On approximation in real Banach spaces

by

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Many authors have studied analytic operations from a complex Banach space to another complex Banach space. In their recent paper Alexiewicz and Orlicz [1] introduced analytic operations defined in real Banach spaces. In this paper I solve the question whether it is possible to approximate uniformly continuous operations by analytic ones.

In section 1 I show that the answer to this question is positive if a certain condition is satisfied. This condition is satisfied if the operations are defined in the space $L^{(p)}$ or $l^{(p)}$, where p is an even positive integer, or in the Cartesian product of these spaces. Section 2 is devoted to the case where the operations are defined in the space $C\langle 0,1\rangle$. In this case the answer is negative and a real-valued continuous functional is not in general the uniform limit of a sequence of differentiable functionals. A regularly differentiable functional[1]) $f(x)$ defined in the unit sphere of the space $C\langle 0,1\rangle$ has the following property:

(A) If ε, r_1 and r_2 are three given positive numbers, $r_1<r_2<1$, then there is always an $x\in C\langle 0,1\rangle$ fulfilling the relations

$$r_1<\|x\|<r_2, \qquad |f(x)-f(\Theta)|<\varepsilon,$$

where Θ is the zero element of the space $C\langle 0,1\rangle$.

Apparently the identical operation defined in the space $C\langle 0,1\rangle$ has not the property (A) and the fact that the space of arguments is richer than the space of values is essential. But the assertion remains true if we replace functionals by operations having their values in a weakly complete space.

In section 3 the results of section 2 are extended to the case where the functionals are defined in the space $L^{(p)}$ or $l^{(p)}$, $p\geqslant 1$ not being an even integer. The main result of this section is that a $\overline{p}$ times regularly differentiable functional has the property (A) if $\overline{p}$ denotes the least integer greater than or equal to $\overline{p}$.

[1]) See section 2, definition 1.

1. Let B be a Banach space. By $\tilde{B}$ we denote the complex Banach space, whose elements are couples of elements of the space B,

$$z=(x,y)=x+iy, \qquad z\in\tilde{B}, \qquad x,y\in B,$$

with the usual definition of operations and with the norm

$$\|z\|=\sup_{0\leqslant\alpha\leqslant 2\pi}\|\cos\alpha\cdot x-\sin\alpha\cdot y\|.$$

Let G be a subset of the space B and let a unique vector $y=F(x)$ from a real Banach space B_1 (a unique number $y=f(x)$) correspond to every $x\in G$. Then we say that $F(x)$ *is an operation* ($f(x)$ *is a functional*).

If $q(x)$ is a real polynomial[2]) in B then there is a polynomial $\tilde{q}(z)$ in $\tilde{B}$ uniquely defined by the condition $\tilde{q}(z)=q(z)$ if $z=x+iy$, $y=\Theta$ (Θ is the zero element of the space B).

THEOREM 1. *Let B be a separable real Banach space. Suppose that there is a real polynomial $q^*(x)$ fulfilling the conditions*

$$q^*(\Theta)=0, \qquad \inf_{x\in B,\|x\|=1} q^*(x)>0.$$

Let G be an open subset of B and let $F(x)$ be a continuous operation defined in G and having its values in an arbitrary Banach space B_1.

Then there exists an operation $H(x)$ analytic[3]) *in G that the inequality*

$$\|F(x)-H(x)\|<1 \tag{1}$$

holds for $x\in G$.

Proof. $q^*(x)$ is a polynomial of degree $m>0$. We write

$$q^*(x)=q_1(x)+q_2(x)+\ldots+q_m(x),$$

where $q_j(x)$ is a homogeneous polynomial of degree j, $j=1,2,\ldots,m$.

Let

$$q(x)=q_1^2(x)+q_2^2(x)+\ldots+q_m^2(x).$$

The polynomial $q(x)$ is non negative, assumes the value zero only if $x=\Theta$, and apparently there is such a positive number η that $\|x\|=1$ implies $q(x)>\eta$.

If $y\in B$, $r>0$, we define the sets $K(y,r)$ and $C(y,r)$:

$$K(y,r)=\mathop{E}_{x}[x\in B, q(x-y)<r],$$

$$C(y,r)=\mathop{E}_{x}[x\in B, q(x-y)>r].$$

[2]) For the definition and properties of polynomials see [1], [2], [3].

[3]) See [1].

We easily prove that $K(y,r)$ is an open bounded set and that for every positive number r' there is such a positive number r that $x \in K(\Theta, r)$ implies that $\|x\| < r'$.

For each point $x_0 \in G$ there is such a positive number $r(x_0)$ that the relation $K(x_0, 2r(x_0)) \subset G$ holds, and that $x \in K(x_0, 2r(x_0))$ implies that

$$\|F(x) - F(x_0)\| < \frac{1}{4}.$$

The sets $K(x, r(x)), x \in G$, cover the set G. As the space B is separable, we can choose a countable covering of the set G:

$$K(x_1, r(x_1)), K(x_2, r(x_2)), \ldots, \qquad x_1, x_2, \ldots \in G.$$

Now we form the sets $D_1, D_2, D_3, \ldots$ which cover the set G in a locally finite manner. We choose a sequence of positive numbers ε_i,

$$3\varepsilon_i < r(x_i), \qquad 1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots,$$
$$\varepsilon_i \to 0 \quad \text{with} \quad i \to \infty,$$

and write

$$D_1 = K(x_1, r(x_1)),$$
$$D_2 = C(x_1, r(x_1) - \varepsilon_2) \cap K(x_2, r(x_2)),$$
$$D_3 = C(x_1, r(x_1) - \varepsilon_3) \cap C(x_2, r(x_2) - \varepsilon_3) \cap K(x_3, r(x_3)),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Let us choose a point $y \in G$. There is such an index k that

$$y \in K(x_k, r(x_k)), y \bar{\in} K(x_1, r(x_1)), \ldots, y \bar{\in} K(x_{k-1}, r(x_{k-1})).$$

Further there is such an index $l > k$ that $y \in K(x_k, r(x_k) - 3\varepsilon_l)$. As

$$K(x_k, r(x_k) - 3\varepsilon_l) \cap C(x_k, r(x_k) - \varepsilon_j) = 0, \qquad j = l, l+1, l+2, \ldots,$$

we get

$$K(x_k, r(x_k) - 3\varepsilon_l) \cap D_j = 0, \qquad j = l, l+1, l+2, \ldots,$$

and we find a neighbourhood $K(x_k, r(x_k) - 3\varepsilon_l)$ of the point y that intersects only a finite number of the sets D_j, $j = 1, 2, 3, \ldots$ As the point y is arbitrary, the covering D_j, $j = 1, 2, 3, \ldots$, is locally finite.

We define another open covering of the set G by means of the relations

$$D_1^* = K(x_1, r(x_1) + 2\varepsilon_1),$$
$$D_2^* = C(x_1, r(x_1) - 3\varepsilon_2) \cap K(x_2, r(x_2) + 2\varepsilon_2),$$
$$D_3^* = C(x_1, r(x_1) - 3\varepsilon_3) \cap C(x_2, r(x_2) - 3\varepsilon_3) \cap K(x_3, r(x_3) + 2\varepsilon_3),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We have $D_j \subset D_j^* \subset G$, $j = 1, 2, 3, \ldots$, and the sets D_j cover the set G. We find again that the set $K(x_k, r(x_k) - 3\varepsilon_l)$, $l > k$, is contained in the complement of the set D_j^*, $j = l, l+1, l+2, \ldots$, and consequently the covering D_j^*, $j = 1, 2, 3, \ldots$, is locally finite.

Let E_n be the n-dimensional Euclidean space. We define the sets $T_1 \subset E_1, T_2 \subset E_2, \ldots$,

$$T_1 = \underset{\tau_1}{E}[-1 \leqslant \tau_1 \leqslant r(x_1) + \varepsilon_1],$$
$$T_2 = \underset{(\tau_1,\tau_2)}{E}[r(x_1) - 2\varepsilon_2 \leqslant \tau_1 \leqslant V_2,\ -1 \leqslant \tau_2 \leqslant r(x_2) + \varepsilon_2],$$
$$T_3 = \underset{(\tau_1,\tau_2,\tau_3)}{E}[r(x_1) - 2\varepsilon_3 \leqslant \tau_1 \leqslant V_3,\ r(x_2) - 2\varepsilon_3 \leqslant \tau_2 \leqslant V_3,\ -1 \leqslant \tau_3 \leqslant r(x_3) + \varepsilon_3],$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where the number V_2 fulfils the condition

$$\text{if} \quad x \in K(x_2, 2r(x_2)), \quad \text{then} \quad q(x - x_1) < V_2 - 1,$$

the number V_3 fulfils the condition

$$\text{if} \quad x \in K(x_3, 2r(x_3)), \quad \text{then} \quad q(x - x_1) < V_3 - 1,\ q(x - x_2) < V_3 - 1,$$

and so on.

Now we define the functionals

$$\varphi_1(z) = (\|F(x_1)\| + 1)\, \nu_1 \cdot \int_{T_1} \exp\{-t_1 a_1 (\tilde{q}(z - x_1) - \tau_1)^2\}\, d\tau_1, \qquad z \in \tilde{B},$$

$$\varphi_2(z) = (\|F(x_2)\| + 1)\, \nu_2$$
$$\cdot \iint_{T_2} \exp\{-t_2 [a_1 (\tilde{q}(z - x_1) - \tau_1)^2 + a_2 (\tilde{q}(z - x_2) - \tau_2)^2]\}\, d\tau_1\, d\tau_2,$$

$$\varphi_3(z) = (\|F(x_3)\| + 1)\, \nu_3 \times$$
$$\times \iiint_{T_3} \exp\{-t_3 [a_1 (\tilde{q}(z - x_1) - \tau_1)^2 + a_2 (\tilde{q}(z - x_2) - \tau_2) + a_3 (\tilde{q}(z - x_3) - \tau_3)^2]\}$$
$$d\tau_1\, d\tau_2\, d\tau_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where

$$\frac{1}{\nu_1} = \int_{E_1} \exp\{-t_1 a_1 \tau_1^2\}\, d\tau_1 = b_1 t_1^{-1/2},$$

$$\frac{1}{\nu_2} = \iint_{E_2} \exp\{-t_2 [a_1 \tau_1^2 + a_2 \tau_2^2]\}\, d\tau_1\, d\tau_2 = b_2 t_2^{-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and $b_1, b_2, \ldots$ are positive numbers depending on the constants $a_1, a_2, \ldots$ The positive constants $a_1, a_2, \ldots, t_1, t_2, \ldots$ will be chosen later.

The functionals $\varphi_j(z)$ are analytic in $\tilde{B}$, $j=1,2,\ldots$ This follows from the fact that the functional $\varphi_j(z)$ is a superposition of the operation

$$\big(\tilde{q}(z-x_1),\tilde{q}(z-x_2),\ldots,\tilde{q}(z-x_j)\big),$$

which is analytic in $\tilde{B}$ and of an analytic function of j complex variables.

We choose the sequence of positive numbers a_n $(n=1,2,\ldots)$ in such a way that the series

$$\sum_{n=1}^{\infty} a_n(1+\|x-x_n\|)^{4m} \tag{2}$$

converges for any $x\in B$. (Here m is the degree of the polynomial $q^*(z)$). This is possible. If, for example,

$$a_n=\frac{1}{n!(1+\|x_n\|)^{4m}},$$

we have

$$a_n(1+\|x-x_n\|)^{4m}\leqslant\frac{(1+\|x\|)^{4m}}{n!},$$

and the series (2) converges for every $x\in B$.

The numbers t_n are chosen sufficiently great for the three conditions (3), (4) and (5) be fulfilled:

$$t_n\geqslant(n!)^2\cdot\frac{1}{b_n}(\|F(x_n)\|+1)^2\cdot|T_n|+1 \tag{3}$$

($|T_n|$ is the Lebesgue measure of the set T_n in E_n),

$$\big|\varphi_n(x)-\|F(x_n)\|-1\big|<\frac{1}{2}\quad\text{for}\quad x\in D_n, \tag{4}$$

$$|\varphi_n(x)|<\frac{1}{2^{n+3}(\|F(x_n)\|+1)}\quad\text{for}\quad x\,\bar{\epsilon}\,D_n^*. \tag{5}$$

This is possible. If $x\in D_n$, we have

$$0\leqslant q(x-x_n)<r(x_n),$$

$$q(x-x_j)>r(x_j)-\varepsilon_n\qquad(j=1,2,\ldots,n-1)$$

and

$$q(x-x_j)<V_n-1\qquad(j=1,2,\ldots,n-1).$$

Recalling the definition of the set T_n we find that for every $x\in D_n$ the sphere in E_n with the centre at the point $\big(q(x-x_1),q(x-x_2),\ldots,q(x-x_n)\big)$ and with the radius ε_n is contained in T_n.

If $x\,\bar{\epsilon}\,D_n^*$, then at least one of the inequalities

$$q(x-x_n)<r(x_n)+2\varepsilon_n,$$

$$q(x-x_j)>r(x_j)-3\varepsilon_n\qquad(j=1,2,\ldots,n-1)$$

is false. Recalling the definition of the set T_n we find that for every $x\in D_n^*$ the sphere in E_n with the radius ε_n and with the centre at the point $\big(q(x-x_1),q(x-x_2),\ldots,q(x-x_n)\big)$ has an empty intersection with the set T_n. Now it is easy to prove that there is a t_n satisfying the required conditions.

Let

$$\varphi(x)=\varphi_1(x)+\varphi_2(x)+\ldots,$$

$$H^*(x)=F(x_1)\varphi_1(x)+F(x_2)\varphi_2(x)+\ldots;$$

we prove that the operations $\varphi(x)$ and $H^*(x)$ are analytic.

This fact is an easy consequence of the following proposition (since the uniform limit of a sequence of analytic operations in complex Banach spaces is analytic[4]):

To every $x_0\in G$ there correspond a positive number δ and an integer n_0 in such a way that the inequality

$$\big(\|F(x_n)\|+1\big)|\varphi_n(x_0+z)|<\frac{1}{2^n} \tag{6}$$

holds for $z\in\tilde{B}$, $\|z\|<\delta$, $n>n_0$.

Let us fix a point $x_0\in G$. There is an index j_0 fulfilling the conditions

$$x_0\in K\big(x_{j_0},r(x_{j_0})\big),\qquad x_0\,\bar{\epsilon}\,K\big(x_j,r(x_j)\big)\qquad(j=1,2,\ldots,j_0-1).$$

Consequently there are a positive number α and an index n' such that the inequality $\tau_{j_0}-q(x_0-x_{j_0})>\alpha$ holds for every point $(\tau_1,\tau_2,\ldots,\tau_n)\in T_n$, $n>n'$.

We find a lower bound of the expression

$$\operatorname{Re}\Big\{\sum_{j=1}^{n}a_j\big(\tilde{q}(x_0+z-x_j)-\tau_j\big)^2\Big\}.$$

As $\tilde{q}(z)$ is a polynomial of degree $2m$, we have

$$q(x_0-x_j+z)=q(x_0-x_j)+Z_j,$$

$$|Z_j|\leqslant M(1+\|x-x_j\|)^{2m}\cdot\|z\|,\qquad\|z\|<1,$$

where M is a constant[5]).

[4]) See [2], p. 83, Theorem 4, 6, 2.
[5]) See [2], p. 69.

It follows

$$(\tilde{q}(x_0+z-x_j)-\tau_j)^2=(q(x_0-x_j)-\tau_j+Z_j)^2$$
$$=(q(x_0-x_j)-\tau_j)^2-2(q(x_0-x_j)-\tau_j)Z_j+Z_j^2,$$
$$\mathrm{Re}\{(\tilde{q}(x_0+z-x_j)-\tau_j)^2\}\geqslant(q(x_0-x_j)-\tau_j)^2-2|q(x_0-x_j)-\tau_j||Z_j|-|Z_j|^2$$
$$=(|q(x_0-x_j)-\tau_j|-|Z_j|)^2-2|Z_j|^2.$$

We write

$$\mathrm{Re}\{\sum_{j=1}^{n}a_j(q(x_0+z-x_j)-\tau_j)^2\}$$
$$\geqslant-2\sum_{j=1}^{n}a_j|Z_j|^2+a_{j_0}(|q(x_0-x_j)-\tau_{j_0}|-|Z_{j_0}|)^2$$
$$\geqslant-2M^2\|z\|^2\sum_{j=1}^{\infty}a_j(1+\|x_0-x_j\|)^{4m}+a_{j_0}(\alpha-M(1+\|x_0-x_{j_0}\|)^{2m}\|z\|)^2,$$

where $(\tau_1,\tau_2,\ldots,\tau_n)\in T_n$ and α is positive. As the series

$$\sum_{j=1}^{\infty}a_j(1+\|x_0-x_j\|)^{4m}$$

converges, we can find such positive numbers β and δ that the inequality

$$\mathrm{Re}\{\sum_{j=1}^{n}a_j(q(x_0+z-x_j)-\tau_j)^2\}>\beta$$

holds for $z\in\tilde{B}$, $\|z\|<\delta$, $(\tau_1,\tau_2,\ldots,\tau_n)\in T_n$, $n>n'$.

We easily get

$$(\|F(x_n)\|+1)|\varphi_n(x_0+z)|\leqslant\frac{t_n^{n/2}}{b_n}(\|F(x_n)\|+1)^2\cdot|T_n|\cdot e^{-\beta t_n}$$
$$\leqslant\frac{(\|F(x_n)\|+1)^2|T_n|}{b_n}\cdot\frac{n!}{\beta^n t_n^{n/2}}\leqslant\frac{(\|F(x_n)\|+1)^2|T_n|n!}{b_n}\frac{1}{\beta^n t_n}\leqslant\frac{1}{n!\,\beta^n},$$

and consequently there is such an index n_0 that the inequality (6) holds for $n>n_0$, $\|z\|<\delta$.

Now let

$$H(x)=\frac{H^*(x)}{\varphi(x)}.$$

The operation $H(x)$ is analytic and we prove that it fulfils the inequality (1). Let us fix a point $x\in G$. We write

$$F(x)-H(x)=F(x)\sum_{i=1}^{\infty}\frac{\varphi_i(x)}{\varphi(x)}-\sum_{i=1}^{\infty}\frac{F(x_i)\varphi_i(x)}{\varphi(x)}$$
$$=\frac{1}{\varphi(x)}\sum_{i=1}^{\infty}(F(x)\varphi_i(x)-F(x_i)\cdot\varphi_i(x)).$$

By I_1 (I_2) we denote the set of the indices j fulfilling the relation $x\in D_j^*$ $(x\,\bar{\in}\,D_j^*)$. We have

$$\|F(x)-H(x)\|\leqslant\frac{1}{\varphi(x)}\sum_{j\in I_1}\|F(x)-F(x_j)\|\varphi_j(x)$$
$$+\frac{\|F(x)\|}{\varphi(x)}\sum_{j\in I_2}\varphi_j(x)+\frac{1}{\varphi(x)}\sum_{j\in I_2}\|F(x_j)\|\varphi_j(x).$$

If $j\in I_1$, then $x\in D_j^*\subset K(x_j,2r(x_j))$ and

$$\|F(x)-F(x_j)\|<\frac{1}{4}.$$

Further, we have $x\in D_l$ for a suitable l,

$$\|F(x)-F(x_l)\|<\frac{1}{4},\qquad\varphi_l(x)>\|F(x_l)\|+\frac{1}{2}$$

and

$$\varphi(x)\geqslant\varphi_l(x)>F(x),\qquad\varphi(x)>\frac{1}{2},\qquad\frac{\|F(x)\|}{\varphi(x)}<1.$$

Finally we use the inequalities

$$\sum_{j\in I_2}\varphi_j(x)\leqslant\sum_{j=1}^{\infty}\frac{1}{2^{j+3}}=\frac{1}{8},\qquad\sum_{j\in I_2}\|F(x_j)\|\varphi_j(x)\leqslant\frac{1}{8}$$

and get the required inequality

$$\|F(x)-H(x)\|\leqslant\frac{1}{4}+\frac{1}{8}+\frac{2}{8}<1.$$

From the theorem 1 we easily deduce the following

THEOREM 2. *Let the space B satisfy the same assumptions as in theorem 1. Let $F(x)$ be an operation defined and continuous in an open set $G\subset B$. Let $\varphi(x)$ be a positive continuous functional in G $(\varphi(x)>0$ if $x\in G)$.*

Then there is an operation $H(x)$ analytic in G and fulfilling the inequality

$$\|F(x)-H(x)\|<\varphi(x).$$

Proof. By theorem 1 there is a functional $\psi(x)$ that is analytic in G and fulfils the inequality

$$\left|\frac{1}{\varphi(x)}+1-\psi(x)\right|<1$$

for $x\epsilon G$. Evidently we have

$$\psi(x)>\frac{1}{\varphi(x)}.$$

By the same theorem there is an operation $H^*(x)$ analytic in G and fulfilling the inequality

$$\|\psi(x)F(x)-H^*(x)\|<1,\qquad x\epsilon G.$$

We write $H(x)=H^*(x)/\psi(x)$ and get

$$\|F(x)-H(x)\|<\frac{1}{\psi(x)}<\varphi(x).$$

Especially, theorem 2 holds if $B=L^{(p)}$ or $B=l^{(p)}$ where p is an even positive integer, as in this case the p-th power of the norm is a polynomial; it holds also if B is the Cartesian product of these spaces. One verifies easily that if the spaces B_1 and B_2 satisfy the assumptions of theorem 1, then their Cartesian product $B_1\times B_2$ satisfies these assumptions too.

Theorem 2 has the following

Corollary. *Suppose that G is an open subset of E_n, that $f(x)$ is a continuous function in G and that $\varphi(x)$ is a positive continuous function in G. Then there is such a function $h(x)=h(\xi_1,\xi_2,\ldots,\xi_n)$ analytic in G that the inequality*

$$|f(x)-h(x)|<\varphi(x)$$

holds for $x\epsilon G$.

This corollary is at the same time a special case of a result due to H. Whitney[6]).

2. We turn to the negative results. In this section we consider operations which are defined in the space $C\langle 0,1\rangle$ and have their values in a weakly complete Banach space.

[6]) See [5], Lemma 6.

By E_1^+ we denote the halfline of non negative numbers.

Let the operation $F(x)$ be defined in an open subset G of a real Banach space B.

Let k be a fixed positive integer. By $P(x,h)$ we denote an operation depending on two variables $x\epsilon G$, $h\epsilon B$ provided that for any fixed $x\epsilon G$, $P(x,h)$ is a polynomial of degree at most k in the variable h and that $P(x,\Theta)=\Theta$. We introduce

Definition 1. *The operation $F(x)$ is k times regularly differentiable, if there is an operation $P(x,h)$ fulfilling the inequality*

$$\|F(x+h)-F(x)-P(x,h)\|\leqslant\alpha(x,\|h\|)\cdot\|h\|^k$$

whenever both sides of the inequality are defined.

We suppose that $\alpha(x,\eta)$ is a non negative functional which is defined in an open subset of the space $C\langle 0,1\rangle\times E_1^+$ and fulfils the following condition: for any given $x_0\epsilon G$ and $\varepsilon>0$ there is such a $\delta>0$ that the inequalities $\|x-x_0\|<\delta$ and $0\leqslant\eta<\delta$ imply that $\alpha(x,\eta)$ is defined and fulfils the inequality $0\leqslant\alpha(x,\eta)<\varepsilon$.

Note. If $k=1$, then the operation $P(x,h)$ is usually called the *differential of $F(x)$* and denoted by $\delta F(x,h)$.

A regularly differentiable operation apparently possesses a Fréchet differential, and it is easy to prove (by means of the extension to the complex case and of the Cauchy formulae) that an analytic operation is k times ($k=1,2,3,\ldots$) regularly differentiable.

Now we state

Theorem 3. *Let the operation $F(x)$ be defined in the sphere $x\epsilon C\langle 0,1\rangle$, $\|x\|<R$ ($R>0$) and regularly differentiable (once) there. Let the values of the operation $F(x)$ belong to a weakly complete space.*

If ε and r are two given positive numbers, $r+\varepsilon\leqslant R$, then there is such a point $x\epsilon C\langle 0,1\rangle$ that the inequalities

$$r\leqslant\|x\|<r+\varepsilon,\qquad \|F(x)-F(\Theta)\|<\varepsilon$$

are satisfied.

From this theorem we deduce that the functional $\|x\|$, $x\epsilon C\langle 0,1\rangle$, is not the uniform limit of a sequence of regularly differentiable functionals in the sphere $\|x\|<1$. Let us suppose that there is such a regularly differentiable functional $f(x)$ that the inequality

$$|f(x)-\|x\||<\frac{1}{4}\tag{7}$$

holds for $x \in C\langle 0,1\rangle$, $\|x\|<1$. In theorem 3 let $r=3/4$, $\varepsilon=1/4$. The theorem states that there is such a point x, that

$$\frac{3}{4}\leqslant\|x\|<1, \qquad |f(x)-f(\Theta)|<\frac{1}{4}.$$

But inequality (7) implies that

$$|f(\Theta)|<\frac{1}{4}, \qquad \big|f(x)-\|x\|\big|<\frac{1}{4}$$

and we arrive at a contradiction.

Proof of theorem 3. If $x \in C\langle 0,1\rangle$, $\|x\|<R$, let $V(x)$ be the set of positive numbers η having the following property: there is an open subset $H(x,\eta)$ of the space $C\langle 0,1\rangle\times E_1^+$, containing all the points (x,ξ), $0\leqslant\xi<\eta$, and such that the inclusion $(x',\xi')\in H(x,\eta)$ implies that $a(x',\xi')$ is defined and that $a(x',\xi')<\varepsilon/2$.

We define

$$\beta(x)=\sup_{\eta\in V(x)}\eta, \qquad \gamma(x)=\min\big(\beta(x),\varepsilon\big)$$

and prove the following

Lemma 1. *The functional $\gamma(x)$ is positive and lower semicontinuous for $x\in C\langle 0,1\rangle$, $\|x\|<R$.*

We prove only that the functional $\beta(x)$ is lower semicontinuous.

Let us fix a point $x_0\in C\langle 0,1\rangle$ and a number $\xi, 0<\xi<\beta(x_0)$. We choose a number ξ', $\xi<\xi'<\beta(x_0)$. The set $H(x_0,\xi')$ is open in the space $C\langle 0,1\rangle\times E_1^+$ and contains all the points (x_0,ζ), $0\leqslant\zeta<\xi'$. As the interval $\langle 0,\xi\rangle$ is a compact set, there is such an open subset U of the space $C\langle 0,1\rangle$ that $x_0\in U$, that all the points (x,ζ), $x\in U$, $0\leqslant\zeta\leqslant\xi$, are contained in the set $H(x,\xi')$ and that $\|x\|<R$ if $x\in U$.

It follows that $\beta(x)\geqslant\xi$ for $x\in U$ and that the functional $\beta(x)$ is lower semicontinuous in the point x_0.

Now we shall prove another lemma. Let us suppose that $Q=Q(x)$ is a homogeneous polynomial of degree one, defined in the space $C\langle 0,1\rangle$ and having its values in a weakly complete Banach space.

Let us fix an $\varepsilon'>0$. By $T(\varepsilon',Q)$ we denote the set of numbers t from the interval $\langle 0,1\rangle$ which have the following property:

if U is an open interval containing the point t, then there is a function $x=x(\tau)\in C\langle 0,1\rangle$ fulfilling the conditions

$$x(\tau)=0 \quad\text{if}\quad \tau\bar{\in} U, \qquad \|x\|<1, \qquad \|Q(x)\|>\varepsilon'.$$

Lemma 6. *The set $T(\varepsilon',Q)$ is finite.*

Proof. Let us suppose that the set $T(\varepsilon',Q)$ is infinite. Then there is a sequence of numbers $t_n\in T(\varepsilon',Q)$, open intervals $U_n\subset\langle 0,1\rangle$, and functions $x_n=x_n(t)\in C\langle 0,1\rangle$, $n=1,2,3,\ldots$, such that the following conditions are fulfilled:

$$t_n\in U_n,$$

$$U_i\cap U_j=0 \quad\text{for}\quad i\neq j,$$

$$\|Q(x_n)\|>\varepsilon',$$

$$x_n(\tau)=0 \quad\text{if}\quad \tau\bar{\in} U_n,$$

$$\|x_n\|\leqslant 1.$$

We apply the following theorem due to W. Orlicz[7]):

If $\bar{x}_1,\bar{x}_2,\bar{x}_3,\ldots$ are elements of a weakly complete Banach space and if there is such a number K that

$$\|\bar{x}_{i_1}+\bar{x}_{i_2}+\ldots+\bar{x}_{i_k}\|\leqslant K$$

for every finite sequence of integers $1\leqslant i_1<i_2<\ldots<i_k$, $k=1,2,3,\ldots$, then the series $\bar{x}_1+\bar{x}_2+\bar{x}_3+\ldots$ converges (conditionlessly).

Let $\bar{x}_n=Q(x_n)$ $(K=\|Q\|)$. According to the theorem of W. Orlicz the series $\sum_{j=1}^{\infty}\bar{x}_j$ converges and we get a contradiction with the assumption that $\|\bar{x}_j\|=\|Q(x_j)\|>\varepsilon'$, $j=1,2,3,\ldots$

We are in a position to prove theorem 3.

We define a sequence $x_n\in C\langle 0,1\rangle$, $x_1=\Theta$. If we have defined a point x_n fulfilling the conditions

$$\|F(x_n)\|\leqslant\varepsilon\|x_n\|, \qquad \|x_n\|<r,$$

then we choose the point x_{n+1} in such a way that

$$\gamma(x_n)>\|x_{n+1}-x_n\|, \tag{8}$$

$$\frac{1}{2}\gamma(x_n)\leqslant\|x_{n+1}\|-\|x_n\|, \tag{9}$$

$$\|F(x_{n+1})\|\leqslant\varepsilon\|x_{n+1}\|. \tag{10}$$

We shall prove that this sequence is necessarily finite.

The point x_{n+1} will be found in the following way (we suppose that $\varepsilon<1$):

[7]) See [4], p. 247, Theorem 3.

We choose such a number $t' \epsilon \langle 0,1\rangle$ that

$$t' \bar{\epsilon} T\left(\frac{1}{8}\varepsilon\gamma(x_n), P(x_n,x)\right)$$

and that

$$|x_n(t')| > \|x_n\| - \frac{1}{4}\gamma(x_n)$$

($P(x_n,x)$ is the differential of the operation $F(x)$, see definition 1).

It follows that there is an open interval U, containing the number t' and such that

$$\|P(x_n,y)\| \leqslant \frac{1}{8}\varepsilon\gamma(x_n),$$

provided that the function $y(t)$ fulfils the conditions

$$|y(t)| \leqslant 1 \quad \text{if} \quad t\epsilon\langle 0,1\rangle, \qquad y(t)=0 \quad \text{if} \quad t \bar{\epsilon} U.$$

Now we fix a function $y(t) \epsilon C\langle 0,1\rangle$ and suppose the following conditions to be satisfied:

$$|y(t)| \leqslant \frac{3}{4}\gamma(x_n) \quad \text{if} \quad t\epsilon\langle 0,1\rangle,$$

$$y(t') = \frac{3}{4}\gamma(x_n)\cdot \operatorname{sgn} x_n(t'),$$

$$y(t)=0 \quad \text{if} \quad t\bar{\epsilon} U.$$

Let $x_{n+1}=x_n+y$. The relations (8), (9) are evidently fulfilled as

$$\|x_{n+1}\| \geqslant |x_{n+1}(t')| = |x_n(t')| + \frac{3}{4}\gamma(x_n) \geqslant \|x_n\| + \frac{1}{2}\gamma(x_n),$$

and from definition 1 we get

$$\|F(x_{n+1})\| \leqslant \|F(x_n)\| + \|P(x_n,y)\| + \alpha(x_n,\|y\|)\|y\|$$

$$\leqslant \varepsilon\|x_n\| + \frac{1}{8}\varepsilon\gamma(x_n) + \frac{1}{2}\varepsilon\frac{3}{4}\gamma(x_n) = \varepsilon\left(\|x_n\| + \frac{1}{2}\gamma(x_n)\right) \leqslant \varepsilon\|x_{n+1}\|.$$

There is necessarily an x_n satisfying $r \leqslant \|x_n\| < r+\varepsilon$. Otherwise there would be an infinite sequence of points x_n fulfilling the relations (8), (9), (10) and $\|x_n\|<r$, $n=1,2,3,\ldots$ From inequalities (8), (9) it follows that the series $\sum_{n=1}^{\infty}\gamma(x_n)$ converges and that x_n is a Cauchy sequence, $x_n \to x$, $\|x\| \leqslant r < R$.

The relations $\gamma(x_n)\to 0$, $\gamma(x)>0$ contradict the fact that the functional $\gamma(x)$ is lower semicontinuous and this contradiction completes the proof of theorem 3.

Note. Theorem 3 remains true if we replace the space $C\langle 0,1\rangle$ by the space of all bounded continuous functions which are defined in a completely regular topological space without isolated points.

3. We shall apply the same method to the case of the spaces $l^{(p)}$ or $L^{(p)}$, $p\geqslant 1$, $p\neq 2,4,6,\ldots$ Therefore the steps that are analogous to those taken in the case of the space $C\langle 0,1\rangle$ will be discussed only briefly.

We suppose that p is a fixed number, $p\geqslant 1$, $p\neq 2,4,6,\ldots$, that $\overline{p}$ is the least integer greater than or equal to p, and that B is the space $l^{(p)}$ or $L^{(p)}$. If $y\epsilon B$, $R>0$, then $S(y,R)$ denotes the sphere in the space B which has the centre at the point y and the radius R. The main result is contained in the following

THEOREM 4. *Let R,r,ε be three positive numbers, $R\geqslant r+\varepsilon$. Let the functional $f(x)$ be defined in $S(\Theta,R)$ and $\overline{p}$ times regularly differentiable there.*

Then there is an $x\epsilon B$ fulfilling the inequalities

$$r \leqslant \|x\| < r+\varepsilon, \qquad |f(x)-f(\Theta)| \leqslant \varepsilon\|x\|.$$

As in the previous case, we can easily deduce from this theorem that the functional $\|x\|$ is not the uniform limit of a sequence of $\overline{p}$ times regularly differentiable functionals in the unit sphere of the space $l^{(p)}$ or $L^{(p)}$.

We shall prove theorem 4 in two steps. First we prove the following theorem 5, then we use theorem 5 to prove theorem 4.

THEOREM 5. *Let R,r,ε be three positive numbers, $R\geqslant r+\varepsilon$ and let the functional $f(x)$ be defined in the sphere $S(\Theta,R)\subset l^{(p)}$, $p\geqslant 1$, $p\neq 2,4,6,\ldots$, and have the following property:*

There is such a functional $w(x,h)$, which is defined for $x\epsilon S(\Theta,R)\subset l^{(p)}$, $h\epsilon l^{(p)}$, and which is a polynomial of degree at most p in the variable h if x is fixed, that the inequality

$$|f(x+h)-f(x)-w(x,h)| < \alpha(x,\|h\|)\|h\|^p$$

is satisfied whenever both sides are defined. We suppose again that the functional $\alpha(x,\eta)$ is defined in an open subset of the space $S(\Theta,R)\times E_1^+$ and has the following property: for any given $x_0\epsilon S(\Theta,R)$ and $\varepsilon'>0$ there is such a $\delta>0$ that the inequalities $\|x-x_0\|<\delta$ and $0\leqslant\eta<\delta$ imply that $\alpha(x,\eta)$ is defined and fulfils the inequality $0\leqslant\alpha(x,\eta)<\varepsilon'$.

Then there is a point $x\epsilon l^{(p)}$ fulfilling the relations

$$r \leqslant \|x\| < r+\varepsilon \qquad |f(x)-f(\Theta)| \leqslant \varepsilon\|x\|.$$

In order to prove theorem 5 we shall need the following

LEMMA 3. *Let ε and Q be two positive numbers and let p be the greatest integer less than or equal to p. Suppose that $q(x)$ is a polynomial in $l^{(p)}$ (with numerical values) of degree at most $\underline{p}$.*

Then there is a point $x \in l^{(p)}$ fulfilling the conditions:

(11) $$\|x\| = Q,$$

(12) $$|q(x) - q(\Theta)| < \varepsilon,$$

(13) *x has only a finite number of coordinates different from zero.*

This lemma is an easy consequence of the following

LEMMA 4. *Let $V(x)$ be a real valued homogeneous polynomial in $l^{(p}$ of degree $v < p$. If $y = \{y_1, y_2, \ldots, y_n, 0, 0, \ldots\}$, then*

$$V(y) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_v \leqslant n} a_{i_1, i_2, \ldots, i_v} y_{i_1} y_{i_2} \cdots y_{i_v}$$

(the numbers $a_{i_1, i_2, \ldots, i_v}$ do not depend on n and are defined uniquely).

The following relations hold:

(14.1) $$a_{i_1, i_2, \ldots, i_{v-1}, j} \to 0 \quad \text{as} \quad j \to \infty,$$

(14.2) $$a_{i_1, i_2, \ldots, i_{v-2}, j, j} \to 0 \quad \text{as} \quad j \to \infty,$$

. . . .

(14.$v-1$) $$a_{i_1, j, \ldots, j} \to 0 \quad \text{as} \quad j \to \infty,$$

(14.v) $$a_{j, j, \ldots, j} \to 0 \quad \text{as} \quad j \to \infty.$$

This lemma is proved by complete induction. Lemma 4 evidently holds if $v = 1$. Let us suppose that lemma 4 holds for polynomials of degree $v-1$, $v < p$. Let

$$h_1 = \{1, 0, 0, 0, \ldots\},$$

$$h_2 = \{0, 1, 0, 0, \ldots\},$$

$$h_3 = \{0, 0, 1, 0, \ldots\},$$

. . . .

The differentials $\delta V(x, h_i)$ (see the note after definition 1) are homogeneous polynomials[8]) of degree $v-1$ in the variable x. Since we have

$$\delta V(y, h_i) = \frac{\partial}{\partial y_i} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_v \leqslant n} a_{i_1, \ldots, i_v} y_{i_1 \ldots i_v} \qquad (1 \leqslant i \leqslant n)$$

[8]) See [2], p. 74, Theorem 4, 2, 9 or [3], II.

the relations (14.1), (14.2), ..., (14.$v-1$) hold. If the relation (14.v) were false, we could find a sequence of indices $j_1, j_2, j_3, \ldots$ having the following property:

If we define the vectors $e_1, e_2, e_3, \ldots$,

$$e_k = \{a_{k,1}, a_{k,2}, a_{k,3}, \ldots\},$$

where

$$a_{k,j_1} = a_{k,j_2} = a_{k,j_3} = \ldots = a_{k,j_k} = \frac{1}{\sqrt[p]{k}},$$

$$a_{k,j} = 0 \quad \text{if} \quad j \neq j_1, j_2, \ldots, j_k,$$

then

$$V(e_k) \to \infty \quad \text{with} \quad k \to \infty.$$

This contradiction completes the proof of lemma 4. Lemma 3 now follows trivially ((22.v)) if p is not an integer. If p is an integer then p is odd and we can write

$$q(x) = q_1(x) + q_2(x),$$

where $q_1(x)$ is a polynomial of degree at most $p-1$ and $q_2(x)$ is a homogeneous polynomial of degree p. Now we again easily prove lemma 3 using the fact that $q_2(x)$ is a continuous odd functional.

This being established we turn to the proof of theorem 5. (We suppose again that $\varepsilon < 1$).

If $x \in l^{(p)}$, $\|x\| < R$, we denote by $V(x)$ the set of positive numbers η having the following property:

There is an open subset $H(x, \eta)$ of the space $l^{(p)} \times E_1^+$, containing all the points (x, ξ), $0 \leqslant \xi < \eta$, and such that the inclusion $(x', \xi') \in H(x, \eta)$ implies that $\alpha(x', \xi')$ is defined and that

$$\alpha(x', \xi') < \frac{\varepsilon}{2p \max(1, \|x\|^{p-1})}.$$

We write

$$\beta(x) = \sup_{\eta \in V(x)} \eta, \qquad \gamma(x) = \min(S(x), \varepsilon).$$

In the same manner as in the preceeding section we prove that the functional $\gamma(x)$ is positive and lower semicontinuous. We again define a sequence $x_n \in l^{(p)}$. Let $x_1 = \Theta$. Having defined an x_n fulfilling the conditions

(15) $$|f(x_n)| \leqslant \varepsilon \|x_n\|, \qquad \|x_n\| < r,$$

(16) $$x_n = \{\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{i_n}^{(n)}, 0, 0, \ldots\},$$

we choose x_{n+1} in such a way that

(17) $$x_{n+1}=\{\xi_1^{(n)},\xi_2^{(n)},\ldots,\xi_{i_n}^{(n)},\xi_{i_n+1}^{(n+1)},\ldots,\xi_{i_{n+1}}^{(n+1)},0,0,\ldots\},$$

(18) $$|f(x_{n+1})|\leqslant\varepsilon\|x_{n+1}\|,$$

(19) $$\|x_{n+1}\|^p=\|x_n\|^p+\left(\frac{1}{2}\gamma(x_n)\right)^p.$$

The point x_{n+1} will be found as follows: we write $x_{n+1}=x_n+h_n$ and have

$$f(x_{n+1})=f(x_n)+w(x_n,h_n)+r(x_n,h_n),$$

$$|r(x_n,h)|<\alpha(x_n\|h\|)\|h\|^p \quad \text{if} \quad \|h\|<\gamma(x_n).$$

By lemma 3 there is a vector h_n fulfilling the conditions

$$h_n=\{0,0,\ldots,0,\xi_{i_n+1}^{(n+1)},\ldots,\xi_{i_{n+1}}^{(n+1)},0,\ldots\},\qquad \|h_n\|=\frac{1}{2}\gamma(x_n),$$

$$|w(x_n,h_n)|<\frac{\varepsilon}{2p\max\{1,\|x_n\|^{p-1}\}}\left(\frac{1}{2}\gamma(x_n)\right)^p.$$

As we have

$$\alpha(x_n,\|h_n\|)<\frac{\varepsilon}{2p\max\{1,\|x_n\|^{p-1}\}},$$

we get

(20) $$|f(x_{n+1})-f(x_n)|<\frac{\varepsilon}{p\max\{1,\|x_n\|^{p-1}\}}\left(\frac{1}{2}\gamma(x_n)\right)^p.$$

The relation (17) gives

(21) $$\|x_{n+1}\|\geqslant\|x_n\|+\frac{1}{p\max\{1,\|x_n\|^{p-1}\}}\left(\frac{1}{2}\gamma(x_n)\right)^p,$$

and from inequalities (15), (20), (21) we get

$$|f(x_{n+1})|\leqslant\varepsilon\|x_{n+1}\|.$$

The point x_{n+1} satisfies all the relations (17), (18), (19).

There is necessarily an x_n satisfying

$$r\leqslant\|x_n\|<r+\varepsilon.$$

Otherwise there would be an infinite sequence of points x_n fulfilling the relations (17), (18), (19) and $\|x_n\|<r$. Relation (19) implies that the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\gamma(x_n)\right)^p$$

converges and from relations (17), (19) it follows that x_n is a Cauchy sequence, $x_n\to x$, $\|x\|\leqslant r<R$. Consequently we have $\gamma(x_n)\to 0$, $\gamma(x)>0$, and the contradiction resulting from the lower semicontinuity of the function $\gamma(x)$ completes the proof of theorem 5.

One verifies easily that every polynomial satisfies the conditions of theorem 5, and consequently we have

LEMMA 5. *Let ε,r_1,r_2 be three positive numbers, $r_1<r_2$. Suppose that $q(x)$ is a polynomial in $l^{(p)}$ with numerical values, its degree being arbitrary ($p\geqslant 1$, $p\neq 2,4,6,\ldots$).*

Then there is a point $x\in l^{(p)}$ satisfying the conditions

(22) $$r_1<\|x\|<r_2,$$

(23) $$|q(x)-q(\Theta)|<\varepsilon,$$

(24) *the point x has only a finite number of coordinates different from zero.*

Proof. It follows from theorem 5 that there is a point $y\in l^{(p)}$ satisfying the conditions (22), (23). As the polynomial $q(x)$ is a continuous functional, there is a point x satisfying all the conditions (22), (23), (24).

Now we shall prove theorem 4 in the case where $B=l^{(p)}$. We repeat line after line the proof of theorem 5 with a slight modification due to the fact that we have to use lemma 5 instead of lemma 3. As the space $L^{(p)}$ contains a subspace isometric to the space $l^{(p)}$, theorem 4 holds in the case where $B=L^{(p)}$ too.

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