On a class of operations over the space of continuous vector valued functions

by

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1. By \( \omega(u) \) we shall denote a non decreasing function, defined for \( u \geq 0 \), positive for \( u > 0 \), vanishing for \( u = 0 \) and such that \( \lim_{u \to 0} \omega(u) = 0 \).

We shall say that the function \( \omega(u) \) satisfies the condition (m) if

(a) \[ \omega(\|v\|) \leq \omega(u) \omega(v) \]

(b) \[ \frac{\omega(u)}{u} \to 0 \quad \text{as} \quad u \to \infty. \]

The condition (m) implies

(b') \[ \frac{\omega(u)}{u} \to \infty \quad \text{as} \quad u \to 0. \]

Indeed, by (a)

\[ \frac{\omega(u)}{u} \geq \frac{\omega(1)}{\omega(1/u)}. \]

The functions \( \omega(u) = u^a \) or \( \omega(u) = u^a ([\ln u] + 1/a) \), where \( 0 < a < 1 \), satisfy the condition (m).

\( X \) will denote a Banach space. \( C(X) \) will stand for the Banach space of continuous \( X \)-valued functions \( x(t) \) defined in \( A = (a,b) \) under the usual definitions of addition and multiplication by scalars and with the norm

\[ \|x\| = \max_{a \leq t \leq b} |x(t)|. \]

By \( C(X)_p \) we shall denote the space of continuous \( X \)-valued functions \( x(t) \) defined for \( -\infty < t < \infty \) and of period \( p \); \( C(X)_p \) may become, as above, Banach space (if we define the norm by the above formula with \( A = (0,p) \)).
Given a function \( a(u) \), we denote for \( \mathbf{a}(t) \) belonging to \( O(X) \) or \( O(X)_p \),
\[
\sup_{x \in \Omega} \| \mathbf{a}(t+h)-\mathbf{a}(t) \| = \mu,
\]
where in the case if \( \mathbf{a} \in O(X) \) the supremum is taken for \( t \in A, a-t \leq h \leq b-t \),
and if \( \mathbf{a} \in O(X)_p \) for arbitrary \( t, h \).

By \( L_0(X) \) or \( L_0(X)_p \) respectively we shall denote the linear space
of functions of \( O(X) \) or \( O(X)_p \) respectively, for which \( \mu < \infty \). Under the usual
definitions of addition and multiplication by scalars, and with the norm
\[
\| \mathbf{a}(t) \| = \max_{x} \| \mathbf{a}(t) \| + \mu,
\]
they are Banach spaces.

If \( a(u)=u^\alpha, 0<\alpha<1 \), we shall write \( L_0(X) \) instead of \( L_0(X) \), and
\( \| \mathbf{a} \| \) instead of \( \| \mathbf{a} \| \). The functions of \( L_0(X) \) with \( 0<\alpha<1 \) are said to
satisfy the Hölder condition with the exponent \( \alpha \), the functions of \( L_0(X) \) are
said to satisfy the Lipschitz condition. In the last case the constant \( k \), such that for \( a-t \leq h \leq b-t \)
\[
\| \mathbf{a}(t+h)-\mathbf{a}(t) \| \leq k|h|,
\]
is called the Lipschitz constant. Analogous terminology will be used for
spaces of periodic functions.

Obviously \( O(X) \supseteq L_0(X) \supseteq L_0(X) \) if \( \alpha < \beta \).

By \( G_0(X) \) or \( G_0(X)_p \) we shall denote a complete subspace of \( O(X) \) or \( O(X)_p \), respectively.
If we restrict the functions \( \mathbf{a}(t) \) to run over the space
\( G_0(X) \) then we obtain a complete subspace \( G_0(X)_p \) of the space \( L_0(X) \).

We shall say that the space \( G_0(X)_p \) is translation-invariant if \( \mathbf{a}(t) \in G_0(X)_p \)
implies, for every \( t, \mathbf{a}(t+\tau) \in G_0(X)_p \),

**Lemma.** If for every \( \tau \in \langle \zeta, \zeta' \rangle \) the function \( \mathbf{a}(\tau; t) \) belongs to \( G_0(X) \)
\( \text{to } G_0(X)_p \) and \( \mathbf{a}(\tau; t) \) depends continuously on the parameter \( \tau \), then the function
\[
y(t) = \int_{\tau}^{\tau'} \mathbf{a}(\tau; t) d\tau
\]
also belongs to \( G_0(X) \) \( \text{to } G_0(X)_p \). The integral is taken in the sense of Riemann-
Graves.

**Proof.** Given a partition \( \pi: \tau_0 < \tau_1 < \ldots < \tau_m = \tau' \), let us write
\[
\mathbf{a}(\pi; t) = \sum_{\pi} \mathbf{a}(\tau_{i+1}; t)/(\tau_i - \tau_{i+1}).
\]
The function of two variables \( \mathbf{a}(\tau; \tau') = \mathbf{a}(\tau, t) \) is uniformly continuous
in \( \langle \zeta', \zeta' \rangle \times A \). Let \( \Omega(\delta) \) be the modulus of continuity of \( \mathbf{a}(\tau, t) \); then,
for every partition \( \pi, \)
\[
\| \mathbf{a}(\tau, \tau') - \mathbf{a}(\tau', \tau'') \| \leq \| \mathbf{a}(\tau', \tau'') \| \Omega(\delta),
\]
if \( | \tau' - \tau'' | < \delta \). Given a normal sequence of partitions \( \{ \pi_n \} \), we see that
\( \pi_n = \tau, \tau_n \) uniformly in \( \langle \zeta, \zeta \rangle \), for \( \mathbf{a}(\pi_n; t) \rightarrow \mathbf{a}(\tau, t) \) as every \( t \in \langle \zeta, \zeta \rangle \),
and the functions \( \mathbf{a}(\pi_n; t) \) are uniformly continuous. Hence
\[
\| \mathbf{a}_n - \mathbf{a} \| \rightarrow 0, \quad \text{if } \mathbf{a} \in G_0(X).
\]

**Theorem 2.** Let the space \( G_0(X)_p \) be translation-invariant; then it is
possible to define linear operations \( T_n(\mathbf{a}) \) from \( G_0(X)_p \) to \( G_0(X)_p \),
is such a manner that:

(a) if \( \mathbf{a} \in G_0(X)_p \), \( \mathbf{a} = a(\mathbf{a}) \) being fixed, the functions
\( \mathbf{a}_n(\cdot) - \mathbf{a}(\cdot) \) satisfy the Lipschitz condition with the constant
\[
\| \mathbf{a}_n - \mathbf{a} \| = Bn a(\mathbf{a}) / \| \mathbf{a} \|,
\]

(b) for \( n = 1, 2, \ldots \),
\[
\| \mathbf{a}_n - \mathbf{a} \| \leq A_n = (1/n). \]

(c) the constants \( A, B \) in (1) and (2) do not depend on \( n \).

It is possible to define \( T_n(\mathbf{a}) \) such that
\[
A = \| \mathbf{a} \|, \quad B = \| \mathbf{a} \|.
\]

The theorem remains true if we remove the condition of translation-
invariance of the space and replace the space \( G_0(X)_p \) by the space \( O(X)_p \).

In this case we may set \( A = \| \mathbf{a} \|, \quad B = \| \mathbf{a} \| \).

**Proof.** The particular case where \( X \) is the space of real numbers
and \( G_0(X)_p \) is identical with \( O(X)_p \) is well known. In this case the operations
\( T_n(\mathbf{a}) \) may be defined in several ways, e.g. by means of singular integrals. This device may be adapted to the space \( G(X)_p \)
satisfying the translation condition.

As an example we present three kinds of the introduction of \( T_n(\mathbf{a}) \),
taking \( p = 2n \).

1. Let us write
\[
T_n(\mathbf{a}) = \frac{1}{2^n} \int_{\zeta}^{\zeta + 2n} \mathbf{a}(\tau, \tau') d\tau = \frac{1}{2^n} \int_{\zeta}^{\zeta + 2n} \mathbf{a}(\tau, \tau') d\tau.
\]
By our Lemma $T_n(x) = C_n(X)_h$. Since $s_n(t) = \left[ \frac{x(t) + \frac{1}{n}}{n} \right] x(t)$, we get

$$\|s_n(t)\| \leq n\|x\|_n \omega \left( \frac{1}{n} \right),$$

$$\|s_n(t)\| \leq 2\|x\|_n \omega \left( \frac{1}{n} \right).$$

Hence $s_n(t)$ satisfies the Lipschitz condition with the constant (1), where $B = \|x\|_n$ and $T_n(x)$ is a continuous operation from $C_n(X)_h$ to $C_n(X)_h$. Moreover,

$$\|x(t) - s_n(t)\| \leq n \int_0^1 \|x(t) - x(t + \tau)\| d\tau \leq \|x\|_n \omega \left( \frac{1}{n} \right).$$

This implies the condition (b) with $A = \|x\|_n$.

2. Set

$$k_n(t) = \sin \frac{\pi t}{2} \cos \frac{\pi t}{2}, \quad y_n = \int_0^\pi k_n(t) dt = \frac{n(2n^2 + 1)}{3}.$$

We define first the Jackson integrals $s_n(t) = S_n(t) = \int_0^\pi x(r) k_n(r-t) dr$.

If $x(t) \in C_n(X)_h$, then, by our Lemma, $s_n(t) \in C_n(X)_h$. The same estimations as in the case of real functions $x(t)$ give

$$\|x - s_n\|_o \leq \|x\|_n \omega \left( \frac{1}{n} \right),$$

$$\|x\|_n \leq \|x\|_o.$$

If $x(t)$ satisfies the Lipschitz condition with the constant $K$, then $s_n(t)$ satisfies this condition with the same constant, for

$$\|s_n(t + h) - s_n(t)\| \leq \int_0^\pi \|x(t + \tau - h) - x(t + \tau)\| k_n(\tau) d\tau \leq K\|x\|_n.$$

Choosing an arbitrary sequence of operations $s_n(t) = T_n(x)$ satisfying the condition (a), (b) with constants $A = \|x\|_n$ and setting

$$T_{n-1}(x) = S_{n-1}(T_n(x)),$$

$$T_n(x) = S_{n-1}(T_{n-1}(x)) \quad n = 1, 2, \ldots,$$

we see that the functions $T_n(x)$ and $T_{n-1}(x)$ satisfy the Lipschitz condition with the constants $\|x\|_n 2\|x\|_n \omega \left( \frac{1}{2n} \right)$ or $\|x\|_n 2\|x\|_n \omega \left( \frac{1}{2(n-1)} \right)$ respectively. Since

$$\|x - T_n(x)\|_o \leq \|x\|_n \omega \left( \frac{1}{2n} \right),$$

we infer by (3) that

$$\|S_{n-1}(x) - T_n(x)\|_o \leq \|x\|_n \omega \left( \frac{1}{2n} \right),$$

and this, together with (3), leads to

$$\|x - T_n(x)\|_o \leq \|x\|_n \omega \left( \frac{1}{2n} \right).$$

Analogously

$$\|x - T_{n-1}(x)\|_o \leq \|x\|_n \omega \left( \frac{1}{2(n-1)} \right).$$

Hence we can set $A = \|x\|_n$, $B = \|x\|_n$. Let us observe that $k_n(t)$ is a trigonometric polynomial of the form

$$k_n(t) = \sum_{m=1}^{2n} \frac{e^{-i m t}}{m^2}.$$

The representation of $T_n(x)$ and $T_{n-1}(x)$ by aid of the Jackson integral shows that these operations may be written as a trigonometrical polynomial of degree $4n - 2$

$$\sum_{m=1}^{2n} \left( a_m \cos m t + b_m \sin m t \right),$$

where

$$a_m = a_m \int_0^\pi T_m(x) \cos m t \, dt, \quad b_m = a_m \int_0^\pi T_m(x) \sin m t \, dt,$$

and are linear operations from $C_n(X)_h$ to $X$.

3. Replacing $C_n(X)_h$ by $C(X)$, we can define $T_n(x)$ as a polygonal function $s_n(t)$ assuming for

$$t_i = a + \frac{i}{n}, \quad i = 0, 1, \ldots, m = \lceil (n - a) \rceil,$$

the value $s(t_i)$, linear in the intervals $(t_i, t_{i+1})$ for $i = 1, \ldots, m - 1$, and in the interval $[a + m/n, b]$ equal to $s(t_{m-1})$ if $a + m/n \neq b$.

In this case we can choose $A = 3\|x\|_n$, $B = 2\|x\|_n$. If $\omega(a)$ satisfies the condition (m), we can replace the coefficient $3/2$ above by $1 + c\omega(1/2)$.

Analogously we can define $s_n(t)$ in the space $C(X)_h$. 

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1) Concerning see, for instance, G. N. Hata, _Kosmogogia i teoria fyzicskoi_, Moscow 1949, p. 111-119.
THEOREM 3. If there exist functions \( x_n(t) \in C_b(X) \) satisfying the Lipschitz condition with the constant (1), and if the inequality (2) is satisfied for \( n=1, 2, \ldots \), then
\[
\| x(t) - x_n(t) \| \leq A \omega \left( \frac{1}{t^n} \right)
\]
and
\[
\| x_n(t') - x_{n-1}(t') \| \leq B 2T \omega \left( \frac{1}{t^n} \right) (t' - t'').
\]
for every \( \omega(t) \); moreover,
\[
\| x(t) \| \leq \| \omega(t) \| + \lambda,
\]
where \( \lambda = 2 \max \{ A + B + B + |A| \} \).

The theorem remains true if we replace \( C_b(X) \) and \( L_\infty (X) \) by \( C_b(X) \) and \( L_\infty (X) \) respectively.

Proof. By our hypothesis
\[
\| x(t) - x_n(t) \| \leq A \omega \left( \frac{1}{t^n} \right)
\]
and
\[
\| x_n(t') - x_{n-1}(t') \| \leq B 2T \omega \left( \frac{1}{t^n} \right) (t' - t'').
\]
Given \( |h| \geq 0 \), let us choose \( n \) so that
\[
\frac{1}{t^n} > |h| \geq \frac{1}{t^n}.
\]
Then the inequality
\[
\| x(t+h) - x_n(t) \| \leq \| x(t+h) - x_n(t+h) \| + \| x(t) - x_n(t) \| + \| x_n(t+h) - x_n(t) \|
\]
\[
\leq 2A \omega \left( \frac{1}{t^n} \right) + B 2T \omega \left( \frac{1}{t^n} \right) |h| \leq 2(A + B) \omega (|h|)
\]
is satisfied. If \( 1 \leq |h| \leq |A| \), the last inequality with \( n = 0 \) leads to
\[
\| x(t+h) - x(t) \| \leq 2(A + B + |A|) \omega (|h|).
\]
Thus setting \( \lambda = 2 \max \{ A + B + B + |A| \} \), we get
\[
\| x(t) \| \leq \| \omega(t) \| + \lambda.
\]

THEOREM 2'. Let \( \omega(t) \) satisfy the condition (m), and let \( C^m= C^m(X) \) be a linear subset of \( C(X) \), whose functions have the following properties:

(1) the class \( C^m \) is contained in \( C^{m+1} \);

(2) the functions of \( C^m \) satisfy the Lipschitz condition with the constant \( K^m = \sum_{n=0}^{m} B_n |w(t)| \), \( B_n \) being independent of \( n \).

If for \( n=1, 2, \ldots \) there exists a function \( x_n(t) \in C^m \) belonging to the given \( C_b(X) \), and satisfying the inequality (2), then
\[
x(t) \in C_b(X) \text{ for } X \subset X,
\]
and
\[
\| x_n(t) \| \leq \| x(t) \| + \lambda,
\]
where
\[
\lambda = A L_\infty (\omega) + A B L_\infty (\omega) + B L_\infty (\omega, x).
\]

Here \( L_\infty (\omega), L_\infty (\omega) \) are constants depending only on \( \omega(t) \), while \( L_\infty (\omega, x) \) depends only on \( \omega(t) \) and \( |\omega(t)| \).

The theorem remains true when we replace \( C_b(X) \) and \( L_\infty (X) \) by \( C_b(X) \) and \( L_\infty (X) \) respectively.

Proof. We shall prove the theorem for the space \( C(X) \); for the space \( C(N) \) the proof runs in the same way.

Suppose there exist functions \( x_n(t) \) satisfying the hypotheses. Given \( a > 1 \), let us set
\[
y_n(t) = y_n(t), \quad y_n(t) = x_n(t) - x_{n-1}(t) \text{ for } n=1, 2, \ldots
\]

By (3)
\[
x(t) = \sum_{n=0}^{\infty} y_n(t)
\]
and the series on the right-hand side converges uniformly in \( \langle \phi, x \rangle \) for sufficiently large \( t \), which results from the estimations below. Let \( m \) be an index; since \( \omega(t) \) satisfies the condition (m), \( n \geq m \) implies
\[
\omega \left( \frac{1}{a^m} \right) = \omega \left( \frac{1}{a^{m-n}} \right) \leq e^{n-n} \omega \left( \frac{1}{a} \right) \leq \frac{1}{a^m},
\]
and for \( n \leq m \)
\[
\omega \left( \frac{1}{a} \right) = \omega \left( \frac{1}{a^{m-n}} \right) \leq e^{n-n} \omega \left( \frac{1}{a} \right) \left( \omega(a) \right) = e^{n-m} \omega \left( \frac{1}{a^{m-n}} \right)
\]
Further, the following inequalities are true:
\[
\| y_n(t) \| \leq \| x(t) - x_{n-1}(t) \| + \| x_{n-1}(t) - x_{n-2}(t) \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right)
\]
and
\[
\| y_n(t') - y_n(t'') \| \leq B \| y_n(t') - y_n(t'') \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
\[
\sum_{n=0}^{\infty} \| y_n(t') - y_n(t'') \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
\[
\| x(t) - x_{n-1}(t) \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
\[
\leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
and
\[
\sum_{n=0}^{\infty} \| y_n(t') - y_n(t'') \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
\[
\| x(t) - x_{n-1}(t) \| \leq 2 A \omega \left( \frac{1}{a^{m-1}} \right) \left| t' - t'' \right|
\]
where \( m = 1, \ldots \) and, finally,
\[
\|y_t(t') - y_t(t'')\| \leq B (A \rho(1) + \|\pi_0\| |t' - t''|)
\]
(5')
\[
\leq B a^m (\omega(a)) \omega \left( \frac{1}{a^m} \right) (A + \frac{\|\pi_0\|}{\omega(1)}) |t' - t''|.
\]
Choose \( a \) so large that
\[
\frac{1}{a^m} < 1, \quad \frac{\omega(a)}{a} < 1, \quad a > |A|;
\]
this is possible in virtue of the postulate (b) in the condition (m). Given \( |\lambda| \in (0,1) \) choose \( m \) so that
\[
\frac{1}{a^{m+1}} > |\lambda| \geq \frac{1}{a^m}.
\]
Then
\[
\|x(t + h) - x(t)\| \leq \sum_{n=1}^{m} \|y_n(t + h) - y_n(t)\| + \sum_{n=m+1}^{\infty} \sum_{k=1}^{n} \|y_k(t)\|
\]
whence the estimations (4), (5), and (5') lead to
\[
\|x(t + h) - x(t)\| \leq B (1 + \frac{\|\pi_0\|}{\omega(1)}) \omega \left( \frac{1}{a^m} \right)
\]
\[+ 2 |\lambda| B a^m (\omega(a)) \omega \left( \frac{1}{a^m} \right) \left( \frac{a}{\omega(a)} \right) - 1 + \frac{4 A \omega(1)}{1 - \omega(1/a)} \frac{1}{\omega(a)} \frac{1}{a},
\]
and since \( \omega(1/a^m) \leq \omega(|\lambda|) \), therefore
\[
\|x(t + h) - x(t)\| \leq B a^m (\omega(a)) \omega \left( \frac{1}{a^m} \right) \left( \frac{a}{\omega(a)} \right) - 1 + \frac{4 A \omega(1)}{1 - \omega(1/a)} \frac{1}{\omega(a)} \frac{1}{a},
\]
Set
\[
I_1(w) = \frac{4}{1 - \omega(1/a)}, \quad I_2(w) = \frac{2 a^2}{1 - \omega(1/a)} + \omega(a), \quad I_3(w) = \frac{\|x_0\|}{\omega(1/a)} \omega(a).
\]
If \( 1 \leq |\lambda| \leq |A| \), then, applying the inequality (4) with \( m = 0 \) we obtain from (5')
\[
\|x(t + h) - x(t)\| \leq \|y_0(t + h) - y_0(t)\| + 4 A \omega(1) \frac{1}{1 - \omega(1/a)} \omega(|\lambda|),
\]
and since
\[
|A| \leq I_1(\omega), \quad \frac{\|x_0\|}{\omega(1)} \leq I_2(\omega, \|x_0\|),
\]
we see that for every \( |\lambda| \leq |A| \) the inequality
\[
\|x(t + h) - x(t)\| \leq \lambda \omega(|\lambda|)
\]
holds with \( \lambda \) defined by the formula (1), whence \( \|x_0\| \leq \|x_0\| + \lambda \).

Remark. The theorem 2 belongs to the domain of the classical approximation problems of D. Jackson and S. Bernstein. In the classical problematics \( x_t(t) \) is supposed to be a real polynomial of degree \( s \) and \( \omega(s) = s^s \), while in our case more general Lipschitzian vector valued functions are admitted and \( \omega(s) \) are slightly more general. Our method does not differ essentially from the classical procedure.

Similarly to the real case we can choose in the space \( O(X)_a \) as \( O^a \) the class of trigonometric polynomials of degree \( \leq a \) and of the form
\[
x(t) = \sum_{a \geq 0} (x_t \cos(at) + y_t \sin(at)),
\]
where \( x_t, y_t \in X_n \). Indeed, as may easily be seen, an analogue of the classical theorem of S. Bernstein holds:
\[
\|x(t)\| \leq n \|x\|_n.
\]
If we choose as \( O^a \) the class of the polynomials of degree \( \leq 2a - 3 \), then they contain the Jackson polynomials \( x_t(t) = s_{2a-3}(x) \) defined in 2, of the proof of theorem 1 and taking \( x_t(t) = s_{2a-3}(x) \) with \( x \in C_0(X)_a \) we obtain as sequence of functions \( x_t(t) \) belonging to \( C_0(X)_a \) (the translation-invariance being assumed), which can be used as a sequence of approximating functions in Theorem 2'.

The method used in the proof of Theorem 2' may also be applied to prove the following theorem:

**Theorem 3.** Let \( \omega(s) \) satisfy the condition (m) and let \( y_n(t) \) denote functions from \( C_0(X)_n \) satisfying the following conditions:

(*) There is a constant \( \Delta > 0 \) such that
\[
\|y_n(t)\| \leq \Delta \omega \left( \frac{1}{n} \right) \quad \text{for} \quad n = 1, 2, \ldots
\]

(**) \( y_n(t) \) satisfy the Lipschitz condition with the constant (1).

Under these hypotheses every lacunary series of form
\[
\sum_{n=1}^{\infty} y_n(t)
\]
converges uniformly in \((0, \rho)\) and represents a function in \(G_1(X)_{p}L_\alpha(X)_p\) if \(a\) is a positive integer satisfying the conditions
\[
\omega\left(\frac{1}{a}\right) < 1, \quad \omega(a) < 1.
\]
An analogous statement holds if we replace \(C_\alpha(X)_{p}\) by \(C_\alpha(X)\).

As an application of Theorem 3 let us consider the following example.

Let \(\varphi(t)\) be a vector valued function with values in \(X\) and of period \(\rho\), satisfying the Lipschitz condition. If \(\varphi(t)=\omega(1/n)\varphi(nt)\), then the conditions (a) and (**) are satisfied, and if \(\alpha\) satisfies the conditions of the theorem, then the series
\[
\sum_{n=1}^{\infty} \omega\left(\frac{1}{n}\right) \varphi(\alpha n t)
\]
represents a function of \(C(X)_{p}L_\alpha(X)_p\), provided that \(\omega(u)\) satisfies the condition (m). In particular, let \(\omega(u)=u^\gamma\), \(0<\gamma<1\); then the condition (m) with the constant \(c=1\) is satisfied and we may apply Theorem 3 with \(a=2, 3, \ldots\)

Choose \(0<h<1, a>1\) and let \(\gamma=-\ln h/\ln a; \) then \(0<\gamma<1\) and \(b^\gamma-(1/a)^\gamma=\omega(1/a)\).

Choose \(0<h<1, a>1\) and let \(\gamma=-\ln h/\ln a; \) then \(0<\gamma<1\) and \(b^\gamma-(1/a)^\gamma=\omega(1/a)\).

belongs to \(C(X)_{p}L_\alpha(X)_p\) for \(0<\delta<\gamma\). This result is in a certain sense the best possible. Indeed, for \(\varphi(t)=\cos t\) the function \(\varphi(t)\) presents for almost every \(t\) the following singularity:
\[
\limsup_{h \to 0} \frac{|\varphi(t+h)-\varphi(t)|}{|h|^\gamma} = \infty
\]
for \(\delta>\gamma\). Analogous singularities may be obtained for more general \(\varphi(t)\) under supplementary conditions imposed upon \(a\) and \(b^\gamma\).

**Theorem 4.** Let \(G_\alpha(X)_{p}\) be translation-invariant, let \(U(x)\) be a linear operation from \(G_\alpha(X)_{p}\) to \(G_\alpha(X)_{p}\). Moreover, let \(U(x)\) map the space \(G_\alpha(X)_{p}L_\alpha(X)_p\) into the space \(G_\alpha(X)_{p}L_\alpha(X)_p\). Under these hypotheses \(U(x)\) has the following properties:

(a) for fixed \(\omega(u)\), \(x \in G_\alpha(X)_{p}L_\alpha(X)_p\) implies \(U(x) \in G_\alpha(X)_{p}L_\alpha(X)_p\),
(b) the operation \(U(x)\) is linear from the space \(G_\alpha(X)_{p}L_\alpha(X)_p\) to the space \(G_\alpha(X)_{p}L_\alpha(X)_p\), and its norm satisfies the inequality
\[
\|U\| \leq \|U(2\pi+1)\| + \|U(2\pi+1)\|\|\alpha\|.
\]

Here \(s=\max(1,p)\) and \(\gamma=\sup_{u \in \omega(u)}\|\omega(u)\|\) is supposed to be finite.

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**On a class of operations**

The theorem remains true if we remove the translation-invariance, replace the spaces \(G_\alpha(X)_{p}L_\alpha(X)_p\), \(L_\alpha(X)_p\) by \(G(X)_{p}L(X)_p\), \(L(X)_p\) respectively, and multiply the right-hand side in the inequality (7) by 2.

**Proof.** We prove first that \(U(x)\) is linear from \(G_\alpha(X)_{p}L_\alpha(X)_p\) to \(G_\alpha(X)_{p}L_\alpha(X)_p\). Indeed, let \(U(x) \to y\) and \(x \to y\) (in the sense of the convergence generated by the norm in \(G_\alpha(X)_{p}L_\alpha(X)_p\)), then \(\|y-x\| \to 0\), \(\|y-x\| \to 0\), whence \(y=U(x)\) and it suffices to apply the closed graph theorem of Banach.

Let \(T_\alpha(x)\) be linear operations of theorem 1 chosen so that \(\alpha=\beta=\alpha\). If \(x \in G_\alpha(X)_{p}L_\alpha(X)_p\), then
\[
\|y-T_\alpha(x)\| \leq \|y\| \alpha\left(\frac{1}{n}\right).
\]

Further, the following inequalities are satisfied:

\[
\|U(x)-U(T_\alpha(x))\| \leq \||U||\|x-T_\alpha(x)\| \leq \|U||\|x\| \alpha\left(\frac{1}{n}\right),
\]

\[
\|T_\alpha(x)\| \leq \|x\| \alpha\left(\frac{1}{n}\right)
\]

\[
\|T_\alpha(x)\| \leq \|T_\alpha(x)\| + \|x\| \alpha\left(\frac{1}{n}\right) \leq \|x\| \alpha\left(\frac{1}{n}\right),
\]

\[
\|U(T_\alpha(x))\| \leq \|U\|\|T_\alpha(x)\|.
\]

By Theorem 2, (8) and (9), (9) imply \(U(x) = C_\alpha(X)_{p}L_\alpha(X)_p\) and

\[
\|U(x)\| \leq \|x\| + \|x\| \alpha\left(\frac{1}{n}\right) + 2\|U\|\|x\| \alpha\left(\frac{1}{n}\right) \leq \|x\| \alpha\left(\frac{1}{n}\right),
\]

and this implies the inequality (7).

**Remark.** We can replace in Theorem 4 the hypothesis of the linearity of \(U(x)\) by the hypothesis that \(U(x)\), as an operation from \(G_\alpha(X)_{p}\) and from \(G_\alpha(X)_{p}L_\alpha(X)_p\) to itself, satisfies the Lipschitz condition of the following form:

\[
\|U(x)\| \leq K \|x\| + \|x\| + K \|x\|\|x\| \|x\|.
\]

Then the assertion of Theorem 4 is to be read:

\(U(x)\), as an operation from \(G_\alpha(X)_{p}L_\alpha(X)_p\) to \(G_\alpha(X)_{p}L_\alpha(X)_p\), satisfies a Lipschitz condition of the form

\[
\|U(x)\| \leq K \|x\|.
\]

---

In formula (7) the norms \( \| U \|_{L_p}, \| U \|_{L_0}, \| U \|_1 \) are to be replaced by 
\( K, \beta_0, K \).

2. We shall consider some cases of the spaces \( C(X), C_p(X), L_0(X), L_p(X) \) with the space \( X \) properly chosen, leading to some classes of functions considered in the approximation theory.

A. Let \( X \) denote the space of real numbers; then \( C(X) \) is the space of continuous functions of period \( p \), and \( C(X)_p, L_0(X), L_p(X) \) is the space of the functions of period \( p \) whose modulus of continuity \( \sigma(u) \) satisfies the inequality
\[
\sigma(u) = O(u) \quad (u \to 0).
\]

B. Let \( X \) be the space \( L \) of functions of period \( p \), integrable with the \( r \)-th power in \( (0, p) \) \( (r \geq 1) \). As \( C_p(X) \), let us choose the space of the functions \( \sigma(t) \) of the form \( \sigma(t) = f(t + v) \) where \( f(v) \in L \); then evidently \( C_2(X) \) is translation-invariant.

Since there exists a linearly-isomorphic correspondence between the functions \( \sigma(t) \) and \( f(v) \) and, moreover, for every \( t \)
\[
||\sigma(t)|| = \left( \int |f(t + v)|^r \, dv \right)^{1/r} = \left( \int |f(v)|^r \, dv \right)^{1/r} = ||f||_r,
\]
therefore it follows that the space \( C_2(X) \) is equivalent to the space \( L \).

The formula
\[
||\sigma(t + h) - \sigma(t)|| = \left( \int |f(v + h) - f(v)|^r \, dv \right)^{1/r}
\]
implies that \( C_2(X), L_0(X), L_p(X) \), is the space of the functions for which the \( L \)-modulus of continuity \( \sigma(u) \) satisfies the condition \( \sigma(u) = O(u) \), whence it is identical with the class \( L(|\sigma|, r) \) of functions occurring in the theory of Fourier series.

B'. Let \( M(u) \) be a monotone, convex and continuous function in \((0, \infty)\), vanishing only for \( u = 0 \) and such that
\[
\lim_{u \to 0} M(u) = 0, \quad \lim_{u \to \infty} M(u) = \infty, \quad M(2u) = O(M(u)).
\]

We choose as \( X \) the space \( L^M \) corresponding to the function \( M(u) \), i.e. the space of measurable functions of period \( p \), for which
\[
M(|f(u)|) \, du
\]
is finite.

The space \( C(X) \) will be defined as in B with \( f(v) \in L^M \).


Now we shall supply an application of theorem 4 to the theory of orthogonal systems. Let \( \Phi \{a_i(t)\} \) be an orthonormal system in \( (a, b) \) and let \( F \) be a class of integrable functions. The sequence \( \{a_i\} \) is called the multiplicator of class \( (F; F) \) if, in case of
\[
a(t) = \sum_{\infty} a_i(t)
\]
being the development of an arbitrary function \( a(t) \) of \( F \) the series
\[
\sum_{\infty} a_i(t)
\]
is also a development of a function \( g(t) \in F \). Let us denote by \( C, L_1, L_2 \) respectively the spaces \( C(X), L_1(X), L_2(X) \), where \( X \) is the space of real numbers.

**Theorem 5.** Let the system \( \Phi \{a_i(t)\} \) be complete in \( C \). If \( \{a_i\} \) is simultaneously a multiplicator of the class \( (C, C) \) and \( (L_1, L_2) \), it is also a multiplicator of the class \( (L_1, L_2) \) with arbitrary \( \omega = \omega(u) \), \( \gamma = \sup \omega \sigma(u) \) being supposed to be finite.

**Proof.** Let \( U(a) \) be an operation in \( C \), transforming the function \( a(t) \) into the function \( y(t) \) whose development is (11). The completeness of the system \( \Phi \) and the closed-graph theorem of Banach imply that \( U \) is a linear operation from \( C \) to \( C \). It suffices to apply Theorem 4.