *complementary* to $M(u)$ (for its definition see the first of the above mentioned papers, p. 8). It is easily seen that $\|x\|\leqslant\|x\|_M$, $\|x\|_M\leqslant 2\|x\|$, whence both norms are equivalent.

If the condition (Δ_2) is satisfied, then for every $x(t)\in L^M$ the integral (1) exists with $k=1$.

Let us denote the differences of the second order of the sequence $M[n]$ by $\Delta^2 M[n]$, *i. e.*

$$\Delta^2 M[n]=M[n]-2M[n-1]+M[n-2],$$

where $M[-1]$ is to be set equal to 0. The convexity of $M[u]$ implies $\Delta^2 M[n]\geqslant 0$. If $M[u]=u^\alpha$, $1<\alpha$, then

$$\Delta^2 M[n]\sim n^{\alpha-2}.$$

The functions $x_n(t)$ being bounded and measurable in $\langle a,b\rangle$, let us write, given an integrable function $x(t)$ in $\langle a,b\rangle$,

$$S_k[x_n]=\sum_{n=1}^{\infty}\Delta^2 M[kn]\int_a^b |x(t)-x_n(t)|\,dt,$$

and let us denote by $I_k[x]$ the integral (1).

L^1 will denote the space of integrable functions with the usual norm $\|x\|_1$, $-M$ will stand for the space of bounded and measurable functions in the interval $\langle a,b\rangle$. The function $y(t)$ being measurable, we shall denote by $\sup^* y(t)$ the infimum of the numbers k such that the set $\underset{t}{E}\{y(t)>k\}$ is of measure 0.

THEOREM 1. 1° *Let the function $x(t)$ be integrable; then there exist measurable and bounded functions $x_n(t)$ such that*

$$\sup{}^*|x_n(t)|\leqslant n \quad \textit{for} \quad n=1,2,\ldots, \tag{2}$$

and for $0<k<1$, $0\leqslant l\leqslant k$, the inequality

$$S_l[x_n]\leqslant (M[1]+M[2]+\ldots+M[n_0-1])(b-a)+I_1[x]$$

is satisfied with

$$n_0=\mathrm{E}\left(\frac{1}{1-k}\right)+1.$$

2° *For every sequence $x_n(t)$ of bounded and measurable functions satisfying the inequality* (2), *and for the integrable function $x(t)$ the inequality*

$$I_k[x]\leqslant (M[1]+M[2]+\ldots M[n_1+1])(b-a)+S_1[x_n] \tag{3}$$

holds for $0<k<1$ with

$$n_1=\mathrm{E}\left(\frac{2k}{1-k}\right)+1.$$

Proof. Let us write

$$E_n=\underset{t}{E}\{(n-1)\leqslant |x(t)|<n\} \quad \text{for} \quad n=1,2,\ldots;$$

then the following inequalities are satisfied:

$$\sum_{n=1}^{\infty} M[k(n-1)]|E_n| \leqslant I_k[x] \leqslant \sum_{n=1}^{\infty} M[kn]|E_n|. \tag{4}$$

Now, for $x(t)$, we define

$$x_n^*(t) = \begin{cases} x(t) & \text{if } |x(t)| < n, \\ n \operatorname{sign} x(t) & \text{elsewhere;} \end{cases}$$

then

$$\sum_{i=0}^{\infty} (i+1)|E_{n+2+i}| \leqslant \int_a^b |x(t)-x_n^*(t)|\,dt \leqslant \sum_{i=0}^{\infty} (i+1)|E_{n+i}|,$$

$$\sum_{n=1}^{\infty} \Big(\sum_{i=1}^{n} (n+1-i)\,w(i)\Big)|E_{n+2}| \tag{5}$$

$$\leqslant \sum_{n=1}^{\infty} w(n) \int_a^b |x(t)-x_n^*(t)|\,dt \leqslant \sum_{n=1}^{\infty} \Big(\sum_{i=1}^{n} (n+1-i)w(i)\Big)|E_n|,$$

where $w(n)$ stands for an arbitrary sequence with non negative terms. We choose $w(n)$ so that

$$\sum_{i=1}^{n} (n+1-i)\,w(i) = M[kn] \quad \text{for} \quad n=1,2,\ldots$$

Since

$$M[kn]-M[k(n-1)] = w(1)+w(2)+\ldots+w(n),$$

we get

$$\Delta^2 M[kn] = w(n).$$

Then (5) implies for $k=l$

$$S_l[x_n^*] \leqslant \sum_{n=1}^{\infty} M[ln]|E_n|.$$

Since for

$$n \geqslant n_0 = \mathrm{E}\left(\frac{1}{1-k}\right)+1, \qquad 0 \leqslant l \leqslant k,$$

the inequality $M[n-1] > M[kn] \geqslant M[ln]$ holds, and

$$S_l[x_n^*] \leqslant \sum_{n=1}^{n_0-1} M[kn]|E_n| + \sum_{n=n_0}^{\infty} M[n-1]|E_n|,$$

we get by (4)

$$S_l[x_n^*] \leqslant \Big(\sum_{n=1}^{n_0-1} M[n]\Big)(b-a)+I_1[x].$$

Now let the condition (2) be satisfied for measurable functions $x_n(t)$. Since $t \in \langle a,b\rangle$ implies

$$|x(t)-x_n^*(t)| \leqslant |x(t)-x_n(t)|,$$

we get

$$S_k[x_n^*] \leqslant S_k[x_n].$$

By (5)

$$\sum_{n=1}^{\infty} M[n]|E_{n+2}| \leqslant S_1[x_n^*].$$

For

$$n \geqslant n_1 = \mathrm{E}\left(\frac{2k}{1-k}\right)+1$$

the inequality

$$M[n] \geqslant M[k(n+2)]$$

is satisfied, whence

$$\sum_{n=1}^{n_1-1} M[n]|E_{n+2}| + \sum_{n=n_1}^{\infty} M[k(n+2)]|E_{n+2}| \leqslant S_1[x_n^*] \leqslant S_1[x_n],$$

$$\sum_{n=1}^{\infty} M[kn]|E_n| \leqslant S_1[x_n] + \Big(\sum_{n=1}^{n_1+1} M[n]\Big)(b-a);$$

this, together with (4), gives

$$I_k[x] \leqslant S_1[x_n] + \Big(\sum_{n=1}^{n_1+1} M[n]\Big)(b-a).$$

THEOREM 2. 1° *Let $x(t)$ be in L^M; then there exist measurable functions $x_n(t)$ such that*

$$\sup{}^* |x_n(t)| \leqslant Kn \quad \textit{for} \quad n=1,2,\ldots, \tag{6}$$

where $K=\|x\|$, satisfying the inequality

$$S_{1/2}[x_n] \leqslant C_1\|x\|, \tag{7}$$

where $C_1 = 1+(M[1]+M[2])(b-a)$.

2° *If, for a sequence of bounded and measurable functions $x_n(t)$, the condition (6) is satisfied with a constant K, and*

$$S_k[x_n] = S < \infty,$$

where $0<k\leqslant 1$, then $x(t)\in L^M$ and

$$\|x\| \leqslant C_2\frac{K}{k}+2\frac{S}{k},$$

where $C_2 = 2\sup\big((M[1]+\ldots+M[4])(b-a),1\big)$.

Proof. *Ad* 1°. If $x(t)\in L^M$, $\bar{x}(t)=x(t)/\|x\|$, then

$$\int_a^b M[|\bar{x}(t)|]\,dt \leqslant 1,$$

whence applying Theorem 1.1^0 to the function $\bar{x}(t)$ with $k=1/2$ we see that there exist measurable functions $\bar{x}_n(t)$ such that $\sup^*|\bar{x}_n(t)|\leqslant n$ and

$$S_{1/2}[\bar{x}_n]\leqslant (M[1]+M[2])(b-a)+I_1[\bar{x}],$$

whence

$$S_{1/2}[\bar{x}_n]\leqslant C_1.$$

This implies immediately that the inequality (7) is satisfied for the functions $x(t)$ and $x_n(t)=\|x\|\bar{x}_n(t)$.

Ad 2^0. Let us apply theorem 1.2^0, with $k=1/2$, replacing the function $M[u]$ by $M[ku]$ and substituting $x(t)/K$ and $x_n(t)/K$ for $x(t)$ and $x_n(t)$ respectively. There follows

$$I_{1/2}\left[\frac{x}{K}\right]\leqslant (M[1]+\ldots+M[4])(b-a)+\frac{S}{K},$$

i. e.

$$\int_a^b M\left[\frac{k|x(t)|}{2K}\right]dt\leqslant (M[1]+\ldots+M[4])(b-a)+\frac{S}{K}.$$

Let

$$\varrho=(M[1]+\ldots+M[4])(b-a)+\frac{S}{K}.$$

If $\varrho\leqslant 1$, then

$$\int_a^b M\left[\frac{k|x(t)|}{2K}\right]dt\leqslant 1,$$

whence

$$\|x\|\leqslant\frac{2K}{k}\leqslant C_2\frac{K}{k}+2\frac{S}{k};$$

if $\varrho>1$, then

$$\int_a^b M\left[\frac{k|x(t)|}{2K\varrho}\right]dt\leqslant 1,$$

whence

$$\|x\|\leqslant 2K\varrho/k,$$

or equivalently

$$\|x\|\leqslant C_2\frac{K}{k}+2\frac{S}{k}.$$

THEOREM 3[2]. *Let the operation $U(x)=U(x,t)$ from L^1 to L^1 satisfy the conditions:*

(a) *$U(x)$ maps the function $x(t)=0$ into itself;*

(b) *$U(x)$ satisfies the condition of Lipschitz:*

$$\|U(x)-U(y)\|_1\leqslant K_1\|x-y\|_1 \quad \textit{for} \quad x,y\in L^1;$$

(c) *if x is a bounded and measurable function, the function $U(x,t)$ is also bounded and measurable,*

(d) $\sup^*|U(x,t)-U(y,t)|\leqslant K\sup^*|x(t)-y(t)|$ *for* $x,y\in M$.

Under these hypotheses:

(α) *$x\in L^M$ implies $U(x)\in L^M$,*

(β) *$U(x)$, as an operation from L^M to L^M, satisfies the Lipschitz condition, i. e.*

$$\|U(x)-U(y)\|\leqslant K_M\|x-y\| \quad \textit{for} \quad x,y\in L^M,$$

(γ) *the constant K_M above may be chosen as*

$$K_M=2C_2K+4C_1K_1.$$

Proof. Let $x\in L^M$ and let $x(t)$ and $x_n(t)$ denote such functions as in theorem 2.1^0. By (a) and (c)

$$\sup^*|U(x_n,t)|\leqslant K\sup^*|x_n(t)|\leqslant K\|x\|n, \tag{8}$$

and since (b) implies

$$\|U(x)-U(x_n)\|_1\leqslant K_1\|x-x_n\|_1,$$

we infer by (7)

$$\begin{aligned}&\sum_{n=1}^{\infty}\Delta^2 M\left[\frac{1}{2}n\right]\int_a^b|U(x,t)-U(x_n,t)|\,dt\\ &\leqslant K_1\sum_{n=1}^{\infty}\Delta^2 M\left[\frac{1}{2}n\right]\int_a^b|x(t)-x_n(t)|\,dt\leqslant K_1C_1\|x\|.\end{aligned} \tag{9}$$

In virtue of (8) and (9) we may apply Theorem 2.2^0 to the functions $U(x,t)$ and $U(x_n,t)$ with $k=1/2$. Thus $U(x)\in L^M$ and

$$\|U(x)\|\leqslant 2C_2K\|x\|+4K_1C_1\|x\|=K_M\|x\|.$$

Given a bounded and measurable function $y(t)$ let us define the operation

$$V(x)=U(x+y)-U(y).$$

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[2]) This is a generalization of a theorem of the author. See W. Orlicz, *Ein Satz über die Erweiterung von linearen Operationen*, Studia Mathematica 5 (1935), p. 127-140; Theorem 1, p. 133.

Since $V(x)$ satisfies the conditions (a)-(d) of our Theorem, therefore, by what has just been proved,

(10) $$\|V(x-y)\| = \|U(x)-U(y)\| \leqslant K_M\|x-y\|.$$

Now we shall prove the validity of the inequality (10) for every $x, y \in L^M$. Let $x_n(t)$, $y_n(t)$ be the functions defined as

$$x_n(t) = \begin{cases} x(t) & \text{if } |x(t)| < n, \\ n \operatorname{sign} x(t) & \text{elsewhere}, \end{cases} \qquad y_n(t) = \begin{cases} y(t) & \text{if } |y(t)| < n, \\ n \operatorname{sign} y(t) & \text{elsewhere}. \end{cases}$$

Since

$$|x_n(t) - y_n(t)| \leqslant |x(t) - y(t)|$$

for $a \leqslant t \leqslant b$, therefore

$$\int_a^b M\left[\frac{|x_n(t)-y_n(t)|}{\|x-y\|}\right] dt \leqslant \int_a^b M\left[\frac{|x(t)-y(t)|}{\|x-y\|}\right] dt \leqslant 1,$$

and this implies

(11) $$\|x_n - y_n\| \leqslant \|x-y\|.$$

Let z_n be functions of L^M converging asymptotically to $z(t)$, and let

$$l = \varliminf_{n\to\infty} \|z_n\|, \qquad \|z_{n_i}\| \to l.$$

If $l > 0$, the asymptotic convergence of $z_{n_i}(t)/\|z_{n_i}\|$ to $z(t)/l$ implies

$$1 \geqslant \varliminf_{i\to\infty} \int_a^b M\left[\frac{|z_{n_i}(t)|}{\|z_{n_i}\|}\right] dt \geqslant \int_a^b M\left[\frac{|z(t)|}{l}\right] dt,$$

whence

(12) $$\|z\| \leqslant l = \varliminf_{n\to\infty} \|z_n\|.$$

This inequality is valid also if $l = 0$, for in this case

$$\int_a^b M[|z_{n_i}(t)|]\, dt \leqslant \|z_{n_i}\|$$

for almost all i's, whence $z(t) = 0$.

By hypothesis (b) the sequence $z_n(t) = U(x_n) - U(y_n)$ converges asymptotically to $z(t) = U(x) - U(y)$; thus (10)-(12) implies

$$\|U(x) - U(y)\| \leqslant \varliminf_{n\to\infty} \|U(x_n) - U(y_n)\| \leqslant K_M \|x-y\|.$$

Remark 1. Let us note that the condition (Δ_2) was not assumed for $M[u]$ in the above Theorem, and this implies in general that

$$\varlimsup_{n\to\infty} \|x - x_n\| > 0, \qquad \varlimsup_{n\to\infty} \|y - y_n\| > 0.$$

Remark 2. If $U(x)$ is a linear operation from L^1 to L^1 mapping bounded functions into bounded functions, then the hypotheses of Theorem 3 are satisfied.

Indeed, by a well-known theorem of Banach[3]), $U(x)$ is linear as an operation from M to M, and this implies the hypothesis (d); $U(x)$ being linear in L^1, the conditions (a) and (b) must also be satisfied.

Theorem 4. *Let $M[u]$ satisfy the condition (Δ_2) and let operations $U_n(x)$ satisfy the hypotheses* (a)-(d) *from theorem* 3, *with common Lipschitz constants K_1 and K. If the sequence $U_n(x,t)$ converges asymptotically for every x belonging to a set R dense in L^M, composed of bounded functions, then for every $x(t) \in L^M$*

(13) $$\|U_n(x) - U_m(x)\| \to 0,$$

as $m, n \to \infty$.

Proof. By theorem 3

(14) $$\|U_n(x) - U_n(y)\| \leqslant K_M \|x-y\| \quad \text{for} \quad n = 1, 2, \ldots$$

By our hypothesis

$$\sup{}^* |U_n(x,t)| \leqslant K^* \sup{}^* |x(t)|$$

for $x \in R$, and $U_n(x,t) - U_m(x,t) \overset{as}{\to} 0$, as $m, n \to \infty$, whence we infer that (13) is satisfied for $x \in R$. Indeed, $x \in R$ implies

(15) $$\int_a^b M[|U_n(x,t) - U_m(x,t)|]\, dt \to 0,$$

and (15) implies (13) if $M[u]$ satisfies the condition (Δ_2). In order to prove (13) for every $x \in L^M$ it suffices to apply the inequality (14) and the following one:

$$\|U_n(x) - U_m(x)\| \leqslant \|U_n(x) - U_n(y)\| + \|U_n(y) - U_m(y)\|$$
$$+ \|U_m(y) - U_m(x)\| \leqslant 2K_M \|x-y\| + \|U_n(y) - U_m(y)\|;$$

here $y \in R$.

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[3]) See S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa 1932; Théorème 7, p. 41.