On the ergodic theorems (III)

(The random ergodic theorem)

by

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1. H. R. Pitt, S. Ulam, J. von Neumann and S. Kakutani have formulated the so-called *random ergodic theorems*[1]). The most general one is that of Kakutani.

In a part of his proof Kakutani uses the theory of Markoff processes with a stable distribution.

E. Marczewski has proposed that a direct proof of Kakutani's theorem should be found which would not use the hypothesis that the considered transformations are 1-1. In this paper I give a brief and direct proof of Kakutani's theorem thus generalized and I also prove that the limit function $\bar{f}$ is essentially independent of the parameters $t_1, t_2, \ldots$ The existence proof of $\bar{f}$ is a reproduction of the first part of Kakutani's proof.

2. Let m be a σ-measure in a σ-field $\boldsymbol{M}$ of subsets of a space X. Let us suppose that $m(X)=1$, and that m is complete (*i. e.* that if $A \epsilon \boldsymbol{M}$, $B \subset A$ and $m(A)=0$, then $B \epsilon \boldsymbol{M}$).

We consider a family $\{\varphi_t\}_{t \epsilon T}$ of transformations of X into itself, which are measurable and preserve m, *i. e.* such that for every $t \epsilon T$ and every $E \epsilon \boldsymbol{M}$ we have $\varphi_t^{-1}(E) \epsilon \boldsymbol{M}$ and $m\varphi_t^{-1}(E)=m(E)$. Let p be a complete σ-measure in a σ-field P of subsets of T. We suppose $p(T)=1$. The family $\{\varphi_t(x)\}$ may be treated as a transformation of $X \times T$ into X; let us suppose that it is measurable with respect to the completed direct σ product $m \times p$.

A *transformation* φ of X into itself is called *indecomposable* if we have $m(E)=0$ or 1 for every set $E \epsilon \boldsymbol{M}$ which is almost φ invariant (*i. e.* such that the symmetric difference $E \dot{-} \varphi^{-1}(E)$ is of m-measure zero). The *family* φ_t *is indecomposable* if $m(E)=0$ or 1 for every set E which is almost φ_t invariant for almost (in the sense of the measure p) all $t \epsilon T$.

[1]) Pitt [2], p. 342, Ulam and von Neumann [4], Kakutani [1].

In the sequel we shall consider the products

$$X \times T_1 \times T_2 \times \ldots \qquad \text{where} \quad T_j = T \quad \text{for} \quad j=1,2,\ldots$$
$$T_n \times T_{n+1} \times \ldots$$

and, in these spaces, the completed direct σ-products of measures enumerated above. These *product measures* are defined for sets which will briefly be called *measurable*.

A *real* (or *complex*) *function* defined on the spaces considered is called *measurable* if the converse image of any open set is measurable. We write $f \equiv g$ if f and g are equal almost everywhere in the sense of the considered measure.

Now we shall prove the

RANDOM ERGODIC THEOREM. *For every m-integrable function $f(x)$ there exists an m-integrable function $\bar{f}(x)$ such that*

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\big(\varphi_{t_k}(x) \ldots \varphi_{t_1}(x)\big) = \bar{f}(x)$$

for almost all $x, t_1, t_2, \ldots$ The limit function $\bar{f}$ is almost invariant with respect to the transformations φ_t, i. e.

$$\bar{f}\big(\varphi_t(x)\big) = \bar{f}(x)$$

for almost all x, t. Hence, if the family $\{\varphi_t\}$ is indecomposable, then $\bar{f}$ is constant.

Proof. Let us consider a transformation ψ of the product $Y = X \times T_1 \times T_2 \times \ldots$ (where $T_j = T$ for $j=1,2,\ldots$) into itself, defined as follows:

$$\psi(x, t_1, t_2, t_3, \ldots) = \big(\varphi_{t_1}(x), t_2, t_3, \ldots\big).$$

From the hypothesis on the family $\{\varphi_t\}$ it follows easily that ψ is measurable and preserves the product measure in Y.

By treating the function f as defined in Y (*i. e.* by putting $f(x, t_1, t_2, \ldots) = f(x)$) and by applying the ordinary individual ergodic theorem[2]) for f and ψ, we obtain the formula $(*)$, where $\bar{f}$ is a priori dependent on all variables $x, t_1, t_2, \ldots$ and ψ invariant.

The remaining part of the theorem results directly from the following

THEOREM 1. *If the function $g(x, t_1, t_2, \ldots)$ is measurable in Y and almost ψ-invariant, i. e.*

$$(1) \qquad g(x, t_1, t_2, \ldots) \equiv g\big(\varphi_{t_1}(x), t_2, \ldots\big),$$

[2]) See e. g. Riesz [3], p. 224.

then g essentially depends only on x, i. e. there is a function $g(x)$, defined in X such that

$$g(x, t_1, t_2, \ldots) \equiv g(x).$$

Obviously

$$g(\varphi_t(x)) \equiv g(x).$$

Proof. We may suppose, without loss of generality, that g is bounded.

Let $h(t_1, t_2, \ldots)$ be a measurable function with $|h| \leqslant M$ and such that

$$\int h(t_1, t_2, \ldots)\, dt_1 dt_2 \ldots = 0. \tag{2}$$

Let us set

$$H(x) = \int g(x, t_1, t_2, \ldots) h(t_1, t_2, \ldots)\, dt_1 dt_2 \ldots$$

We shall prove that

$$H(x) \equiv 0. \tag{3}$$

It follows, by iterations, from (1) that

$$g(x, t_1, t_2, \ldots) \equiv g(\varphi_{t_{n-1}}(x) \ldots \varphi_{t_1}(x) \ldots, t_n, t_{n+1}, \ldots),$$

whence, putting

$$H_n(x, t_1, t_2, \ldots) \equiv \int g(x, t_1, t_2, \ldots) h(t_n, t_{n+1}, \ldots)\, dt_n dt_{n+1} \ldots,$$

we obtain

$$\begin{aligned} H_n(x, t_1, t_2, \ldots) &\equiv \int g(\varphi_{t_{n-1}}(x) \ldots \varphi_{t_1}(x), t_n, t_{n+1}, \ldots) h(t_n, t_{n+1}, \ldots)\, dt_n dt_{n+1} \ldots \\ &\equiv H(\varphi_{t_{n-1}}(x) \ldots \varphi_{t_1}(x)). \end{aligned} \tag{4}$$

We choose a function $g_\varepsilon(x, t_1, t_2, \ldots, t_N)$ such that

$$\int |g - g_\varepsilon|\, dx\, dt_1 dt_2 \ldots < \frac{\varepsilon}{M} \tag{5}$$

where N depends on ε.

If $n > N$, then in view of (2),

$$H_n(x, t_1, t_2, \ldots) \equiv \int (g - g_\varepsilon) h(t_n, t_{n+1}, \ldots)\, dt_n dt_{n+1} \ldots$$

whence it follows from (5) that

$$\int |H_n(x, t_1, t_2, \ldots)|\, dx\, dt_1 dt_2 \ldots < \frac{\varepsilon}{M} M = \varepsilon.$$

The identity (4) implies

$$\begin{aligned} \int |H(x)|\, dx &= \int |H(\varphi_{t_{n-1}}(x) \ldots \varphi_{t_1}(x))|\, dx\, dt_1 dt_2 \ldots \\ &= \int |H_n(x, t_1, t_2, \ldots)|\, dx\, dt_1 dt_2 \ldots < \varepsilon \end{aligned}$$

and, consequently, (3).

Applying the auxiliary theorem (see Section 3) for $y = (t_1, t_2, \ldots)$, we obtain the proposition of our theorem.

The random ergodic theorem is thus proved.

Finally let us observe that it follows from this theorem that the family φ_t is indecomposable if and only if the transformation ψ is such[3]).

3. We shall prove the above mentioned auxiliary theorem. It concerns measurable functions of a pair of variables (x, y) running on the direct product of two σ-measure spaces (with normed measures).

THEOREM. *If*

$$\iint |g(x, y)|\, dx\, dy < \infty$$

and if

$$\int g(x, y) h(y)\, dy = 0$$

almost everywhere for every bounded function h such that $\int h(y)\, dy = 0$, then g essentially depends only on x, i. e. there is a function $g^*(x)$ of one variable such that $g = g^*$ almost everywhere.

Let $f(x)$ and $h(y)$ be arbitrary bounded functions. By applying the hypothesis to the function $h(y) - \int h(y)\, dy$, we obtain the identity

$$\int g(x, y) h(y) dy = \int g(x, y)\, dy \int h(y)\, dy$$

almost everywhere, whence

$$\iint [g(x, y) - \int g(x, y)\, dy] f(x) h(y)\, dx\, dy = 0.$$

It follows from the arbitrariness of f and h that

$$g(x, y) - \int g(x, y)\, dy = 0$$

almost everywhere, and consequently we can put

$$g^*(x) = \int g(x, y)\, dy.$$

References

[1] S. Kakutani, *Random ergodic theorems and Markoff processes with a stable distribution*, Proceedings of the Second Berkeley Symposium on mathematical statistics and probability 1950 (1951), p. 247-261.

[2] H. R. Pitt, *Some generalizations of the ergodic theorem*, Proceedings of the Cambridge Philosophical Society 38 (1942), p. 325-343.

[3] F. Riesz, *Sur la théorie ergodique*, Commentarii Mathematici Helvetici 17 (1944-5), p. 221-239.

[4] S. M. Ulam and J. von Neumann, *Random ergodic theorem*, Bulletin of the American Mathematical Society 51 (1954), p. 660.

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[3]) Cf. Kakutani [1], p. 258, Theorem 3, the equivalence of (a) and (f).