de composition et 2° de ce que le produit de composition de deux fonctions monotones est encore une fonction monotone. On peut donner à (iv) un énoncé modifié, mais équivalent, moyennant le passage aux dérivées, à savoir: En posant

$$\varphi(\lambda)=\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,dt,$$

il vient, pour toute fonction $f \in L(-\infty,\infty)$,

$$\lim_{n} \sqrt[n]{W_n^*} = \max_{\lambda} |\varphi(\lambda)|,$$

où

$$W_n^* = \int_{-\infty}^{\infty} |\Phi_n(t)|\,dt, \qquad \Phi_1(t)=f(t),$$

$$\Phi_2(t)=\int_{-\infty}^{\infty} f(t-s) f(s)\,ds, \qquad \Phi_3(t)=\int_{-\infty}^{\infty} \Phi_2(t-s) f(s)\,ds \qquad \text{etc.}$$

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Fourier transforms on perfect sets

by

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It is a classical fact that not every continuous periodic function has an absolutely convergent Fourier series. The purpose of this note is to establish a stronger theorem of the same kind. For convenience we shall consider Fourier *integrals* instead of Fourier *series*. Let P be a bounded perfect set on the line, and $f(x)$ a summable function. The Fourier transform of $f(x)$

$$\varphi(x)=\int_{-\infty}^{\infty} f(y) e^{ixy}\,dy$$

is a continuous function, and its restriction to the set P considered as a topological space is continuous on P. Our result states for perfect sets P of a certain type that not every continuous function on P is thus obtained as the restriction of a Fourier transform.

The theory of the Fourier transform has been extended to arbitary locally compact abelian groups [5]. The result above on Fourier series, which can be considered as a theorem about the compact circle group, was stated and proved for arbitrary compact abelian groups by Segal [4]. Interesting generalizations of Segal's work by E. Hewitt and by R. E. Edwards are to appear in the near future. In a different direction, but still on a general type of group, H. Reiter discusses in a forthcoming paper the restrictions of Fourier transforms to a set P composed of denumerably many linearly independent group elements.

The theorem to be proved could be stated in a more general context; since, however, it is new for the line and its main interest is for that case, we shall not indulge in greater generality.

THEOREM. *Let P be a bounded perfect set on the line, such that for every function φ defined and continuous on P, there exists a summable function f with*

$$\varphi(x)=\int_{-\infty}^{\infty} f(y) e^{ixy}\,dy$$

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for all x in P. For any function of bounded variation μ consider the Fourier-Stieltjes transform

$$\hat{\mu}(y)=\int_{-\infty}^{\infty} e^{-ixy}\,d\mu(x).$$

If μ is constant on the intervals contiguous to P, or in other words the variation of μ is concentrated on P, then $\hat{\mu}$ does not tend to zero as $|y|$ tends to infinity.

First we interpret the theorem in case P is an interval. Choose a summable function vanishing outside P and let μ be its indefinite integral. By the Riemann-Lebesgue theorem the Fourier-Stieltjes transform of $d\mu$ tends to zero, and so the conclusion of the theorem fails for P. We conclude that some continuous function on P does not coincide there with a Fourier-Stieltjes transform.

This case was easy, and the theorem is sharper if P is thin in some sense. Now the usual way of proving that a perfect set is a set of multiplicity for Fourier series or integrals is to construct a function μ of bounded variation, with variation concentrated on the set, whose transform tends to zero [2,3]. We can assert that not every continuous function on such a set coincides there with a Fourier transform, and the result seems to be new for all these sets.

In the proof we shall refer to the following Banach spaces:

$\boldsymbol{L}^1$ is the algebra of summable functions on the line;

$\boldsymbol{L}^\infty$ is the space of bounded measurable functions on the line, and is the dual of $\boldsymbol{L}^1$;

$\boldsymbol{C}(\boldsymbol{P})$ is the space of continuous functions on $\boldsymbol{P}$;

$\boldsymbol{M}(\boldsymbol{P})$ is the space of bounded complex Borel measures vanishing outside $\boldsymbol{P}$, and is the dual of $\boldsymbol{C}(\boldsymbol{P})$;

$\boldsymbol{C}_\infty$ is the space of continuous functions on the line tending to zero.

We take the usual definitions of the norms in these spaces. Since $\boldsymbol{M}(\boldsymbol{P})$ is less well-known than the other cases, we recall that the norm is given as

$$\|\mu\|=\int|d\mu(x)|,$$

which represents the total variation of the measure. We shall identify an element of $\boldsymbol{M}(\boldsymbol{P})$ with the corresponding function of bounded variation, constant on intervals contiguous to $\boldsymbol{P}$.

LEMMA. *Suppose the hypothesis of the theorem holds for a perfect set $\boldsymbol{P}$. Then the set of functions which are the Fourier-Stieltjes transforms of measures in $\boldsymbol{M}(\boldsymbol{P})$ is closed as a subspace of $\boldsymbol{L}^\infty$.*

Proof. Let $\boldsymbol{A}$ be the closed subspace of $\boldsymbol{L}^1$ consisting of those functions f for which

$$\varphi(x)=\int_{-\infty}^{\infty} f(y)\,e^{ixy}\,dy$$

vanishes for all x in $\boldsymbol{P}$. Then the Fourier transform carries the quotient space $\boldsymbol{L}^1/\boldsymbol{A}$ one-one *onto* $\boldsymbol{C}(\boldsymbol{P})$, and the transformation is continuous. By the inversion theorem of Banach, the inverse operator is also continuous.

Let us calculate the linear functionals on $\boldsymbol{L}^1/\boldsymbol{A}$ in two ways. First they are exactly those functionals on $\boldsymbol{L}^1$ which vanish on $\boldsymbol{A}$; these are functions α in $\boldsymbol{L}^\infty$ with

$$\alpha[f]=\int_{-\infty}^{\infty} f(y)\,\alpha(-y)\,dy \quad \text{for every } f \text{ in } \boldsymbol{L}^1.$$

On the other hand $\boldsymbol{L}^1/\boldsymbol{A}$ is isomorphic to $\boldsymbol{C}(\boldsymbol{P})$ and so has the same functionals. Thus we can also write

$$\alpha[f]=\int_P \varphi(x)\,d\mu(x)$$

for a uniquely determined μ in $\boldsymbol{M}(\boldsymbol{P})$. Insert the definition of φ in this formula:

$$\alpha[f]=\int_P\int_{-\infty}^{\infty} f(y)\,e^{ixy}\,dy\,d\mu(x)=\int_{-\infty}^{\infty} f(y)\int_P e^{ixy}\,d\mu(x)\,dy=\int_{-\infty}^{\infty} f(y)\,\hat{\mu}(-y)\,dy.$$

Thus for every f in $\boldsymbol{L}^1$

$$\int_{-\infty}^{\infty} f(y)\left[\alpha(-y)-\hat{\mu}(-y)\right]dy=0,$$

from which we conclude almost everywhere that

$$\alpha(y)=\hat{\mu}(y).$$

That is to say, the transforms of the measures in $\boldsymbol{M}(\boldsymbol{P})$ coincide with the functions in $\boldsymbol{L}^\infty$ orthogonal to $\boldsymbol{A}$. Since this is a closed (and even weakly closed) subspace, the lemma is proved.

Assuming the theorem is false, let $\boldsymbol{M}_0(\boldsymbol{P})$ be the linear set of measures μ in $\boldsymbol{M}(\boldsymbol{P})$ for which

$$\hat{\mu}(y)\to 0 \quad \text{for} \quad |y|\to\infty.$$

By the lemma, a uniform limit of such functions $\hat{\mu}_n$ is again the transform of a measure in $\boldsymbol{M}(\boldsymbol{P})$, which must moreover tend to zero. Consequently the transforms of measures in $\boldsymbol{M}_0(\boldsymbol{P})$ form a closed subspace of $\boldsymbol{L}^\infty$, which we shall denote by $\boldsymbol{S}$. Observe at the same time that $\boldsymbol{M}_0(\boldsymbol{P})$ is complete in its own norm; for if a sequence of measures in $\boldsymbol{M}_0(\boldsymbol{P})$ converges, the corresponding transforms converge uniformly to a limit function which is still in $\boldsymbol{S}$, so that the limit of the sequence of measures belongs to $\boldsymbol{M}_0(\boldsymbol{P})$. By the inversion theorem $\boldsymbol{M}_0(\boldsymbol{P})$ and $\boldsymbol{S}$ are equivalent Banach spaces.

Now S is a closed subspace of C_∞ as well as of L^∞. A linear functional λ on S can be extended to all of C_∞ and so has the form

$$\lambda[g]=\int_{-\infty}^{\infty} g(-y)\,d\lambda(y),$$

where λ is a bounded complex-valued Borel measure on the line. If g is the transform of the measure μ in $M_0(P)$, this can be written

$$\lambda[g]=\int_{-\infty}^{\infty}\int_P e^{ixy}\,d\mu(x)\,d\lambda(y)=\int_P\int_{-\infty}^{\infty} e^{ixy}\,d\lambda(y)\,d\mu(x)=\int_P \hat\lambda(x)\,d\mu(x).$$

This shows that the most general linear functional in $M_0(P)$ is given by integration with a continuous function. We shall derive a contradiction by finding a functional which cannot be expressed in this form.

Let us fix a non-trivial measure μ in $M_0(P)$. Since $\hat\mu$ tends to zero, μ is not a purely discrete measure. We assert that for some point x in P, the open intervals of the form $(x-\delta, x)$ and $(x, x+\delta)$ for $\delta>0$ all contain part of the mass of μ. Indeed, form the maximal open set on the line in which μ vanishes identically. Its complement in non-denumerable, since the mass of μ is not concentrated on any countable set. Because it is open it is the union of countably many open intervals. Choose a point in P which is not the end-point of any such interval, nor contained in any of them; evidently every one-sided neighbourhood contains mass, as we had to show.

Define the function

$$\alpha(z)=\begin{cases}0 & \text{for } z\leqslant x,\\ 1 & \text{for } z>x,\end{cases}$$

and consider the functional in $M_0(P)$ given by integration with α:

$$\alpha[\mu]=\int_P \alpha(x)\,d\mu(x).$$

If there is a continuous function γ defining the same functional, then

$$\int_P [\alpha(x)-\gamma(x)]\,d\mu(x)=0 \qquad \text{for every } f \text{ in } A.$$

But for any fixed real number y, the measure defined by

$$d\nu(x)=e^{-ixy}\,d\mu(x)$$

belongs to $M_0(P)$ (since it is concentrated on P and its transform tends to zero). Hence for all y

$$\int_P e^{-ixy}[\alpha(x)-\gamma(x)]\,d\mu(x)=0.$$

By the unicity theorem for Fourier-Stieltjes transforms, the two measures $\alpha(x)\,d\mu(x)$, $\gamma(x)\,d\mu(x)$ are the same.

Now μ has a mass in every neighbourhood $(x-\delta, x)$, whereas $\alpha\,d\mu$ vanishes to the left of x. Consequently γ assumes zero values at points arbitrarily close to x on the left. By the same kind of reasoning, γ assumes the value one at points as close as we please to x on the right. This is impossible, as γ is supposed to be continuous. So the functional in $M_0(P)$ given by integration with α is not the same as that defined by any continuous function, and this completes the proof of the theorem.

It may be of interest to remark that the lemma used in the proof can be completed to the following assertion:

For a bounded perfect set P, three conditions are equivalent:

a) The transforms of functions in L^1, when restricted to P, cover $C(P)$;

b) The transforms of measures in $M(P)$ form a closed subspace of L^∞;

c) The transforms of measures in $M(P)$ form a *weakly closed* subspace of L^∞ considered as the dual of L^1.

The remaining parts of the proof can be carried through using theorems about Banach spaces. A related result is proved in a paper of Carleson [1] by different methods.

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