A theorem on the structure of linear operations
by
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In a previous paper [1] I proved a general theorem concerning linear operations depending on a parameter\(^1\). This theorem contains as particular cases some theorems of Saks [7], [8] concerning the structure of the sequences of operations with values in the space of measurable functions, i.e., those which deal with the behaviour of the sequences at individual points. Some new theorems of Saks's type were also obtained in [1] as applications. However, the theorems of Saks, dealing with the behaviour of the sequences in the mean, were not obtainable from the results of [1].

It is the purpose of this paper to generalize the results of [1] so as to fill the above-mentioned gap.

I am very obliged to Mr. R. Sikorski who has called my attention to an error in the first draft of this paper.

1. Preliminary definitions

\(T\) will denote an abstract set in which a \(\sigma\)-algebra \(\mathcal{G}\) of subsets (Halmos [3], p. 28) is defined. We suppose that \(\mu\) is a \(\sigma\)-measure in \(\mathcal{G}\), such that \(\mu(T) < \infty\). Under these circumstances the measure space \((T, \mathcal{G}, \mu)\) is defined, namely on introducing the distance of two sets \(e_1, e_2 \in \mathcal{G}\) by the formula

\[\delta(e_1, e_2) = \mu(e_1 + e_2) - e_1 e_2\]

we get a pseudometric space. Identifying two sets \(e_1, e_2 \in \mathcal{G}\) if \(\delta(e_1, e_2) = 0\) we get a metric space which is also denoted by \((T, \mathcal{G}, \mu)\); this space is complete. We shall suppose in the sequel that the space \((T, \mathcal{G}, \mu)\) is separable (it is usual to call the measure separable in this case). By \(e, h,\)

\(^1\) This paper has many points in common with the paper [6] of Orlicz which also aims at deducing some theorems of Saks' type from a general theorem. The methods of Orlicz are different from ours and are suitable for systems of operations depending only on a discrete measurable parameter. Professor Orlicz has been the first to introduce the operation \(U(e, h)\) (see below).

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The main result of this paper (Theorem 1) states that in the case of the operation $U$ being linear the set $T$ may be decomposed, $T = \rho + h$, in such a manner that for every $x$ the element $U(x|x)$ is "nearby contained" in $B$ and that no set $k = h$ of positive measure has this property, unless $x$ belongs to a set of the first category.

2. Properties of the operation $U(x)$

Lemma 1. The operation $U(x|x)$ is continuous in the space $X \times (T, \mathcal{G}, \mu)$.

Proof. The operation $U(x|x)$ is continuous for fixed $x$. This follows from (a1). Let $x_n \to x$, $e_n \to e$ and write $V_n(x|x_n) = U(x|x_n)$. Then $V_n(x)$ is a sequence of linear operations from $X$ to $Y$, convergent everywhere, whence by a theorem of Mazur and Orlicz ([5], p. 153-154)

$$\|V_n(x_n) - V(x|x_0)\| \to 0,$$

which implies

$$U(x_n|x_n) \to U(x|x_0).$$

The following condition $(B, h, e)$ will be needed:

There is a set $\varepsilon$ such that $\mu(h - e) < \varepsilon$ and $U(x|x)eB$.

The set of the elements $x$ for which this condition is satisfied will be written $P(B, h, e)$.

Lemma 2. For every $h$ and $e > 0$ the set $P(B, h, e)$ is analytic.

Proof. The set $\mathcal{H}$ of the elements $e(x(T, \mathcal{G}, \mu))$ for which $\mu(x|e) < e$, is obviously open. The set $Q$ of the couples $(x, e)$ such that $U(x|x)eB$ is analytic, for it is the inverse image under $U$ of the analytic set $B$. Using the symbolical notation ([Kuratowski] [4], p. 123) we can write

$$P(B, h, e) = \left( \bigcup_{x} (x|e) \in Q(x \times \mathcal{B}) \right),$$

that is, $P(B, h, e)$ is the projection on $X$ of the analytic set $Q(x \times \mathcal{B})$; hence it is analytic too.

Lemma 3. Let the set $B$ be linear. If the set $P(B, h, e)$ is of the second category, then the set $P(B, h, 2e)$ is residual.

Proof. Since $P(B, h, e)$ is an analytic set, it fulfils the condition of Baire; hence it contains a sphere except a set of the first category. Hence the set $W$ of the differences of the elements of $P(B, h, e)$ contains a set $B = K - N$ where $K$ is a sphere with centre 0 and $N$ is of the first category.

We notice now that $x_n \to x$ in $P(B, h, e)$ implies $x_n \to x$ in $P(B, h, 2e)$, for there are sets $e_1, e_2$ such that $\mu(h - e) < e_1, \mu(h - e) < e_2, U(x|x_1)eB, U(x|x_2)eB$; hence for the set $e = e_1 e_2$ we have $\mu(h - e) < 2e$ and $U(x|x)eB$. 

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2) By an analytic set we mean any set which is the result of the operation $\mathcal{A}$ ([Kuratowski] [4], p. 4) performed upon open sets.
$U(\sigma_1\sigma) = U(\sigma_1) U(\sigma) \in B$. It follows now that $K - NP(B, h, 2e)$ implies $2e \in P(B, h, 2e)$, whence

$$\sum_{n=1}^{\infty} n(K - NP(B, h, 2e)) = X - \sum_{n=1}^{\infty} nNP(B, h, 2e),$$

$nA$ denoting in these formulae the set of the elements $nA$ with $a \in A$. The set $nA$ is obviously of the first category.

3. Decomposition theorems

In this paragraph the set $B$ will be supposed to be linear.

**Theorem 1.** There exists a decomposition $T = e + h$ and a residual set $R$ in $X$ such that

(i) for every $a$ and every $e > 0$ there exists a set $e'$ such that $\mu(e - e') < e$ and $U(a|e') \in B$,

(ii) for every $e \in R$ and every set $h \subset A$ of positive measure $U(a|h)$ non-$e$.

**Proof.** Let $\mathcal{F}$ be the class of the sets $h$ for which the condition

$(B, h, e)$ is satisfied for every $x$ and every $e > 0$, and let $a$ denote the supremum of the measures of the sets in $\mathcal{F}$. There exists a set $e_0 \in \mathcal{F}$ such that $\sigma - 1/n \leq \mu(e_0)$. Let us write

$$e = \sum_{n=1}^{\infty} e_n, \quad h = T - e.$$

The condition (i) is then evidently satisfied.

Now consider the following condition:

(n) there exists a set $h \subset A$ such that $\mu(h) > 0$ and $U(x|h) \in B$.

To prove (ii) it suffices to show that the set $Z$ of the elements $x$ satisfying the condition (n) is of the first category. Suppose the contrary, and denote by $Q_n$ the set of the elements $x$ for which there exists a set $h \subset A$ such that $\mu(h) > 1/n$ and $U(x|h) \in B$.

Clearly

$$Z = \sum_{n=1}^{\infty} Q_n,$$

whence one of the sets $Q_n$, say $Q_n$, must also be of the second category. In the class $\mathcal{F}$ of the sets $h \subset A$ of measure not less than $a = 1/n$, there exists a sequence $h_n$ composing a dense set. Let us write $X_{h_n} = P(B, h_n, 2^{-n})$, then

$$Q_n \subset \sum_{n=1}^{\infty} X_{h_n},$$

hence for every $m$ there is an $n_m$ such that the set $X_{h_{n_m}}$ is of the second category. By Lemma 3 the set $P(B, h_{n_m}, 2^{-m-1})$ is residual. Now write

$$W = \prod_{n=1}^{\infty} P(B, h_{n_m}, 2^{-m-1}), \quad e' = \lim_{n \to \infty} h_{n_m}.$$

Then the set $W$ is residual, and

$$\mu(e') \geq \lim_{n \to \infty} \mu(h_{n_m}) \geq a.$$

Let $x \in W$, then for every $m$ there exists a set $e_m(x)$ such that

$$\mu(h_{n_m} - e_m(x)) < 2^{-m-1} \quad \text{and} \quad U(x|e_m(x)) \in B.$$

We may suppose freely that $e_1(x) C e_1(x) \ldots$ Then

$$\mu(e' - e_m(x)) = \lim_{n \to \infty} \mu(e' - e_m(x)) = 0,$$

for we have

$$\mu(e' - e_m(x)) \leq \mu(\sum_{m=1}^{\infty} h_{n_m} - e_m(x)) \leq \sum_{m=1}^{\infty} \mu(h_{n_m} - e_m(x)) \leq 2^{-m-2}.$$

Thus $W \subset P(B, e', e)$ for every $e > 0$, whence

$$W \subset \prod_{n=1}^{\infty} P(B, e', 1/n) = V.$$

The set $V$ is evidently linear, it satisfies the condition of Baire (being analytic) and is residual since it includes the set $W$. This implies $X = V$.

Now for every $x \in V$, $e > 0$ the condition $(B, e', e)$ is satisfied, hence $e' \in \mathcal{F}$. This, however, leads to a contradiction, since $(e' + e') \notin \mathcal{F}$ and $\mu(e + e') = \mu(e) + \mu(e') = a > a$, contrary to the definition of the number $a$.

Now the question arises whether or not the set $e'$ in the assertion (i) of Theorem 1 might be chosen independently of $x$, i.e. whether (i) might be replaced by the following assertion:

(i') for every $e > 0$ there exists a set $e'$ such that $\mu(e - e') < e$ and $U(x'|e') \in B$ for every $x$.

We shall show by a counterexample that the answer is negative.

Let $X = Y$ be the well-known space $L$ of the Lebesgue measurable functions in $[a, b]$, $U(x) = x$. By $B$ we shall denote the subset of $Y$ composed of essentially bounded functions. This set is linear and of $F_\sigma$ type. By well-known theorems (i) is true with $e = [a, b]$; however, a set of positive measure on which all functions of $L$ are simultaneously essentially bounded, does not exist.
Theorem 2. Let the set \( B \) satisfy the condition (b3). Then there exists a decomposition \( T = e + h \) and a residual set \( ECX \) such that:

(i') \( U(x|e) \in B \) for every \( x \),

(ii') \( U(x|h') \) non \( e \) for every \( x \in B \) and every set \( h' \in h \) of positive measure.

The proof is obvious.

4. Applications

Now we shall present some applications of the above theorems.

Let us denote by \( \mathfrak{S} \) the space of the sequences \( y = [y_n(t)] \) of real-valued \( \mu \)-measurable functions defined on \( T \). The elements of this space may be considered as functions defined in \( T \) with values in the space \( \mathbb{R} \) of the sequences of real numbers (Banach [2], p. 10). We define the addition and multiplication by scalars in \( \mathfrak{S} \) as usual, and the norm as

\[
\|y\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{t \in I_n} |y_n(t)| dt;
\]

then \( \mathfrak{S} \) becomes an \( F \)-space. Upon setting \( L = \mathbb{R}, \mathbb{Y} = \mathfrak{S} \), we see that the conditions (a1)-(a3) are satisfied. A sequence \( y_n = [y_n(t)]_{n=1}^{\infty} \) of elements of \( \mathfrak{S} \) converges to \( y = [y(t)] \) if and only if

\[
\lim_{n \to \infty} \sup_{t \in I_n} |y_n(t) - y(t)| = 0.
\]

Denote by \( B_1, \ldots, B_m \) the sets of the elements \( y = [y_n(t)] \) of \( \mathfrak{S} \) for which the following conditions are satisfied:

1. The sequence \( [y_n(t)] \) is asymptotically bounded (i.e., \( \lambda \to 0 \) implies \( \lim_{n \to \infty} \lambda_n y_n(t) = 0 \));
2. The sequence \( [y_n(t)] \) converges a.e. (almost everywhere);
3. The sequence \( [y_n(t)] \) converges a.e. for every \( t \in T \). Then

\[
\sum_{n=1}^{\infty} |y_n(t)|^2 < \infty \quad \text{a.e. (a > 0)},
\]

4. The sequence \( [y_n(t)] \) converges in \( L^2 \) (a > 0);
5. The sequence \( [y_n(t)] \) converges in \( L^2 \) (a > 0);
6. The sequence \( [y_n(t)] \) converges in \( L^p \) (a > 0).

These sets are obviously linear; we shall prove that they are measurable (B).

Ad B1. The set

\[
A_{\text{amp}} = E[y = [y_n(t)], \mu(E[|y_n(t)| > m]) < 1/p]
\]

is evidently closed and

\[
B_1 = \left( \prod_{p=1}^{m} \left( \sum_{n=1}^{m} A_{\text{ampp}} \right) \right).
\]

Ad B2. The sets

\[
B_{\text{amp}} = E[y = [y_n(t)], \mu(E[|y_n(t) - y(t)| > 1/p]) < 1/q]
\]

are closed and

\[
B_2 = \left( \prod_{p=1}^{m} \left( \sum_{n=1}^{m} B_{\text{amp}} \right) \right).
\]

Ad B3. Given any element \( y = [y_n(t)] \) let us write

\[
o_n(g) - o_n(g, t) = \max_{n \in \mathbb{N}, t \in I_n} |y_n(t)|.
\]

Then \( \|y_n - y\| \to 0 \) implies

\[
\lim_{n \to \infty} \sup_{t \in I_n} |y_n(t) - y_n(t)| = 0.
\]

The sequence \( [o_n(t)] \) is bounded if and only if the sequence \( [o_n(t)] \) is asymptotically bounded, hence

\[
B_3 = \left( \prod_{p=1}^{m} \left( \sum_{n=1}^{m} A_{\text{amp}} \right) \right),
\]

where

\[
A_{\text{amp}} = E[y = [o_n(t)], |o_n(t)| < m] < 1/p.
\]

We shall prove that the sets \( A_{\text{amp}} \) are closed. Let \( y_n \in A_{\text{amp}}, y_n \to y \), and write

\[
o_n = E[o_n(t) > m],
\]

then \( \mu(o_n) < 1/p \). Since

\[
\lim_{n \to \infty} \sup_{t \in I_n} |o_n(t) - o_n(t)| = 0,
\]

there exists a sequence \( k_n \) such that \( o_n(y_n(t)) \to o_n(y(t)) \) a.e.

Let us write

\[
o_n = \lim_{k \to \infty} e_{k_n}.
\]
then \(\mu(\alpha)\leq 1/p\) and \(t\in T_0\) implies \(t\in T_0\) (the sequence \([k]\) being extracted from \([k]\)) whence \(\omega_0(\alpha, t)\leq m\). It follows that \(\omega_0(\alpha, t)\leq m\) a.e. in \(T_0\), thus

\[ E[\omega_0(\alpha, t) > m] \subseteq C_\alpha \]

and \(\mu = \mathbb{A}_\alpha^{\text{emp}}\).

Add \(B_1\). Write

\[ \omega_0(\alpha, t) = \max_{\eta_0(t) = \eta_1(t)} |\eta_1(t) - \eta_0(t)| \]

and

\[ D_\alpha = E\{\mu(E[\omega_0(\alpha, t) > m]) \leq 1/n\} \]

We can prove, as above, that the sets \(D_\alpha\) are closed; then we apply the formula

\[ B_1 = \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \sum_{k=1}^{\infty} \prod_{\eta_0(t) = \eta_1(t)}^{\infty} D_\alpha \]

Add \(B_1\). We write

\[ E_{\alpha} = E\{\eta = (\eta_1(t), \eta_0(t))|\eta_0(t) = \eta_1(t)\} \subseteq \mathbb{E}_{\alpha} \]

This set is closed. For, if \(\eta_0 = ((\eta_1(t), \eta_0(t))|\eta_0(t) = \eta_1(t))\), then there exists a sequence \([k]\) such that \(\eta_0(t) \rightarrow \eta(t)\) a.e. whence by Fatou's lemma

\[ \int \eta_1(t) \, dt \leq \lim \inf \int \eta_0(t) \, dt , \]

i.e. \(\eta \in E_{\alpha}\). \(\mathbb{E}_{\alpha}\)-measurability of the set \(B_1\) follows by formula

\[ B_1 = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \prod_{\eta_0(t) = \eta_1(t)}^{\infty} D_\alpha \]

The proofs in the remaining cases are similar.

All the sets \(B_1-B_3\) have the properties \((b_1)\) and \((b_2)\). The sets \(B_4\), \(B_5\), \(B_6\), \(B_7\) also have the property \((b_3)\).

Let us denote by \(\mathbb{E}\) the space of measurable functions \(\eta = \eta(t)\) defined in \(T\), with the norm

\[ ||\eta|| = \int \frac{|\eta(t)|}{1 + |\eta(t)|} \, dt \]

It is an \(F\)-space (Banach [2], p. 9). An operation \(V(x) = V(x, t)\) from \(X\) to \(S\) is linear if it is additive and \(x_0 \rightarrow 0\) implies

\[ \lim_{n \rightarrow \infty} V(x_0, t) = 0 \]

We shall consider any sequence \([V_n(x, t)]\) of linear operations from \(X\) to \(S\) as an operation \(U(x, t) = [V_n(x, t)]\) from \(X\) to \(S\). Taking \(Y\) the space \(\mathbb{E}\) we derive from Theorems 1 and 2 the

**Theorem 3.** Given any sequence \([V_n(x, t)]\) of linear operations from \(X\) to \(S\) there exist decompositions \(T = \alpha + h_1 + \ldots + h_k\) and a residual set \(E\) such that:

(iA) \(V_n(x, t)\) is asymptotically bounded on \(E\) for every \(x\),

(iiA) \(V_n(x, t)\) is not asymptotically bounded on every set \(h \in h_1\) of positive measure and every \(x \in E\),

(iB) \(V_n(x, t)\) converges asymptotically on \(E\) for every \(x\),

(iiB) \(V_n(x, t)\) does not converge asymptotically on every set \(h \in h_1\) of positive measure and every \(x \in E\),

(iiiA) \(V_n(x, t)\) is bounded a.e. in \(E\) for every \(x\),

(iiII) \(V_n(x, t)\) is bounded a.e. in \(h_1\) for every \(x \in E\),

(iiiB) \(V_n(x, t)\) converges a.e. in \(E\) for every \(x\),

(iiII) \(V_n(x, t)\) converges a.e. in \(h_1\) for every \(x \in E\),

(ivA) \(\sum_{n=1}^{\infty} |V_n(x, t)|^\alpha < \infty\) a.e. in \(E\) for every \(x\),

(iiII) \(\sum_{n=1}^{\infty} |V_n(x, t)|^\alpha \leq \infty\) a.e. in \(h_1\) for every \(x \in E\),

Moreover, for every \(x\) and \(e > 0\) there exist sets \(\eta', \eta'', \eta'''\) such that

\(\mu(\eta_0 - \eta') < e, \mu(\eta - \eta''') < e, \mu(\eta'' - \eta''') < e\)

and

(iA) \(\sup_{n} \int |V_n(x, t)|^\alpha \, dt < \infty,\)

(iiA) \(\sup_{n} \int |V_n(x, t)|^\alpha \, dt = \infty\) for every set \(h \in h_1\) of positive measure and every \(x \in E\),

(iiiB) \(\lim_{n \rightarrow \infty} \int |V_n(x, t) - V_m(x, t)|^\alpha \, dt = 0,\)

(iiIII) \(\lim_{n \rightarrow \infty} \int |V_n(x, t) - V_m(x, t)|^\alpha \, dt > 0\) for every set \(h \in h_1\) of positive measure and every \(x \in E\),

(ivB) \(\sum_{n=1}^{\infty} \int |V_n(x, t)|^\alpha \, dt < \infty\)

(iiIII) \(\sum_{n=1}^{\infty} \int |V_n(x, t)|^\alpha \, dt = \infty\) for every set \(h \in h_1\) of positive measure and every \(x \in E\).

\[^{*}\text{Orlicz [6] deduces also all the cases considered here from a general theorem.}\]
Now denote by $\mathcal{G}$, the space of the functions $y = \eta(t)$ depending on the parameter $\lambda \varepsilon(a, b)$, which are continuous in $\lambda$ for fixed $t$, and $\mu$-measurable for fixed $\lambda$. The norm is defined as

$$\|y\| = \frac{\lambda}{\max_{\lambda \in \mathbb{R}} \max_{t \in [0, 1]} |\eta(t)|} \int_0^1 \max_{\lambda \in \mathbb{R}} |\eta(t)| dt,$$

where $a_n \to b$; this space is complete. The elements of $\mathcal{G}$, will be regarded as sequences depending on the continuous parameter $\lambda$. Choosing as $L$ the functions which are continuous in $[a, b]$ we easily see that we can consider $\mathcal{G}$, as the space $Y$ of type described in section 1. Denote by $B_1, \ldots, B_k$ the sets of the elements of $\mathcal{G}$, for which respectively

1. the sequence $\eta(t)$ is asymptotically bounded when $\lambda \to b$,
2. the sequence $\eta(t)$ converges asymptotically when $\lambda \to b$,
3. $\int_a^b \int \eta(t) dt$ exists,
4. $\lim_{\lambda \to b} \int |\eta(t)| dt = 0$ ($a > 0$),
5. $\int \lim_{\lambda \to b} |\eta(t)| dt < \infty$ ($a > 0$).

All these spaces are linear, measurable $(B)$, and satisfy the conditions $(b_1), (b_2)$; the sets $B_1, B_2$ satisfy also the condition $(b_2)$.

Similarly to Theorem 3 we can deduce now

**Theorem 4.** Let $V_1(x, t)$ denote for fixed $\varepsilon(a, b)$ a linear operation from $X$ to $S_i$, suppose it to be continuous in $\lambda$ for fixed $x$ and $t$. Then there exist decompositions $T = e_1 + e_2 + \cdots + e_k + h$ and a residual set $R$ such that

(i) the sequence $V_1(x, t)$ is asymptotically bounded on $e_i$ for every $x$, as $\lambda \to b$,
(ii) for every set $h \subset \mathcal{H}$, its measure and every $x \in R$ the sequence $V_2(x, t)$ is not asymptotically bounded on $h$, as $\lambda \to b$,
(iii) $\lim_{\lambda \to b} V_2(x, t)$ exists on $e_i$ for every $x$,
(iv) $\lim_{\lambda \to b} V_2(x, t)$ does not exist on every set $h \subset \mathcal{H}$, of positive measure and every $x \in R$.

Moreover, for every $x$ and $\varepsilon > 0$ there exist sets $e', e'', e'''$ such that

$$\mu(e') < \varepsilon, \mu(e'' - e') < \varepsilon, \mu(e'' - e''') < \varepsilon$$

and

$$\int \frac{\mu(|x|)}{dl} = \int \frac{\mu(|t|)}{dl}.$$
Sur les fonctionnelles multiplicatives

par

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Introduction

Ce travail est une continuation de mon article précédent [2]. Nous y considérons un sous-espace linéaire fermé $\mathcal{E}$ de l'espace $X$ conjugué à un espace $X$ du type $B$, un espace linéaire fermé $\mathcal{F}$ d'opérations linéaires de $\mathcal{E}$ à $X$; enfin un espace linéaire $\mathcal{M}$ de fonctionnelles linéaires dans $\mathcal{F}$, qui satisfont à l'axiome qui était désigné dans [2] par (F). Cet axiome sera cité plus loin sous la condition (12). A toute fonctionnelle $\mathcal{F}$ qui appartient à $\mathcal{M}$, nous faisons correspondre une opération $T_\mathcal{F}$ linéaire de $\mathcal{E}$ à $\mathcal{E}$, notamment

$$T_\varphi \varphi = F_\varphi [\varphi \varphi \varphi]$$

($\varphi \in \mathcal{E}$, $\varphi \in X$)

(voir [2], Introduction).

Nous étudions ensuite l'équation $\varphi + T_\varphi \varphi = \psi$ ($\varphi, \psi \in \mathcal{E}$), en faisant correspondre à l'opération $T_\varphi$ un nombre $D(F)$ qu'on appelle le déterminant de cette équation.

En général, on ne peut pas demander que le nombre correspondant à l'équation $(I + T_\varphi)(I + T_\psi) = \psi$ soit égal à $D(F_\varphi)D(F_\psi)$, vu que la fonctionnelle $F$ et, par conséquent, $D(F)$ ne sont pas déterminées par $T_\varphi$.

Nous introduisons ici une sorte de „multiplication“ des éléments de $\mathcal{M}$, de manière que l'on ait

$$T_{F_0 + F_0} = T_{F_0} \cdot T_{F_0} \quad \text{pour} \quad F_0 \in \mathcal{M} \quad (i = 1, 2)$$

nous démontrerons que la fonctionnelle $D(F)$ vérifie l'équation

$$D(F_0)D(F_0) = D(F_0 + F_0 + F_0)$$

pour tout couple $F_0, F_0$ d'éléments permutable de $\mathcal{M}$.)

I. Considérations générales

Soit $\mathfrak{A}$ un anneau du type $(B)$, c'est-à-dire un anneau linéaire avec une norme homogène $|A|$, satisfaisant à l'inégalité $|A-B| \leq |A| + |B|$ pour $A, B \in \mathfrak{A}$; regardé comme espace linéaire, cet anneau est un es-

1) M. R. Sikorski a remplacé la condition de permutilalité d'éléments $F_0, F_0$ par une autre, moins restrictive.