par exemple

\[
f = \left\{ \begin{array}{l}
t \sin \frac{\pi \log t}{\log \lambda} \\
(\lambda \text{ positif et } \neq 1),
\end{array} \right.
\]

Si (2), l'équation

\[
a^3 = f
\]

n'est pas résolvable dans Q. En effet, supposons, au contraire, qu'il existe un opérateur \( \sigma \) satisfaisant à cette équation. Cet opérateur est réel ou purement imaginaire, c'est-à-dire il est de la forme \( f/g \) ou bien de la forme \( i f/g \), où les fonctions \( f \) et \( g \) (appartenant à C) sont réelles. On a

\[
T_\lambda (a^3) = (T_\lambda f)^3 = T_\lambda f = -i^3 f = -i^3 a^3,
\]

d'où

\[
T_\lambda a = \pm i \lambda a.
\]

Or, ce n'est pas possible, car on voit aussitôt que l'opérateur \( T_\lambda a \) est réel ou imaginaire, suivant que \( a \) est réel ou imaginaire.

Cette contradiction prouve que l'équation (3) n'est pas résolvable.

(Reçu par la Rédaction le 27.12.1958)

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**Singular integrals and periodic functions**

by

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1. The purpose of this note is to extend to periodic functions some of the results about singular integrals known for the non-periodic case. We shall be more specific later and begin by recalling basic facts.

Let \( x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k), \ldots \) denote points in the \( k \)-dimensional Euclidean space \( \mathbb{R}^k \). By \( x \) we shall also denote the vector joining the origin \( 0 = (0, \ldots, 0) \) with the point \( x \). The length of the vector \( x \) will be denoted by \( |x| \). If \( x \neq 0 \), by \( x' \) we shall mean the projection of \( x \) onto the unit sphere \( S \) having \( 0 \) for centre. Thus

\[
x' = \frac{x}{|x|}, \quad |x'| = 1.
\]

We shall consider kernels \( K(x) \) of the form

\[
K(x) = \frac{\Omega(x')}{|x|^r} = \frac{\Omega(x')}{r^s},
\]

where \( \Omega \) is a scalar (real or complex) function defined on \( S \) and satisfying the following conditions:

1. \( \Omega(x') \) is continuous on \( S \) and its modulus of continuity \( \omega(\delta) \) satisfies the Dini condition

\[
\int_0^1 \frac{\omega(\delta)}{\delta} \, d\delta < \infty;
\]

2. The integral of \( \Omega(x') \) extended over \( S \) is zero.

Condition 1 is certainly satisfied if the function \( \Omega \) satisfies a Lipschitz condition of positive order. It could be considerably relaxed in very important special cases, but for the problems discussed in this paper it is the most suitable one. On the other hand, condition 2 is absolutely essential, as explained in [2]^{1}.

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1) Numbers in brackets refer to the bibliography at the end of the paper.
We shall denote by $K_\varepsilon(x)$, and call it the truncated kernel, the expression defined by the equations
\[ K_\varepsilon(x) = \begin{cases} K(x) & \text{for } |x| \geq \varepsilon, \\ 0 & \text{elsewhere}. \end{cases} \]

Let $f(x) \in L^p$, $p \geq 1$. The convolution
\[ \tilde{f}(x) = \int f(y) K_\varepsilon(x-y) dy = \int K_\varepsilon(y) f(x-y) dy \]
exists as an absolutely convergent integral for all $x$. We write here $dy$ for $dy_1 \ldots dy_n$. The limit of $\tilde{f}(x)$ as $\varepsilon \to 0$, if it exists, will be denoted by $\tilde{f}(x)$ and called the Hilbert transform, associated with the kernel $K_\varepsilon$ of $f(x)$. The classical Hilbert transform corresponds, except for a numerical constant, to $k=1$ and $K(x)=1/x$.

We state without proof a number of known results concerning these transforms.

**Theorem A.** The function $\tilde{f}(x)$ exists almost everywhere if $f \in L^p$, $p \geq 1$. More generally, if $F$ is a totally additive function of set in $E^n$, the limit
\[ \lim_{\varepsilon \to 0} \int K_\varepsilon(x-y) F(dy) \]
exists almost everywhere.

**Theorem B.** If $f(x)$ belongs to $L^p$, $1 < p < \infty$, so does $\tilde{f}(x)$. Moreover,
\[ \|\tilde{f}\|_p \leq A_p\|f\|_p. \]

Here and hereafter constants like $A_p$ are positive and depend not only on the parameters shown explicitly but also on the kernel $K$ (i.e., on the function $B$). By $A$ (or $B_1, B_2, \ldots$) we shall denote constants depending at most on the kernel $K$, and we may use identical notation for different constants. By $\|f\|_p$ we mean the norm
\[ \left( \int |f|^p dx \right)^{1/p}. \]

Theorem B breaks down for $p=1$ and $p=\infty$. If $f \in L^1$, the function $\tilde{f}$ need not be integrable, even locally. Nor does the boundedness of $\tilde{f}$ imply that of $f$ (even if we assume the existence of $f$, which is not guaranteed here by Theorem A). For these extreme cases we have the following results:

1) Theorem A-F. as well as part 1 of Theorem G, are proved in [2]. Cases 2 and 3 of Theorem G are established by Mr. Cotlar [3].

**Theorem C.** If $\|f\|_p(1+\log^2|f|)$ is integrable over $E^n$, then $\tilde{f}$ is integrable over every set $S$ of finite measure and
\[ \frac{1}{S} \int_S |\tilde{f}| \leq A_{1/p} f \|f\|_p. \]

**Theorem D.** If $f \in L^1$, then $\tilde{f}$ is integrable over every set $S$ of finite measure if $0 < a < 1$. Moreover,
\[ \frac{1}{S} \int_S |\tilde{f}| \leq A_{1-a} |S|^{1-a} \left( \int |f| \right)^a. \]

**Theorem E.** If $f \in L^1$, then the measure of the set $\{x: f(x) \geq y > 0\}$ satisfies the inequality
\[ |\{x: f(x) \geq y > 0\}| \leq A_y \|f\|_1. \]

**Theorem F.** Suppose that $|f(x)| \leq 1$ for all $x$, and let
\[ g(x) = \lim_{\varepsilon \to 0} \int K_\varepsilon(x-y) f(y) dy. \]

Then, for any bounded set $S$,
\[ \frac{1}{S} \int_S |\exp \lambda y| dx \leq A_{1/\lambda}, \]
provided $0 < \lambda \leq \lambda_0 = \lambda_0(S, E)$. If $f$ is, in addition, continuous, the last integral is finite for every $\lambda > 0$.

**Theorem G.** Given any $f$, let
\[ \Phi(x) = \sup_{x \in S} \left| \int K_\varepsilon(x-y) f(y) dy \right|. \]
Then in the three cases
\[ f \in L^p, \quad p > 1; \]
\[ |f(1+\log^2|f|)| \in L^1; \]
\[ f \in L^1, \]
the inequalities (1.2), (1.3), (1.4) hold, respectively, with $\tilde{f}$ replaced by $\Phi$.

2. Let us now consider a system $e_1, e_2, \ldots, e_\infty$ of independent vectors in $E^n$ and let $a_0 = a_1, a_2, \ldots$ be the sequence of all lattice points generated by these vectors:
\[ a_n = m_1 e_1 + m_2 e_2 + \cdots + m_n e_n, \]
where $m_1, m_2, \ldots, m_n$ are arbitrary integers. If we define a function $K^*(a)$ by the formula
\[ K^*(a) = K(a) + \sum_{a_0}^\infty |K(a_0) - K(a)|, \]
the series on the right converges absolutely and uniformly over any bounded set provided we discard the first few terms. (Of course we can also replace the terms in curly brackets by \(K(x, s_x) - K(-x, s_x)\).) For \(s \in S\) and \(r\) large enough, then

\[
|K(x - s_x) - K(-x, s_x)| \leq \frac{1}{|x|^r} \sum_j A_j \left| A_j \right|
\]

(see [1], p. 95), and condition 1 st imposed on the kernel \(K\) implies that the terms on the right here form a convergent series.

It is easy to see that the function \(K^*\) has periods \(e_1, e_2, \ldots, e_k\). For let \(x\) be the value 0 at the origin. We may then write the series \(\sum_{j=1}^n |K(x, s_x) - K(-x, s_x)|\). Hence

\[
K^*(x) - K^*(x + e_i) = \sum_{j=1}^n |K(x, s_x) - K(x, s_x)|,
\]

where \(s_x^* = s_x - e_i\). The last series converges absolutely. Let us find the values of \(m_1, \ldots, m_k\) and consider the sum of the terms with variable \(m_k\). This sum is zero due to the cancellation of "adjoining" terms and the fact that \(K(x)\) tends to zero as \(|x|\) tends to infinity. Hence the sum of the whole last series is zero. It follows that \(K^*(x)\) has the period \(e_1\), and so also periods \(e_1, \ldots, e_k\).

Of course, the passage from \(K(x)\) to \(K^*(x)\) is a classical one. If \(k = 1\),

\[
e_1 = 2\pi, \quad K(x) = 1/|x|,
\]

then

\[
K^*(x) = \frac{1}{2} \cos \frac{1}{2} \pi.
\]

If \(k = 2\), \(K(x) = |x|^2\), then \(K^*(x)\) is the classical \(p\) function of Weierstrass, etc.

Let us now suppose for the sake of simplicity that the vectors \(e_1, e_2, \ldots, e_k\) are all mutually orthogonal and of length 1. We may assume that they are situated on coordinate axes. Let \(f(x) = f(x_1, x_2, \ldots, x_k)\) be a function of period 1 in each \(x_i\) and integrable over every bounded set. We shall consider the convolution

\[
\int f(x) K^*(x - y) dy = \int K^*(y) f(x - y) dy,
\]

where \(R\) is the "fundamental" cube

\[
|\xi_j| \leq \frac{1}{2} \quad (j = 1, 2, \ldots, k)
\]

and the integral is taken in the principal value sense. For such functions \(f\) we easily obtain results analogous to Theorems A-G.

In what follows, \(f(x) = f(x_1, x_2, \ldots, x_k)\) will systematically denote a function of period 1 in each variable (for simplicity, we shall use the abbreviation "periodic function"). Integrable over \(R\). By \(\mathfrak{M}_p(f)\) we shall denote the norm

\[
\left( \frac{1}{R} \int |f|^p dx \right)^{1/p}.
\]

Theorem 1. The integral \(\int K^*(x - y) f(y) dy\) exists for almost every value of \(x\). The same holds for the integral

\[
\int \frac{K^*(x - y) P(dy)}{R}
\]

for any totally additive function of set \(P\).

Theorem 2. If \(f \in L^p\), \(1 < p < \infty\), then \(f^*\) also belongs to \(L^p\) and

\[
\mathfrak{M}_p[f^*] \leq \mathfrak{M}_p[f].
\]

Theorem 3. If \(\|f\|\log^{1/2}\|f\|\) is integrable over \(R\), then

\[
\int \left( \frac{1}{R} \int |f|^p dx \right)^{1/2} \frac{|f|}{R} \int |f|^p dx - B.
\]

Theorem 4. If \(f\) is merely integrable over \(R\), then for every \(0 < \alpha < 1\) and every set \(S \in \mathfrak{R}\),

\[
\int \left( \frac{1}{R} \int |f|^p dx \right)^{1/2} \frac{|f|}{R} \int |f|^p dx = A(\alpha) \int \left( \frac{1}{R} \int |f|^p dx \right)^{1/2}.
\]

Theorem 5. If \(f\) is integrable over \(R\), then the measure of the set \(E_x\) of points \(x \in \mathfrak{R}\) at which \(|f^*| \geq y > 0\) satisfies the inequality

\[
|E_x| \leq \frac{A}{y} \mathfrak{M}_p[f].
\]

Theorem 6. If \(\|f\| < 1\), then

\[
\int \exp \lambda |f|^p dx \leq A,
\]

provided \(\lambda\) is small enough, \(0 < \lambda < \lambda_0 (K^*).\) If \(f\) is also continuous, then the last integral is finite for every \(\lambda > 0\).

Given a function \(f_i\), let

\[
\psi(x) = \sup \{ \int f(x - y) K^*_i(y) dy \},
\]

where \(K^*_i(x)\) is the function equal to \(K^*(x)\) except in the \(\epsilon\)-neighbourhoods of the lattice points \(x\), in which it is equal to zero.
Theorem 7. According as
(i) \(f \in L^p, \quad p > 1,\)
(ii) \(|f| \log^+ |f| \in L^1,\)
(iii) \(f \in L^1,\)
we have respectively the inequalities (2.6), (2.7), (2.8), with \(f^*\) replaced by \(f.\)

3. These results can be easily obtained from Theorems A-G. For let \(R\) denote the cube
\[(R), \quad |x_j| < 1 \quad (j = 1, 2, \ldots, k),\]
and let \(f_j(x)\) be the function equal to \(f(x)\) in \(R\) and to zero elsewhere.

The difference \(K^* - K\) being bounded in \(R,\)
\[|K^*(x) - K(x)| < B \quad \text{for} \quad x \in R,\]
and the function \(f(x - y)\) being identical with \(f_j(x - y)\) for \(x\) and \(y\) in \(R,\)
we see immediately that the second integral (2.4) converges for \(x \in R\) if and only if the integral
\[\int_R K(y)f_j(x - y) dy \quad (x \in R)\]
does, and that the difference between these two integrals is numerically less than
\[B \int_R |f(x - y)| dy = B \int_R |f(y)| dy.\]

On the other hand, if we replace \(R\) in (3.1) by \(E^k\) we obtain the function \(f_j^*(x),\) and the error committed will not exceed
\[C \int_{E^k} |f(x - y)| dy \leq C \int_R |f_j(y)| dy = C \int_R |f(y)| dy.\]

Collecting results we see that, for \(x\) in \(E,\) the integral \(f_j^*(x)\) exists if and only if \(f_j(x)\) does, and that
\[|f_j^*(x) - f_j(x)| \leq A \mathcal{M}_p[f]. \quad (x \in E).\]

In particular, \(f_j^*(x)\) exists almost everywhere in \(E.\) A similar argument applies to the integral (2.3) and Theorem 1 is established.

Passing to Theorem 2 we observe that (3.2) leads to
\[\mathcal{M}_p[f] \leq \mathcal{M}_p[f^*] + A \mathcal{M}_p[f] \leq \mathcal{M}_p[f] + A \mathcal{M}_p[f^*] = A \mathcal{M}_p[f],\]
which completes the proof.

If \(|f| \log^+ |f|\) is integrable, (3.3) and Theorem C, with \(S = E,\) give
\[\int_R |f^*| dx \leq \int_R |f| dx + A \mathcal{M}_p[f],\]
\[\leq A \left( \int_R |f^*| dx + A + A \mathcal{M}_p[f] \right) \leq A \left( |f| \log^+ |f| dx + B,\right)\]
and (2.7) is established.

Passing to Theorem 4, let us integrate the inequality
\[|f^*|^n \leq |f| + A \mathcal{M}_p[f],\]
which follows from (3.2), over \(S.\) We get
\[\int_S |f^*|^n dx \leq \int_S |f| dx + |S| A \mathcal{M}_p[f],\]
\[\leq \frac{A}{1 - a} \left( \int_S |f| dx \right)^a + \left| S \right| A \mathcal{M}_p[f],\]
\[\leq \frac{A}{1 - a} \left( |S|^a \int_S |f| dx + |S| A \mathcal{M}_p[f] \right) \leq \frac{A}{1 - a} |S|^a \mathcal{M}_{pa}[f],\]
since \(|S| \leq 1.\) This gives (2.9).

For the proof of Theorem 5 we may suppose that \(\mathcal{M}_p[f] = 1.\) Let us denote the constants \(A,\) and \(A^*\) by \(A,\) and \(A^*\) respectively. Let us also temporarily assume that \(y/2 \geq A.\) From (3.2) we see that the set of points \(x \in E\) at which \(|f^*| \geq y\) is contained in the set of points \(x\) at which \(|f| \geq y/2\) and so, by Theorem E, has a measure not exceeding
\[A \left( \frac{1}{2} \right)^{n-1} \int_E |f| dx = A (2^{n-1})^{-1}.\]

This gives (2.9), with \(A = 2^{n-1} A^*,\) provided \(y > 2A.\) Increasing the constant \(A\) so that \(A^* \leq 1\) for \(0 < y \leq 2A,\) we shall have the inequality (2.9) trivially satisfied for such \(y's.\) Thus (2.9) holds for all positive \(y.\)

Suppose now that \(|f| \leq 1.\) The inequality (3.2) shows that \(|f^* - f| \leq A\) in \(E.\) We denote by \(g_j(x)\) the function (1.6) corresponding to \(f_j,\) then clearly \(|f_j - g_j| \leq A\) in \(E.\) Hence, by (3.5),
\[|f^*(x) - g_j(x)| \leq A \quad (x \in E).\]

Using this and (1.7) for \(S = E,\) we obtain
\[\int_E |f^*(x)| dx \leq \int_E |f^*| dx + A \mathcal{M}_p[f^*] dx \leq A \lambda,\]
which is (2.10). Similarly we obtain the result concerning continuous functions \(f.\)
The proof of Theorem 7 is analogous to the proofs of the preceding results provided we use Theorem G and the inequality
\[ |f_k(x) - f_k(x_0)| \leq A M \| f \| \],
analogous to (3.2) and established similarly.

Remarks. 1° The best possible constant \( A_p \) in (1.2) satisfies inequalities
\[ A_p \leq \frac{A}{p-1} \quad 1 < p \leq 2, \quad A_p \leq A \quad p > 2, \]
Moreover, if \( A_p \) is the constant corresponding to the kernel \( K(-x) \), then
\[ A_p = A_p \text{ for } p = \frac{p}{p-1}, \quad (1 < p < \infty) \]
(see [2]). It follows from the proof of Theorem 2 that the same results hold for the constant \( A_p \) in (3.6).

That the constant \( A_p \) in part (i) of Theorem G satisfies inequality (3.3) is proved in [3]. Therefore this inequality also holds in part 1° of Theorem 7.

2° Using the second inequality (3.3) we can deduce (2.10) from (2.6) by a familiar argument. For, if \( |f| \leq 1 \), then
\[ \int \exp \left| f' \right| ds < \frac{2}{k} \exp \left( \int \exp \left| f' \right| ds \right) \]
\[ \leq 2 + 2\sum_{n=1}^{\infty} \frac{(2\lambda)^n}{(2n)!} \int \left| f' \right|^2 ds \leq 2 + 2\sum_{n=1}^{\infty} \frac{(2\lambda)^n}{(2n)!}, \]
and the last series is finite provided \( 2\lambda < e^{-1} \).

Since the second estimate (3.3) holds if we replace \( f' \) by \( f \), it follows that also in (2.10) we can replace \( f' \) by \( f \).

3° Theorem D has an analogue for the functions (1.1a), with \( \int f_0 \) in (1.4) replaced by the total variation of \( F \) over \( B^k \) (see [2]). A corresponding result holds for the function (2.5).

4. We now pass to a different group of theorems. Let us consider a periodic function \( f(x) = f(\xi_1, \xi_2, \ldots, \xi_k) \) and its Fourier series
\[ f(x) \sim \sum_{m, \lambda} \left( \sum_{\xi_1, \xi_2, \ldots, \xi_k} e^{2\pi i (m_1 \xi_1 + m_2 \xi_2 + \cdots + m_k \xi_k)} \right), \]
where \( m = (m_1, m_2, \ldots, m_k) \), and \( (m, x) \) is the scalar product \( m_1 \xi_1 + \cdots + m_k \xi_k \) of \( m \) and \( x \).

If \( f \in L^p, p > 1 \), or if only \( |f| \log^* |f| \) is integrable, then the periodic function \( f'(x) \) is also integrable and we may consider its Fourier series
\[ f'(x) \sim \sum_{m, \lambda} e^{2\pi i (m_1 \xi_1 + m_2 \xi_2 + \cdots + m_k \xi_k)}, \]
where the integral is taken in the principal value sense. That these coefficients exist, is clear if we observe that in the neighbourhood of the origin \( K(0) \) differs from \( K(x) \) by a bounded function, that for \( |x| \) small \( K(x) e^{-2\pi i (m, x)} = K(x) [1 + O(|x|)] = K(x) + O(|x|^{-1}), \)
and that the integral of \( K \) over \( R \), in the principal value sense, exists owing to condition 2° imposed on \( K \).

Next, we show that for \( m = (0, 0, \ldots, 0) \) we have
\[ \gamma_m = \frac{1}{2\pi} K(y) e^{-2\pi i (m, y)} dy = \lim_{\epsilon \to 0} \int K(y) e^{-2\pi i (m, y)} dy, \]
i.e., the Fourier coefficient of \( K^* \) is equal to the corresponding Fourier transform of \( K \).

One remark is indispensable here. The function \( K \), being in \( L^2 \), has by the theorem of Plancherel a Fourier transform, also in \( L^2 \). This transform is, however, defined almost everywhere only, while here we insist on its existence at the lattice points \( m \). We must therefore show that under the conditions imposed on \( K \) the last integral in (4.3) does exist.

\[ \lim_{\epsilon \to 0} \int K(y) e^{-2\pi i (m, y)} dy = \int K(y) e^{-2\pi i (m, y)} dy, \]
exists for each \( m \). We will follow, as we are going to show, from the existence — already established — of the \( \gamma_m \). We may assume that \( m = (0, 0, \ldots, 0) \), since in the remaining case the limit (4.4) clearly exists and is zero.

Let \( K \) be the cube with centre \( x \), and congruent to \( R \); thus \( R = K \). Let \( T \) denote the sphere with centre at the origin and radius \( \epsilon, 0 < \epsilon < \frac{1}{2} \).

Using (2.5) we have
\[ \int_{R^k} K^*(y) e^{-2\pi i (m, y)} dy = \sum_{\xi_1, \xi_2, \ldots, \xi_k} \int_{\epsilon} K(y - \xi_1) e^{-2\pi i (m, y)} dy, \]
where
\[ \int_{T} K(y) e^{-2\pi i (m, y)} dy = \sum_{\xi_1, \xi_2, \ldots, \xi_k} \int K(y - \xi_1) e^{-2\pi i (m, y)} dy. \]
Owing to the convergence of the series of the right sides in (3.3) we see that if in the last sum in (4.5) we replace the domain of integration \( R = \Gamma(x) \) by \( R_\epsilon \), we commit an error \( O(\epsilon) \). Suppose that in the new series we only retain terms with \( |x_\nu| \leq \rho \). The contribution of the omitted terms tends to zero as \( \epsilon \to 0 \), and the sum retained is

\[
\sum_{\epsilon < |x_{\nu}| \leq \rho} (K(y - x_{\nu}) - K(-x_{\nu})) e^{-\epsilon \rho (m, x)} dy = \sum_{\epsilon < |x_{\nu}| \leq \rho} K(y) e^{-\epsilon \rho (m, x)} dy.
\]

Observing that the measure of the union of the sets \( R_\epsilon \), with \( |x_\nu| \leq \rho, \nu \neq 0 \), differs from the measure of the sphere \( |y| \leq \rho \) by \( O(\rho^{k-1}) \), and that \( K(y) = O(|y|^{-k}) \) for \( |y| \to \infty \), we can write (4.5) in the form

\[
 \int_{R = \Gamma(0)} K^*(y) e^{-\epsilon \rho (m, x)} dy = \int_{|y| \leq \rho} K(y) e^{-\epsilon \rho (m, x)} dy + O(\epsilon) + o_\epsilon(1),
\]

where \( o_\epsilon(1) \) is a quantity tending to zero as \( \epsilon \to 0 \). This formula not only proves the existence of (4.4) but also the equations (4.3), provided \( m \neq (0, 0, \ldots, 0) \).

If \( m = (0, 0, \ldots, 0) \), the preceding argument shows that

\[
\gamma_m = -\lim_{\epsilon \to 0} \sum_{\epsilon < |x_{\nu}| \leq \rho} K(-x_{\nu}) = \lim_{\epsilon \to 0} \sum_{\epsilon < |x_{\nu}| \leq \rho} K(x_{\nu}).
\]

The latter quantity need not be zero and the formula (4.3) fails then. However, \( \gamma_x \) is zero, for example in the case when the sum of the \( K(x_{\nu}) \) extended over the \( x_{\nu} \) situated in the circle \( |x_{\nu}| = \rho \) is zero. It is clearly so in the classical case \( n = 1, K(x) = 1/|x| \). If \( k = 2 \) and

\[
K(x) = \frac{e^{i \pi m}}{|x|^2} = \frac{2^m}{|x|^{2m}} \quad (\pi = \arg \alpha, m = \pm 1, \pm 2, \ldots),
\]

we have \( \gamma_0 = 0 \) if \( m \) is not divisible by 4. In particular, \( \gamma_0 = 0 \) for \( K(x) = 1/|x|^2 \).

Remarks. 1. An argument similar to the proof of (4.3) shows that if the integral of \( f \) over \( R \) is zero, i.e. if \( \alpha \) is zero, then

\[
\gamma_0 = \int_{R = \Gamma(0)} f(y) K(x - y) dy = \lim_{\epsilon \to 0} \left\{ \int_{|y| \leq \rho \epsilon} K(y) f(x - y) dy \right\},
\]

the inner limit existing for all \( x \) and the outer one if and only if the integral (2.4) converges. Also

\[
\gamma_0 = \int_{R = \Gamma(0)} f(y) K(x - y) dy = \lim_{\epsilon \to 0} \left\{ \int_{|y| \leq \rho \epsilon} f(y) K(x - y) dy \right\},
\]

the new kernel \( K^* \) differs from the old one by a constant only and we now have \( \gamma_0 = 0 \).

5. Theorem 8. If \( f(x) \) is integrable (in particular if \( f \in L^p \), \( p > 1 \)) then the Fourier series (4.3) of \( f \) has coefficients

\[
\gamma_m = \sum_{n=0}^{\infty} \gamma_n \gamma_m, \tag{5.1}
\]

where \( \gamma_m \) is given by (4.2a) or, for \( m \neq (0, 0, \ldots, 0) \), by (4.3).

Given any trigonometric series \( \sum \alpha_n e^{2\pi i n \cdot \mathbf{x}} \), we may call it the conjugate of the former, corresponding to the kernel \( K^* \). Thus the result may be stated that if the conjugate function \( f^* \) is integrable, the Fourier series of \( f^* \) is the conjugate of the Fourier series of \( f \).

This theorem is very well known in the one-dimensional case (see (8), pp. 165, 165). It is obtained there either through complex methods or by considering a certain definition of integral more general than that of Lebesgue. While the first approach fails for general \( k \), the second is applicable straightforwardly.

Let \( f(x) = f(x_1, \ldots, x_k) \) be measurable and periodic. Let us consider any partition \( P \) of the cube \( R \) into a finite number of parallelipipedes with edges parallel to the axes. These parallelipipdes will be denoted by \( A_1, A_2, \ldots, A_N \) and their measures by \( |A_1|, \ldots, |A_N| \). Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_k) \) be any point of \( A_j \) and let \( t = (\tau_1, \tau_2, \ldots, \tau_k) \) be any vector with \( 0 \leq \tau_j \leq 1 \). Let us consider the sum

\[
\sum_{j=1}^{N} f(\tau_j + \xi_i) |A_j|. \tag{5.2}
\]

It will be denoted by \( S \) or \( S(f, P) \), \( S(f, t) \); of course it also depends on \( P \) and the points \( \xi_i \).

If \( S \) converges in measure to a limit \( I \) as the norm of the partition \( P \), i.e. the largest diameter of the \( A_j \) tends to zero, we shall say that the function \( f(x) \) is integrable over \( R \) and that

\[
\int_{R} \gamma(x) dx = I.
\]
This is an immediate extension of the definition familiar in the case \( k = 1 \) (see [8], p. 151) and the same proof as there shows that, if \( f \) is integrable over \( E \), then \( \varphi \) is integrable over \( E \) and both integrals have the same value. Therefore, Theorem 8 will be a corollary of the following result:

**Theorem 9.** Suppose that \( f(x) \) is periodic and \( E \)-integrable. Then the function \( f(x) \), as well as the functions \( f(x) e^{-2\pi i m \cdot \xi} \), are \( E \)-integrable over \( E \), and the Fourier coefficients of \( f \), in the \( E \)-sense, satisfy the equation (5.1).

We first observe that, if \( f \) is a trigonometric monomial \( e^{2\pi i m \cdot \xi} \), then the conjugate function is (see (2.4))

\[
\int_E \varphi(x) e^{2\pi i m \cdot \xi} \, dx = \varphi(x) e^{-2\pi i m \cdot \xi}.
\]

Hence, for any trigonometric polynomial \( \sum c_m e^{2\pi i m \cdot \xi} \), the conjugate function is \( \sum c_m e^{-2\pi i m \cdot \xi} \).

We shall now show that for any \( f \in L \) and for any lattice point as the function \( f(x) e^{-2\pi i m \cdot \xi} \) is \( E \)-integrable over \( E \) and the integral is \( c_m g(x) \). In view of the preceding remark we may assume that \( c_m = 0 \). Let us set \( f = f' + f'' \), where \( f' \) is a trigonometric polynomial and \( M \), \( f'' \) is small. Without loss of generality we may assume that the coefficient of \( e^{2\pi i m \cdot \xi} \) in \( f'' \) is zero. Thus \( f = f' + f'' \) and

\[
S(f) = \sum_{m \in \mathbb{Z}} S(f) e^{2\pi i m \cdot \xi} = S(f) e^{2\pi i m \cdot \xi} + S(f) e^{-2\pi i m \cdot \xi}.
\]

The first term in the right tends to zero as the norm of the partition approaches zero. The absolute value of the second term is

\[
\left| \sum_{m \in \mathbb{Z}} S(f) e^{2\pi i m \cdot \xi} \right| \leq \sum_{m \in \mathbb{Z}} |S(f) e^{2\pi i m \cdot \xi}| = \sum_{m \in \mathbb{Z}} |S(f) e^{2\pi i m \cdot \xi}| = \sum_{m \in \mathbb{Z}} \left| \int_E f(x) e^{2\pi i m \cdot \xi} \, dx \right|.
\]

and so is small. It follows that \( g \), and so also the last term in (5.3), is small except for \( \xi \)'s belonging to a set of small measure, no matter what is the partition \( P \). Therefore the left side of (5.3) is small except for \( \xi \)'s in a small set, provided the norm of \( P \) is small enough. This shows that \( f(x) e^{-2\pi i m \cdot \xi} \) is \( E \)-integrable over \( E \) and the integral is zero. Thus the proof of Theorem 9 is completed.

6. We shall now give a few illustrations for the results obtained.

Suppose that the function \( \varphi \) in the numerator of the kernel \( K \) is a spherical harmonic \( Y_n \) of order \( n \). In other words, \( Y_n(x) / x^k \) is a homogeneous polynomial \( P \) of degree \( n \) in the variables \( \xi_1, \xi_2, \ldots, \xi_k \), satisfying in these variables Laplace's equation \( \Delta P = 0 \). Thus

\[
K(x) = \frac{Y_n(x)}{x^k} \quad (n = 1, 2, \ldots).
\]

In the simplest case \( k = 2 \), we have \( Y_n = e^{i\omega x} \).

It is well known that the Fourier transform of \( K \) in (6.1) is a numerical multiple of \( Y_n \); more precisely,

\[
\hat{Y}_n(y) = \frac{2\pi}{y} \int_{\mathbb{R}^2} Y_n(x) e^{-2\pi i x y} \, dx = \frac{\Gamma\left(\frac{1}{2}\right)}{y^{\frac{n}{2} + 1}} \frac{F \left( \frac{1}{2}, \frac{1}{2}; y \right)}{\pi}.
\]

Combining this with previous results we have the following theorem:

**Theorem 10.** Let \( P(x) = P_n(x) \) be a homogeneous polynomial of degree \( n \) in the variables \( \xi_1, \xi_2, \ldots, \xi_k \), satisfying Laplace's equation \( \Delta P = 0 \). Let \( f(x) \) be an \( E \)-integrable function.

**Proof.**

Consider the series

\[
\sum_{m \in \mathbb{Z}} P_m(x) e^{2\pi i m \cdot \xi} = \sum_{m \in \mathbb{Z}} P_m(x) e^{2\pi i m \cdot \xi} |
\]

and satisfy (2.7);

(i) If \( f \in L^1 \), \( f \) is integrable, (6.4) is the Fourier series of an \( f \) of the class \( L^1 \), and \( f \) satisfies (2.6);

(ii) If \( f \in L^2 \), \( f \) is integrable, (6.4) is the Fourier series of an \( f \) of the class \( L^2 \), and satisfies (2.7);

(iii) If \( f \in L^r \), \( f \) is integrable, (6.4) is the Fourier series of an \( f \) of the class \( L^r \), and satisfies (2.7).

Thus, (iii) follows from (ii) by taking the limit as \( r \to \infty \).
(iii) If \( f \) is merely integrable, (6.4) is the Fourier series, in the \( B \)-sense, of a function \( f^* \) satisfying (2.8).
(iv) If \( |f| \leq 1 \) the function \( f^* \) satisfies (2.10).

The simplest cases here are

\[
P_n(x) = P_{n,1}(x) = \xi_j,
\]

or

\[
P_n(x) = P_{n,2}(x) = \xi_i \xi_j, \quad (i \neq j).
\]

In these cases part (i) of Theorem had been proved by Marcinkiewicz [5]. His method does not yield the remaining parts of Theorem 9 since it consists of repeated application of a result from the case \( k = 1 \) and somewhat loses strength as \( k \) increases. His proof works for more general cases than (6.5) or (6.6). Of course, also the argument given above applies to general series (5.1a), provided we know that the multipliers \( \gamma_n \) are the Fourier coefficients of a suitable kernel \( K \) of our type. To the problem what properties of the \( \gamma_n \) guarantee that assertion we shall return elsewhere.

7. We shall now prove results concerning the behaviour of the conjugate series in the case the function \( f \) satisfies a Lipschitz (Hölder) condition

\[
|f(x+h) - f(x)| \leq O(|h|^\alpha),
\]

with \( C \) independent of \( x \) and \( h \). If this condition is satisfied we shall write

\[
f \in L_{\alpha}.
\]

If we disregard constant functions, only the case \( 0 < \alpha \leq 1 \) need be considered.

In the one-dimensional case there is a familiar result, due to Privalov, asserting that if \( f \) is in \( L_{\alpha} \), \( 0 < \alpha < 1 \), so is \( f^* \) (see e.g. [3], p. 156). The result is false for \( \alpha = 1 \) (see below). The theorem which follows is an extension of Privalov's result to the \( k \)-dimensional case:

**Theorem 11.** Suppose that \( f(x) \) is periodic and of the class \( L_{\alpha} \), \( 0 < \alpha < 1 \), and that \( \Omega \in L_{\beta} \), \( \beta > \alpha \), on \( \Sigma \). Then \( f^* \in A_\alpha \).

**Proof.** Suppose that \( |h| \) is small. We may assume that the integral of \( K^* \) over \( R \) is zero. Then, denoting by \( \Gamma(x,r) \) the sphere with centre \( x \) and radius \( r \), we may write \( f^* \) in the form

\[
f^*(x) = \int_R \left[ f(x + t) - f(x) \right] K^*(t) \, dt
\]

or

\[
= \int_{R - R(h,3|h|)} \left[ f(x + t) - f(x) \right] K^*(t) \, dt + O(|h|^\alpha),
\]

since the integrand here is \( O(|t|^\alpha) \cdot O(|h|^{-\beta}) \), and the integral of the latter function over \( \Gamma(0,3|h|) \) is \( O(|h|^\alpha) \). Thus

\[
f^*(x+h) - f^*(x) = \int_{R - R(h,3|h|)} \left[ f(x + t) - f(x) \right] K^*(t) \, dt + O(|h|^\alpha),
\]

where \( R(-h) \) denotes the cube \( R \) translated by \(-h\). A simple calculation shows that if we replace in the last integral the domain of integration by \( R - R(0,3|h|) \) we commit an error \( O(|h|^\alpha) \cdot O(|h|^\alpha) = O(|h|^{2\alpha}) \), for in the neighbourhood of the boundary of \( R \) the integrand is \( O(1) \), and in the shell \( |h| < |t| < 3|h| \) the integrand is \( O(|h|^\alpha) \cdot O(|h|^{-\beta}) \).

Thus

\[
f^*(x+h) - f^*(x) = O(|h|^{\alpha}) + \int_{R - R(h,3|h|)} \left[ f(x + t) - f(x) \right] K^*(t + h) - K^*(t) \, dt.
\]

The first factor in the integrand here is \( O(|h|^\alpha) \). In estimating the second factor, in which \( |h| \leq |t| \), we use the series (2.2) and the formula

\[
O\left[ \frac{|t|}{|t+h|} \right] = O\left( \frac{|t|}{|t+h|^\beta} \right) = O\left( \frac{|t|}{|t+h|^\beta} \right) + O\left( \frac{1}{|t+h|} \right)^{1-\alpha}.
\]

The contribution of the term \( K(t) \) on the right is

\[
O\left( \frac{|h|^{\alpha}}{|h|^{\beta-\alpha}} \right) + O\left( \frac{|h|^{\alpha}}{|h|^{\beta-\alpha} \cdot |h|^{2\alpha}} \right) = O\left( \frac{|h|^{\alpha}}{|h|^{\beta-\alpha}} \right),
\]

and the contribution of the remaining terms is

\[
O(|h|^{\alpha}) + O(|h|) = O(|h|^{\alpha}).
\]

Thus, collecting results, we see that the last integral does not exceed

\[
O(|h|^{\alpha}) \int_{R - R(h,3|h|)} \frac{dt}{|t+h|^\beta} = O(1) \cdot O(|h|^{\alpha}) + O(|h|^{\alpha}) = O(|h|^{\alpha}),
\]

which shows that \( f^* \in A_\alpha \).

The following result is an obvious corollary of Theorem 11:

**Theorem 12.** Suppose that the function \( f(x) \) given by (6.3) is of the class \( A_\alpha \), \( 0 < \alpha < 1 \), and that \( f^* \) is the same as in Theorem 10. Then the series (6.4) is the Fourier series of a function in \( A_\alpha \).

Let now \( f(x) \) denote any, not necessarily periodic, function defined in a domain \( D \subset \mathbb{R}^k \). We shall say that the function \( f(x) \) satisfies condition \( A_\alpha \), and write

\[
f \in A_\alpha,
\]
if \( f(a) \) is continuous and if

\[
(f(x+h)+f(x-h)-2f(x)) = O|a|
\]

for any \( a \) and \( h \) such that \( x, h \leq k \) are in \( D \), with \( h \) independent of \( x \) and \( h \).

Clearly, \( A \subset A_x \). If the left side in (7.4) is \( O|h| \) as \( |h| \to 0 \), uniformly in \( x \in D \), we shall write \( f \in A_x \).

In a number of problems the class \( A \) seems to be a more natural one to consider than \( A \). For example, the theorem of Privalov quoted above does not hold for \( \gamma = 1 \), but it can be shown that if \( f(a_{x}) = \sum \alpha_{y} e^{i\omega y} \) is in \( A \) (in particular, if \( f \in A \)), then the conjugate function

\[
\sum c_{m}(-i \text{ sign } m) e^{im\omega y}
\]

is also in \( A \) (see [7]). An analogous result could be established for the \( k \)-dimensional case, but the proof then is decidedly more difficult and unlike the proof of Theorem 11 is not an imitation of the argument in the one-dimensional case. For this reason we shall confine our attention here to a rather special result which is of interest on account of certain applications.

Let us suppose that \( f \) is periodic and of the class \( A \) and that \( A \) is merely bounded. We then have (1) and (2). For \( f'(x+h)+f'(x-h)-2f'(x) \) we get an expression analogous to the right side of (7.3) with \( K(t+h)+K(t-h)-2K(t) \). Suppose that for \( |h| \to 0 \) we have an inequality

\[
|K(t+h)+K(t-h)-2K(t)| \leq A \frac{|h|^\gamma}{|h|^{\gamma+\gamma}} \quad (\gamma > 0)
\]

valid for some \( \gamma \) and \( A \) independent of \( t \) and \( h \). Then, using formula (2.2), we find, as before, that

\[
K'_{x+y}+K'_x(t-h)-2K'_x(t) = O \left( \frac{|h|^\gamma}{|h|^{\gamma+\gamma}} \right) + O(|h|^\gamma)
\]

and

\[
f(x+h)+f(x-h)-2f(x) = O(|h|^\gamma) \int_{|t| \geq A} \frac{dt}{|t|^{\gamma+\gamma}} + O(|h|^\gamma) \int_{0}^{1} O(1) dt = O(h),
\]

so that \( f \in A_x \).

A similar argument shows that if \( f \) is continuously differentiable (by this we mean that all derivatives of \( f \) of order 1 exist and are continuous) then \( f \in A_x \). For we may write \( f = f_1 + f_2 \), where \( f_1 \) is a finite polynomial and \( |f_2(x+\varepsilon)-f_2(x)| \leq \varepsilon |x| \) for \( |x| \) small enough. Then clearly,

\[
|f_2(x+h)+f_2'(h)-2f_2'(x)| \leq C |h|,
\]

for \( |h| \) small enough, and since \( f_2 \) is a trigonometric polynomial and so satisfies condition \( \Lambda_x \), the function \( f' := f_1' + f_2' \) also satisfies condition \( \Lambda_x \).

Inequality (7.5) is certainly valid, with \( \gamma = 1 \) if \( \Omega \) has bounded second derivatives with respect to the spherical coordinates. For assuming, as we may, that \( \Omega \) is real-valued, by the Mean-Value Theorem

\[
K(t+h)+K(t-h)-2K(t) = h^2 K''(t+\theta h) = O(h^2)
\]

for \( |h| \geq 2|h| \), \( K'' \) denoting here the second directional derivative and \( -1 < \theta < 1 \). Thus

**Theorem 13.** Suppose that \( K \) satisfies (7.5). Then the assumption \( f \in A \) implies \( f' \in A_x \), and if \( f \) is continuously differentiable, \( f' \) satisfies condition \( \Lambda_x \). The conclusion holds, in particular, if \( f' \) is defined by the series (6.4) and \( P(m) \) is the same as in Theorem 10.

In the concluding section of this paper we shall give a few observations about functions \( f \in A_x \).

8. In this section we prove a few results about discrete analogues of the Hilbert transform.

Let \( X = (\ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots) \) be any two-way infinite sequence of real or complex numbers. For any \( p > 0 \) we shall denote by \( \|X\|_{p} \) the \( p \)-th norm of \( X \):

\[
\|X\|_{p} = \left( \sum_{\lambda=0}^{\infty} |x_{\lambda}|^p \right)^{1/p}.
\]

The class of sequences \( X \) with \( \|X\|_{p} \) finite will be denoted by \( P \).

Let \( \bar{X} = (\ldots, x_{-1}, x_0, x_1, \ldots) \) denote the sequence

\[
x_{m} = \sum_{n=-\infty}^{\infty} x_n, \quad m = n,
\]

the prime indicating that the term \( x_{-m} \) is omitted in summation. This is a discrete analogue of the Hilbert transform

\[
f(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy,
\]

and it is very well known that

\[
\|\bar{X}\|_{p} \leq A \|X\|_{p}, \quad p > 1,
\]

a result which was extended by Riesz [9] to

\[
\|\bar{X}\|_{p} \leq A_{\rho} \|X\|_{p}, \quad p > 1.
\]

The best value of \( A_{\rho} \) is \( \pi \); for other \( p \)'s the best value of \( A_{\rho} \) is unknown.
Riesz [6] deduced (8.3) from the inequality \[ \|f\|_p \leq A_p \|f\|_p, \]
valid for the function (8.2). His argument is applicable to a more general class of discrete transforms which we are going to introduce now.

Let us again consider the space \( L^p \) and a kernel \( K(x) = O(|x|^{-d}) \) with properties described in section 1. Let \( e_1, e_2, \ldots, e_k \) be a system of \( k \) linearly independent vectors in \( L^2 \); thus, in particular, all the \( e_i \) are different from zero. Set \( p_0 = p_1, p_2, \ldots \) be the sequence of all lattice points in \( L^2 \) generated by this system, i.e., the \( p_i \)'s are of the form \( \mu e_1 + \mu_2 e_2 + \cdots + \mu_k e_k \), where the coefficients \( \mu_j \) are arbitrary real integers. For any sequence \( X = (e_1, e_2, \ldots) \) of real or complex numbers we define the transform \( \bar{X} = (e_1, e_2, \ldots) \) by the formula

\[
\bar{X}_m = \sum_{n \geq m} e_n K(p_m - p_n).
\]

For such sequences \( \bar{X} \) we have the following result generalizing (8.3):

**Theorem 14.** If \( X \in L^p \), \( p > 1 \), so is \( \bar{X} \), and

\[
\|\bar{X}\|_p \leq A_p \|X\|_p,
\]

where \( A_p \) depends only on \( p \) and the kernel \( K \).

The series on the right are all absolutely convergent if \( X \in L^p, p \geq 1 \). For \( p = 1 \) this is immediate, and for \( p > 1 \) follows by an application of Holder's inequality; for since \( K(x) = O(|x|^{-d}) \) as \( |x| \to \infty \), the series

\[
\sum_{n \geq m} |e_n| |K(p_m - p_n)|^q
\]

is finite for every \( q > 1 \).

For the sake of simplicity we assume that the vectors \( e_1, e_2, \ldots, e_k \) are all mutually orthogonal, of length 1, and situated on the coordinate axes. The proof in the general case remains essentially the same. Let \( R_m \) denote the cube with centre \( p_m \) and edges of length 1, parallel to the axes. By \( R_m \) we shall denote the concentric similarly situated cube with edges \( 1/2 \). Given a sequence \( X = (e_1, e_2, \ldots, e_m) \) let \( f(x) \) denote the function taking the value \( e_m \) at the points of \( R_m \) (\( m = 0, 1, 2, \ldots \)) and equal to zero elsewhere in \( L^2 \). The function \( f \) is in \( L^p \) if and only if \( X \) is in \( L^p \), and the ratio \( \|X\|_p/\|f\|_p \) depends on \( k \) but not on \( p \), hence on an account of Theorem B of Section 1.

\[
\sum_{m \geq 0} \|f(x)\|^p dx = \int |f(x)|^p dx \leq \|f\|_p^p = A_p^p \sum_{m \geq 0} |e_m|^p.
\]

For \( \sigma \in K_m \) we may write

\[
\bar{f}(x) = \sum_{m \geq 0} e_m \int K(x - y) dy + \sigma_m \int K(x - y) dy.
\]

Let \( \omega(\delta) \) be the modulus of continuity of \( \phi \) on \( \Sigma \). Without loss of generality we may assume that \( \omega(\delta) \geq \delta \), since otherwise in the inequalities

that follow we replace \( \omega(\delta) \) by \( \min(\omega(\delta), \delta) \), and (8.7) may be written

\[
\int |\bar{f}(x)|^p dx \leq \|\bar{f}\|_p^p = A_p^p \sum_{m \geq 0} |e_m|^p.
\]

Let us integrate these expressions over the cube \( K_m \) and sum the results over all \( m \). It is easy to show that all the three terms are majorized by a fixed multiple of \( \sum |e_m|^p \).

This is certainly the case for the sum involving \( |f| \) (see (8.6)). Since

\[
\sum_{m \geq 0} |e_m| |K(p_m - p_n)|^{\alpha} \leq A_p^{\alpha} \sum_{m \geq 0} |e_m|^p \leq A_p^{\alpha},
\]

also the second sum satisfies the condition. Finally, setting

\[
\alpha_m = |e_m|^d \omega \left( \frac{1}{|p_m|^{\alpha}} \right) \quad \text{for} \quad m > 0, \quad \alpha_0 = 0,
\]

and observing that the Dini condition imposed on \( \omega \) implies that \( \Sigma \alpha_m = \omega \) is finite, we may write the following inequalities, in which \( p' \) is the exponent conjugate to \( p \):

\[
\sum_{m \geq 0} |e_m|^d \alpha_m |p_m - p_n|^{\alpha} \leq \sum_{m \geq 0} |e_m|^d |\alpha_m|^d |p_m - p_n|^{\alpha} \leq \alpha_0^{\alpha} \sum_{m \geq 0} |e_m|^p |p_m - p_n|^{\alpha}
\]

\[
= \alpha_0^{\alpha} \sum_{m \geq 0} |e_m|^p |p_m - p_n|^{\alpha}.
\]

This completes the proof of Theorem 14.
Of course, Theorem 14 can be restated in the language of bilinear (or quadratic) forms:

\[ |\sum \hat{a}_m \hat{b}_n K(p_m - p_n)| \leq \|a\| \|b\|_p, \quad p > 1.\]

9. Let us now change our notation slightly and let \( \mathbf{m} \) and \( \mathbf{n} \), similarly \( \mathbf{m} \) and \( \mathbf{n} \), denote the general lattice point in \( \mathbb{E}^3 \), i.e., \( m = \sum m_k e_k \), \( n = \sum n_k e_k \), where the \( \mu \)'s are arbitrary integers and \( e_1, \ldots, e_k \) are unit vectors mutually orthogonal.

**Theorem 15.** The series

\[ \sum_n \mathcal{K}(\mathbf{m}) e^{i\mathbf{m} \cdot \mathbf{x}} \]

is the Fourier series of a bounded function \( \chi(x) \). The number

\[ M = \text{ess sup } |\chi(x)| \]

is the norm of the linear transformation

\[ \mathcal{T}_m = \sum_n \mathcal{K}(\mathbf{m} - \mathbf{n}) \hat{a}_n, \]

considered as a transformation from \( L^p \) into \( L^p \).

We already know that the transformation (9.2) is bounded and that this fact will enable us to deduce the boundedness of the function (9.1). Since \( \sum |\mathcal{K}(\mathbf{m})|^2 \) is finite, (9.1) is in any case the Fourier series of a function \( \chi \in L^2 \). Similarly, if \( \sum |\hat{a}_n|^2 \) converges,

\[ \sum_n \mathcal{K}(\mathbf{m} - \mathbf{n}) \hat{a}_n \sim \hat{\chi}(\mathbf{x}) e^{i\mathbf{x} \cdot \mathbf{m}}. \]

By (9.3), \( \sum \hat{a}_n e^{i\mathbf{x} \cdot \mathbf{m}} \) is the Fourier series of the integrable function

\[ \hat{\chi}(\mathbf{x}) = \hat{\chi}(\mathbf{x}) e^{i\mathbf{x} \cdot \mathbf{m}}. \]

Since \( \sum |\hat{a}_n|^2 \) is finite, the function \( \hat{\chi} \) is quasicontinuously integrable and we can write

\[ \int |\hat{\chi}|^2 = \int |\hat{\chi}|^2 |\mathcal{K}(\mathbf{m})|^2 \leq M^2 \int |\hat{\chi}|^2 \leq M^2 \sum |\hat{a}_n|^2, \]

so that the number \( M(\leq \infty) \) is no less than the norm of the transformation (9.2). Moreover, one immediately sees that \( M \) is actually equal to the norm of the transformation. Since the transformation is bounded, the theorem follows.

An interesting illustration is provided in the two-dimensional case by the transformation

\[ \hat{x} = \sum_n \frac{\hat{a}_n}{(m - n)^4}, \]

where \( \hat{a}_n \) and \( \hat{x} \) denote complex integers. This seems to be the most natural extension of the classical Hilbert transformation (8.1) to the two-dimensional case. The norm of (9.4) is the upper bound of the modulus of the function given by the Fourier series

\[ \sum_n \frac{\hat{a}_n e^{i\mathbf{x} \cdot \mathbf{m}}}{(m - n)^4}, \]

The latter function occurs already in the work of Kronecker on elliptic functions and is expressible in terms of elliptic theta functions.

10. We conclude by a few remarks concerning smooth functions.

Suppose a function \( f(x) \) is determined in the neighbourhood of a point \( x_0 \in \mathbb{E}^3 \). We say that \( f \) is smooth at \( x_0 \) if

\[ f(x_0 + h) + f(x_0 - h) - 2f(x_0) = o(h) \quad \text{as } |h| \to 0. \]

If \( f \) is smooth and continuous at every point of an open set \( D \), \( f \) will be called smooth in \( D \). The latter notion has close connection with condition \( \lambda_k \) introduced in Section 7, the only difference being that the notion of smoothness in \( D \) does not presuppose the uniformity of the "o" in (10.1) with respect to \( x_0 \in D \).

Clearly, if \( f \) is differentiable at \( x_0 \) (i.e., if it has a total differential at \( x_0 \)), then \( f \) is smooth at \( x_0 \), but the converse need not be true. Thus smooth functions may be considered as a generalization of differentiable functions. Similarly, functions of the class \( \lambda_k \) may be considered as a generalization of continuously differentiable functions (i.e., functions with continuous first partial derivatives).

The notion of smoothness of functions is familiar in the simplest case \( k = 1 \) (see [7]), and the definition (10.1) seems to be a natural extension of that special case to general \( k \). In what follows we shall prove a few simple results concerning smooth functions. We shall not presuppose any longer that the functions \( f \) considered are periodic, and the results themselves will have little connection with the previous discussion.

(a) If \( f \) is smooth in \( D \) and real-valued, and if \( f \) has a maximum (or minimum) at \( x_0 \), then \( f \) is differentiable at \( x_0 \) and the partial derivatives of \( f \) at \( x_0 \) with respect to the coordinates are zero.

This is immediate since (10.1) can be rewritten

\[ |f(x_0 + h) - f(x_0)| + |f(x_0 - h) - f(x_0)| = o(|h|), \]

and since for \( |h| \) small enough both terms in curly brackets are of the same sign, we get \( f(x_0 + h) - f(x_0) = o(|h|) \), which is the desired result.

(b) If \( f \) is smooth in \( D \) and real-valued, then the set \( S \) of the points of differentiability of \( f \) is dense in \( D \); indeed, it is of the power of the continuum in every sphere \( K \) totally contained in \( D \).
We may suppose that the closure of \(K\) is in \(D\). Let \(g(x)\) be a real-valued and continuously differentiable function vanishing on the boundary of \(K\), positive inside \(K\) and taking a large value at the centre of \(K\). Then the sum \(h = f + g\) certainly has a maximum at a point \(a_k \in K\), and so is differentiable at \(a_k\). Hence, also \(f\) is differentiable at \(a_k\), which shows that \(S\) is dense in \(D\).

Let now \(f(x) = a_1 \xi_1 + a_2 \xi_2 + \ldots + a_k \xi_k\) be a real-valued linear function with coefficients \(a_1, a_2, \ldots, a_k\) numerically small but otherwise quite arbitrary. The function \(h(x) = f(x) + g(x)\) will then still have a maximum at a point \(a_k (\xi_1, \ldots, \xi_k) \in K\), and so will be differentiable at that point. Moreover, the first partial derivatives of \(h(x)\) at \(a_k (\xi_1, \ldots, \xi_k)\) will be zero, and so the first partial derivatives of \(f(x) + g(x)\) at that point will be \(-a_1, \ldots, -a_k\). Thus the point \(a_k (a_1, \ldots, a_k)\) varies with the system \((a_1, \ldots, a_k)\). It follows that the set of points \(a_k (a_1, \ldots, a_k)\) is of the power of the continuum, \(h = f + g + t\) is differentiable in a subset of \(K\) of the power of the continuum, and the same holds for \(f\).

(i) The partial derivatives \(u_k (\xi_1, \ldots, \xi_k)\) of the potential

\[
u(k > 2)
\]

\[
\frac{f(y)}{|x-y|^{k-1}}
\]

corresponding to a continuous density \(f\), satisfy condition \(A_k\) in every finite sphere.

Without loss of generality we may suppose that \(f(y)\) vanishes for \(|y|\) large. It is a classical fact that under the assumption of continuity of \(f\) the partial derivatives \(u_k\) exist everywhere, are continuous and given by the formulae

\[
u(k - 2)
\]

\[
\frac{f(y) - \xi_k}{|x-y|^{k-1}}
\]

It is also very well known that the second partial derivatives of \(u\) need not exist at individual points, and statement (c) is a substitute for the existence of these derivatives.

It is enough to give a sketch of proof since the whole argument follows familiar lines. On the right of the last formula we have a convolution of \(f\) with the kernel \(K(x) = -(k-2) \xi_i |x|^k\). In the integrals

\[
\int f(y) K(x-y) dy,
\]

we consider separately the parts extended over the sphere \(|y - x| \leq 3 |h|\) and over the remainder of the space \(E^p\). Since \(f\) is bounded, and