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Subspaces of the Bourgain–Delbaen space

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Abstract. It is shown that every infinite-dimensional closed subspace of the Bourgain–Delbaen space $X_{a,b}$ has a subspace isomorphic to some ℓ^p.

Introduction. In 1980, Bourgain and Delbaen [5, 4] introduced some separable $\mathcal{L}^\infty$ spaces with surprising properties: all have the Radon–Nikodym property, and so certainly do not have subspaces isomorphic to c_0; some of them (the spaces of "Class $\mathcal{X}$") have the Schur property; the others ("Class $\mathcal{Y}$") have dual spaces isomorphic to ℓ^1. Despite their importance, these spaces were not much studied subsequently, and it became habitual to remark that they were "not well-understood". There has been some renewed interest recently, partly because these spaces are interesting test-cases for questions about uniform homeomorphisms [9, 6] and smooth surjections [2, 7]. Alspach [1] has investigated their Szlenk index. This paper is an attempt to understand a bit better the subspace structure of the spaces of Class $\mathcal{Y}$, that is to say, in Bourgain's notation, the spaces $X_{a,b}$ with $b < 1/2 < a < 1$ and $a + b > 1$. Bourgain and Delbaen showed that every infinite-dimensional subspace of such a space has an infinite-dimensional reflexive subspace; however, they did not characterize which reflexive spaces occur as subspaces of $X_{a,b}$; Bourgain [4, p. 46] raised the question of whether $X_{a,b}$ has a subspace with no unconditional basic sequence. The main result of the present paper answers these questions by showing that each infinite-dimensional subspace of $X_{a,b}$ has a subspace isomorphic to ℓ^p. The p in question is determined by $1/p + 1/p' = 1$ where $a^{p'} + b^{p'} = 1$.

Our notation and terminology are mostly standard. In particular, we follow modern practice by saying that vectors $x_1, x_2, \ldots$ are *successive* linear combinations (or blocks) of a sequence (y_n) if there are integers $m_1 \le n_1 < m_2 \le n_2 < m_3 \le \ldots$ and scalars $\alpha_1, \alpha_2 \ldots$ such that $x_k = \sum_{j=m_k}^{n_k} \alpha_j y_j$.

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The unconditional sequence spaces $U_{a,b}$. Closely associated with the Bourgain–Delbaen spaces are some spaces with unconditional basis, which we shall denote by $U_{a,b}$. In this section we shall study these spaces, eventually showing that they are just ℓ^p-spaces with equivalent norms. The norm $\|\cdot\|_{a,b}$ is defined by a recursion similar to (but simpler than!) the one that leads to the Tsirelson space [8]. We fix real numbers a, b with $a, b < 1$, $a + b > 1$. For a vector $x \in \mathbb{R}^d$, or a finitely-supported vector $x \in \mathbb{R}^{(\mathbb{N})}$, we define (recursively)

$$\|x\|_{a,b} = \max\{\|x\|_\infty, \max_{l\in\mathbb{N}}(a\|x\restriction_{[0,l]}\|_{a,b} + b\|x\restriction_{[l+1,\infty)}\|_{a,b})\}.$$

That is to say that the norm $\|(x_0, x_1, \ldots, x_d)\|_{a,b}$ of a vector in $\mathbb{R}^{d+1}$ is whichever is greater of $\max_i |x_i|$ and $\max_l(a\|(x_0, x_1, \ldots, x_l)\|_{a,b} + b\|(x_{l+1}, \ldots, x_d)\|_{a,b})$. It is an elementary exercise to see that this is indeed an unambiguous definition. We then define $U_{a,b}$ to be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. It should be noted that in the definition of the space $U_{a,b}$ we do not need to suppose that $b < 1/2$ (a condition essential for the Bourgain–Delbaen construction). However, it will be convenient in all that follows to assume that $b \le a$. The symmetry of the definition of the norm $\|\cdot\|_{a,b}$ means that the main result of this section, Theorem 1, remains true when $a < b$, though with a replacing b in the final estimates.

The recursive calculation of norms in the space $U_{a,b}$ leads naturally to the construction of a finite dyadic tree of intervals of natural numbers, and it will be useful to have a standard notation for such trees. We write $\Sigma = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ for the set of all finite strings of 0's and 1's, including the empty string (). In our intended application, a "0" in a string σ will always be associated with a move to the left and a "1" with a move to the right. We shall accordingly denote the number of 0's and the number of 1's in a string σ by $l(\sigma)$ and $r(\sigma)$ respectively. For $\sigma, \tau \in \Sigma$ we write $\sigma \preccurlyeq \tau$ and say that σ *precedes* τ if σ is an initial segment of τ. Each element σ of Σ has two immediate successors, which we may denote by $\sigma 0$ and $\sigma 1$. By an *admissible subtree* of Σ we shall mean a non-empty, finite subset Υ of Σ having the property that, whenever $\sigma \in \Upsilon$, all predecessors of σ are also in Υ and, of the two immediate successors of σ, either both are in Υ, or else neither is. Those σ with no successors in Υ form the set $\max\Upsilon$ of maximal elements of Υ.

A *dyadic tree of intervals* is a family $I(\sigma)$ of non-empty intervals in $\mathbb{N}$, indexed by some admissible subtree Υ, with the property that whenever $\sigma \in \Upsilon$ is non-maximal, the interval $I(\sigma)$ is the disjoint union of its subintervals $I(\sigma 0)$ and $I(\sigma 1)$, with $I(\sigma 0)$ lying to the left of $I(\sigma 1)$. We note that the intervals $i(\tau)$ corresponding to $\tau \in \max\Upsilon$ form a partition of the original interval $I()$.

If x is a finitely supported vector in $\mathbb{R}^{(\mathbb{N})}$, and $I(\sigma)$ ($\sigma \in \Upsilon$) is any dyadic tree of intervals, it is clear from the recursive definition of the norm that

$$\|x\|_{a,b} \ge \sum_{\tau\in\max\Upsilon} a^{l(\tau)} b^{r(\tau)} \|x\restriction_{I(\tau)}\|_{a,b}.$$

Moreover, for a suitably chosen tree, we have

$$\|x\|_{a,b} = \sum_{\tau\in\max\Upsilon} a^{l(\tau)} b^{r(\tau)} \|x\restriction_{I(\tau)}\|_\infty.$$

Notice that in the case where $\|x\|_{a,b} = \|x\|_\infty$ this latter equality holds for the trivial tree $\Upsilon = \{()\}$.

We shall now proceed to establish the inequality $\|x\|_{a,b} \le \|x\|_p \le C\|x\|_{a,b}$ for an arbitrary finitely-supported vector x in $\mathbb{R}^{(\mathbb{N})}$, thus showing that $\|\cdot\|_{a,b}$ is equivalent to the ℓ^p-norm, where $1/p + 1/p' = 1 = a^{p'} + b^{p'}$.

A few naïve remarks will perhaps help to clarify the calculations that follow. The inequality $\|x\|_{a,b} \le \|x\|_p$ is easy to establish by induction on the size of the support of x. Indeed, $\|x\|_{a,b}$ is equal either to $\|x\|_\infty$ or to $a\|x\restriction_{[0,k)}\|_{a,b} + b\|x\restriction_{[k,\infty)}\|_{a,b}$, and this latter quantity is at most

$$(a^{p'} + b^{p'})^{1/p'}(\|x\restriction_{[0,k)}\|_{a,b}^p + \|x\restriction_{[k,\infty)}\|_{a,b}^p)^{1/p} \le (\|x\restriction_{[0,k)}\|_p^p + \|x\restriction_{[k,\infty)}\|_p^p)^{1/p} = \|x\|_p,$$

by Hölder's inequality and our inductive hypothesis. There are, of course, some vectors for which $\|x\|_{a,b} = \|x\|_p$; they may be characterized using the condition for equality to occur in Hölder's inequality. Indeed, they are exactly those vectors where a norm calculation of the kind described above leads to a dyadic tree of intervals with the property that the ratio $\|x\restriction_{I(\sigma 0)}\|_p : \|x\restriction_{I(\sigma 1)}\|_p$ is precisely $a^{p'-1} : b^{p'-1}$ for every non-maximal σ, and such that $\|x\restriction_{I(\tau)}\|_p = \|x\restriction_{I(\tau)}\|_\infty$ for each maximal τ.

If we are thinking of $\|\cdot\|_{a,b}$ as an approximation to $\|\cdot\|_p$ then, every time that we are obliged to split an interval other than in the ratio $a^{p'-1} : b^{p'-1}$ with respect to the ℓ^p-norm, we introduce an underestimate. The proof we give proceeds by constructing a certain dyadic tree and keeping fairly careful accounts of the accumulated underestimation. It will be convenient to write $\alpha = a^{p'}$ and $\beta = b^{p'}$, so that $\alpha + \beta = 1$. As already remarked, we lose no generality in supposing that $a \ge b$.

LEMMA 1. *Let $y \in \mathbb{R}^{(\mathbb{N})}$ be a non-zero vector, with support contained in the finite interval J. Assume that y satisfies*

$$\|y\|_\infty^p \le \frac{2\beta}{5}\|y\|_p^p.$$

We may choose a natural number k, not an end-point of the interval J, and a natural number l (equal either to k or to $k-1$) in such a way that

$$\|y\|_p \le \exp\left[\frac{1}{5pp'}\frac{|y_k|^p}{\|y\|_p^p}\right][a\|y\restriction_{[0,l]}\|_p + b\|y\restriction_{[l+1,\infty)}\|_p].$$

That is to say, either

$$\|y\|_p \le \exp\left[\frac{1}{5pp'}\frac{|y_k|^p}{\|y\|_p^p}\right][a\|y\restriction_{[0,k-1]}\|_p + b\|y\restriction_{[k,\infty)}\|_p],$$

or

$$\|y\|_p \le \exp\left[\frac{1}{5pp'}\frac{|y_k|^p}{\|y\|_p^p}\right][a\|y\restriction_{[0,k]}\|_p + b\|y\restriction_{[k+1,\infty)}\|_p].$$

Notice that in either case k is an end-point of the subinterval $J \cap [0,k]$ or $J \cap [k,\infty)$ which contains it.

Proof. It will simplify notation to suppose that the interval J is $[1,n]$. We choose k to be the unique natural number that satisfies

$$\sum_{j=1}^{k-1} |y_j|^p < \alpha\|y\|_p^p \le \sum_{j=1}^{k} |y_j|^p.$$

Our assumption implies that $\|y\|_\infty^p < \beta\|y\|_p^p$, and hence that $|y_1|^p < \beta\|y\|_p^p \le \alpha\|y\|_p^p$ and $\sum_{j=1}^{n-1} |y_j|^p = \|y\|_p^p - |y_n|^p > (1-\beta)\|y\|_p^p = \alpha\|y\|_p^p$. Thus k cannot be either of the end points $1, n$ of the supporting interval J. By choosing l to be either $k-1$ or k, we may arrange that

$$\left|\sum_{j=1}^{l} |y_j|^p - \alpha\|y\|_p^p\right| \le \frac{1}{2}|y_k|^p.$$

So if we write $w = y\restriction_{[0,l]}$ and $z = y\restriction_{[l+1,\infty)}$, we have

$$\|w\|_p^p = (\alpha+\varepsilon)\|y\|_p^p, \qquad \|z\|_p^p = (\beta-\varepsilon)\|y\|_p^p,$$

where $|\varepsilon| \le \frac{1}{2}(|y_k|/\|y\|_p)^p$. We can now calculate as follows:

$$\begin{aligned} a\|w\|_p + b\|z\|_p &= [a(\alpha+\varepsilon)^{1/p} + b(\beta-\varepsilon)^{1/p}]\|y\|_p \\ &= [\alpha(1+\varepsilon/\alpha)^{1/p} + \beta(1-\varepsilon/\beta)^{1/p}]\|y\|_p. \end{aligned}$$

Of course, for small values of ε,

$$\alpha(1+\varepsilon/\alpha)^{1/p} + \beta(1-\varepsilon/\beta)^{1/p} \approx \exp\left[-\frac{1}{2pp'}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\varepsilon^2\right],$$

and it is an elementary exercise to see that

$$\alpha(1+\varepsilon/\alpha)^{1/p} + \beta(1-\varepsilon/\beta)^{1/p} > \exp\left[-\frac{1}{pp'}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\varepsilon^2\right],$$

whenever $|\varepsilon| < \beta/5$. In our case, since we are assuming that $\|y\|_\infty^p < (2\beta/5)\|y\|_p^p$, the quantity ε as defined above is indeed smaller than $\beta/5$. We are thus led to the inequality

$$\begin{aligned} \|y\|_p &\le \exp\left[\frac{1}{pp'}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\varepsilon^2\right][a\|w\|_p + b\|z\|_p] \\ &\le \exp\left[\frac{1}{5pp'}\frac{|y_k|^p}{\|y\|_p^p}\right][a\|w\|_p + b\|z\|_p], \end{aligned}$$

using once again the fact that

$$|\varepsilon| \le \frac{1}{2}\frac{|y_k|^p}{\|y\|_p^p} \le \frac{\beta}{5}.$$

THEOREM 1. *Let a, b be real numbers satisfying $a, b < 1$, $a+b > 1$ and let p, p' be determined by $1/p + 1/p' = 1 = a^{p'} + b^{p'}$. The norm $\|\cdot\|_{a,b}$ is equivalent to the usual ℓ^p-norm.*

Proof. As in the preceding lemma, we may suppose that $b < a$ and we retain the notation $\alpha = a^{p'}$, $\beta = b^{p'}$. We consider an arbitrary non-zero $x \in \mathbb{R}^{(\mathbb{N})}$ and give a recursive definition of an admissible tree Υ, a dyadic tree of intervals $(I(\sigma))_{\sigma\in\Upsilon}$, and elements $i(\sigma)$ of $I(\sigma)$, which we shall use to estimate $\|x\|_{a,b}$. We start by taking $I()$ to be any finite interval that contains the support of x. If a string τ is already in Υ and $I(\tau)$ has already been defined we need to specify whether τ is going to be a maximal element of Υ and, if not, what the two "daughter" intervals $I(\tau 0)$ and $I(\tau 1)$ are going to be.

There will be two criteria involved in deciding if τ is maximal. First, τ will be declared to be maximal if the following condition holds:

(A) $$\|x\restriction_{I(\tau)}\|_\infty \ge (2\beta/5)^{1/p}\|x\restriction_{I(\tau)}\|_p.$$

If this condition does not hold, then of course Lemma 1 is applicable to the vector $y = x\restriction_{I(\tau)}$. We let $i(\tau)$ be the unique $i \in I(\tau)$ such that

$$\sum_{j\in I(\tau)\cap[0,i)} |y_j|^p < \alpha\|y\|_p^p \le \sum_{j\in I(\tau)\cap[0,i]} |y_j|^p.$$

Thus $i(\tau)$ is the "k" corresponding to $y = x\restriction_{I(\tau)}$ in Lemma 1. We recall from that lemma that $i(\tau)$ is not an end-point of $I(\tau)$. The second condition for maximality is that τ will be maximal if

(B) $$\sum_{\sigma\preccurlyeq\tau} \frac{|x_{i(\sigma)}|^p}{\|x\restriction_{I(\sigma)}\|_p^p} > 5.$$

The effect of this criterion is to ensure that

$$\sum_{\sigma\prec\tau} \frac{|x_{i(\sigma)}|^p}{\|x\restriction_{I(\sigma)}\|_p^p} \le 5$$

for every τ in the tree. Indeed, otherwise the recursive construction would have been terminated (by criterion (B)) at a predecessor of τ.

In the event that neither (A) nor (B) holds, we choose l as in Lemma 1 and define the daughter intervals by $I(\tau 0) = I(\tau) \cap [0, l]$, $I(\tau 1) = I(\tau) \cap [l+1, \infty)$. We notice that $i(\tau)$ *is* an end-point of one or other of these intervals, and hence also of any interval $I(\upsilon)$, with $\upsilon \succ \tau$, which contains it.

This completes the recursive construction of Υ, $I(\sigma)$ and $i(\sigma)$. The set $\max \Upsilon$ of maximal elements may be partitioned as $A \cup B$, where A is the set of τ for which condition (A) holds. We notice that the natural numbers $i(\tau)$, defined for $\tau \in \Upsilon \setminus A$, are all distinct. Indeed, if υ and τ are incomparable elements of Υ, then $i(\upsilon)$ and $i(\tau)$ are elements of the disjoint intervals $I(\upsilon)$ and $I(\tau)$; on the other hand, if $\tau \prec \upsilon$ and $i(\tau) \in I(\upsilon)$ then $i(\tau)$ is an end-point of $I(\upsilon)$ while $i(\upsilon)$ is not.

It follows from Lemma 1 that, whenever σ is a non-maximal element of Υ,

$$\|x\restriction_{I(\sigma)}\|_p \le \exp\left[\frac{1}{5pp'}\frac{|x_{i(\sigma)}|^p}{\|x\restriction_{I(\sigma)}\|_p^p}\right][a\|x\restriction_{I(\sigma 0)}\|_p + b\|x\restriction_{I(\sigma 1)}\|_p].$$

We deduce from this inequality, together with the remark we made following the introduction of criterion (B), that

$$\begin{aligned}\|x\|_p &\le \sum_{\tau\in\max\Upsilon} a^{l(\tau)}b^{r(\tau)}\exp\left[\sum_{\sigma\prec\tau}\frac{1}{5pp'}\frac{x_{i(\sigma)}^p}{\|x\restriction_{I(\sigma)}\|_p^p}\right]\|x\restriction_{I(\tau)}\|_p\\ &\le e^{1/(pp')}\sum_{\tau\in\max\Upsilon} a^{l(\tau)}b^{r(\tau)}\|x\restriction_{I(\tau)}\|_p\\ &\le e^{1/(pp')}\left[\sum_{\tau\in A} a^{l(\tau)}b^{r(\tau)}\left(\frac{5}{2\beta}\right)^{1/p}\|x\restriction_{I(\tau)}\|_\infty + \sum_{\tau\in B} a^{l(\tau)}b^{r(\tau)}\|x\restriction_{I(\tau)}\|_p\right]\\ &= e^{1/(pp')}[H_A + H_B],\end{aligned}$$

in an obvious notation. It follows from the relationship between trees and norm calculations that $H_A \le (5/(2\beta))^{1/p}\|x\|_{a,b}$. On the other hand, we may use Hölder's inequality and the fact that $a^{p'} + b^{p'} = 1$ to show that

$$H_B \le \Big(\sum_{\tau\in B}\|x\restriction_{I(\tau)}\|_p^p\Big)^{1/p} = \Big(\sum_{j\in J}|x_j|^p\Big)^{1/p},$$

where

$$J = \bigcup_{\tau\in B} I(\tau) = \Big\{ j \in I(): \sum_{\sigma \text{ with } j\in I(\sigma)} \frac{|x_{i(\sigma)}|^p}{\|x\restriction_{I(\sigma)}\|_p^p} > 5\Big\}.$$

We thus have

$$\begin{aligned}5H_B^p &\le \sum_{j\in J}|x_j|^p \sum_{\sigma \text{ with } j\in I(\sigma)}\left(\frac{|x_{i(\sigma)}|}{\|x\restriction_{I(\sigma)}\|_p}\right)^p\\ &\le \sum_{j\in I()}|x_j|^p \sum_{\sigma \text{ with } j\in I(\sigma)}\left(\frac{|x_{i(\sigma)}|}{\|x\restriction_{I(\sigma)}\|_p}\right)^p\\ &= \sum_{\sigma\in\Upsilon}|x_{i(\sigma)}|^p\|x\restriction_{I(\sigma)}\|_p^{-p}\sum_{j\in I(\sigma)}|x_j|^p = \sum_{\sigma\in\Upsilon}|x_{i(\sigma)}|^p\end{aligned}$$

and this is at most $\|x\|_p^p$, since as we noted before, the $i(\sigma)$ are all distinct.

We have finally obtained the following inequalities:

$$e^{-1/(pp')}\|x\|_p \le H_A + H_B \le (5/(2\beta))^{1/p}\|x\|_{a,b} + 5^{-1/p}\|x\|_p$$

whence

$$(e^{-1/p} - 5^{-1/p})\|x\|_p \le (5/(2\beta))^{1/p}\|x\|_{a,b},$$

which leads to a final estimate of the form

$$\|x\|_p \le Cpb^{-p'/p}\|x\|_{a,b},$$

with C a constant independent of p and b (and smaller than 50).

The author would like to thank Mark Boddington for carrying out some numerical experiments which suggested that Theorem 1 might be true. A study of various generalizations of the spaces $U_{a,b}$ will appear in [3].

The Bourgain–Delbaen construction. In this section we shall recall the construction of the spaces $X_{a,b}$, using a notation consistent with the original, but differing somewhat from it. As well as seeming (to the author at least!) somewhat clearer, this notation appears to be better suited to possible generalization. The ingredients needed in a construction of this kind are a sequence of sets $\Delta_0, \Delta_1, \ldots$ and linear mappings that we shall denote by u_m. The next paragraph sets out the properties that these sets and mappings have to satisfy.

We suppose that the sets $\Delta_0, \Delta_1, \ldots$ are disjoint and finite, and that the union $\Gamma = \bigcup_{n\in\mathbb{N}}\Delta_n$ is infinite. For $n \ge 0$, we write $\Gamma_n = \bigcup_{m\le n}\Delta_m$. For each $n \ge 0$, we need to have a linear operator $u_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Delta_{n+1})$ and we define $i_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_{n+1})$ by setting

$$(i_n f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \Gamma_n,\\ (u_n f)(\gamma) & \text{if } \gamma \in \Delta_{n+1}.\end{cases}$$

We define $i_{m,n} : \ell^\infty(\Gamma_m) \to \ell^\infty(\Gamma_n)$ to be the composition $i_{m,n} = i_{n-1} \circ i_{n-2} \circ \ldots \circ i_m$ and note that, for $m < n < p$ and $f \in \ell^\infty(\Gamma_m)$, we have

$$(i_{m,p} f)\restriction_{\Gamma_{n+1}} = i_{m,n} f.$$

It follows that we may well-define a linear mapping $j_m : \ell^\infty(\Gamma_m) \to \mathbb{R}^\Gamma$ by setting

$$(j_m f)(\delta) = (i_{m,n} f)(\delta) \quad (\delta \in \Gamma_n).$$

We now make the further assumption that the mappings u_m have been defined in such a way that the norms of all the compositions $i_{m,n}$ are bounded by some constant λ. This tells us that the mappings j_m take values in $\ell^\infty(\Gamma)$ and that, for each m, the operator j_m is an isomorphism from $\ell^\infty(\Gamma_m)$ onto a finite-dimensional subspace $X_m = \operatorname{im} j_m$ of $\ell^\infty(\Gamma)$ with

$$\|f\| \le \|j_m f\| \le \lambda \|f\|.$$

Finally, we take X to be the closure in $\ell^\infty(\Gamma)$ of the union of the increasing sequence of subspaces X_m. Since the subspaces X_m are λ-isomorphic to $\ell^\infty(\Gamma_m)$, the space X is a separable $\mathcal{L}^\infty_\lambda$-space, whose properties are determined (in a way that is not always straightforward to decide) by the operators u_n. The tricky part of the construction lies in finding u_n's which are such that the norm condition on the $i_{m,n}$'s is satisfied.

However the u_n are defined, the space X obtained in this way has some useful structure. Each of the subspaces X_n is the range of a projection S_n, defined by $S_n x = j_n(x\restriction_{\Gamma_n})$. If we set $P_0 = S_0$ and $P_n = S_n - S_{n-1}$ (for $n \ge 1$), then the subspaces $M_n = \operatorname{im} P_n$ form a finite-dimensional decomposition of X. When, later on in the paper, we refer to the *support* of a vector $x \in X$, we shall be thinking in terms of this f.d.d. Thus, if $x = \sum_m z_m$ with $z_m \in M_m$, then $\operatorname{supp}(x)$ will mean the set of m for which $z_m \ne 0$. Similarly, we shall say that the vectors $x_1, x_2, \ldots$ are *successive* if there exist natural numbers $m_1 \le n_1 < m_2 \le n_2 < m_3 \le \ldots$ such that $\operatorname{supp} x_k \subseteq [m_k, n_k]$. There is a relationship between this notion of support and the more obvious one where we are thinking of the vector x as a function on Γ; namely, $\operatorname{supp} x \cap [0,n] = \emptyset \Leftrightarrow x\restriction_{\Gamma_n} = 0$. It is also worth noting that, since the spaces $M_n = \{j_n(x) : x \in \ell^\infty(\Gamma_n) \text{ and } x\restriction_{\Gamma_{n-1}} = 0\}$ are λ-isomorphic to $\ell^\infty(\Delta_n)$ and so have uniformly bounded basis constant, the space X has a basis. Such basis vectors occur as the w_γ in Proposition 2 below (though the fact that they form a basis is not crucial there).

We now pass to the details of the Bourgain–Delbaen construction. Let a, b be real constants with $0 < b < 1/2 < a < 1$ and $a + b > 1$. We shall show how to construct the space $X_{a,b}$ by defining (recursively) the sets Δ_n and the mappings u_n. We start by taking Δ_0 to be a set with just one element, say $\Delta_0 = \{0\}$. Now we define

$$\Delta_{n+1} = \{n+1\} \times \bigcup_{0 \le k < n} \{k\} \times \Gamma_k \times \Gamma_n \times \{\pm 1\}.$$

So an element of Δ_n is a 5-tuple of the form

$$\delta = (n, k, \xi, \eta, \pm 1).$$

This notation replaces the explicit enumeration that appears in [4] and [5]. It will be convenient to have names for the five coordinates of δ:

$$n = \operatorname{rank}(\delta), \quad k = \operatorname{cut}(\delta), \quad \xi = \operatorname{base}(\delta), \quad \eta = \operatorname{top}(\delta), \quad \pm 1 = \operatorname{sign}(\delta).$$

The mapping $u_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Delta_{n+1})$ is defined by

$$(u_n f)(n, k, \xi, \eta, \pm 1) = a f(\xi) \pm b(f(\eta) - (i_{k,n}(f\restriction_{\Gamma_k}))(\eta)).$$

It is shown in [4, 5] that with the above definitions, the composite mappings $i_{m,n}$ are indeed uniformly bounded with

$$\|i_{m,n}\| \le \lambda = a/(1-2b).$$

It is perhaps worth repeating the original argument in our modified notation. We assume inductively that, for some n, all the mappings $i_{m,n}$ $(m \le n)$ have norm at most λ. We now consider some $f \in \ell^\infty(\Gamma_m)$ and some $\gamma = (n+1, k, \xi, \eta, \pm 1) \in \Delta_{n+1}$. By definition,

$$\begin{aligned} |(i_{m,n+1} f)(\gamma)| &= (u_n i_{m,n} f)(\gamma) \\ &\le a|(i_{m,n} f)(\xi)| + b|(i_{m,n} f)(\eta) - (i_{k,n}((i_{m,n} f)\restriction_{\Gamma_k}))(\eta)|. \end{aligned}$$

If the cut k is greater than m, then $i_{m,n} f = i_{k,n}(i_{m,k} f) = i_{k,n}((i_{m,n} f)\restriction_{\Gamma_k})$ so that the second term above vanishes, leaving $|(i_{m,n+1} f)(\gamma)| \le a\|i_{m,n} f\|$, which is at most $a\lambda\|f\|$ by our inductive hypothesis. If, on the other hand, $k \le m$, it must be that $\xi \in \Gamma_k \subseteq \Gamma_m$, so that $|(i_{m,n} f)(\xi)| = |f(\xi)| \le \|f\|$. Also, $(i_{m,n} f)\restriction_{\Gamma_k} = f\restriction_{\Gamma_k}$, an element of $\ell^\infty(\Gamma_k)$ satisfying $\|f\restriction_{\Gamma_k}\| \le \|f\|$. Applying our inductive hypothesis to the two mappings $i_{m,n}$ and $i_{k,n}$ we obtain

$$|(i_{m,n+1} f)(\gamma)| \le a|(i_{m,n} f)(\xi)| + b|(i_{m,n} f)(\eta) - (i_{k,n} f\restriction_{\Gamma_k})(\eta)| \le a\|f\| + 2b\lambda\|f\|.$$

Since $a = (1-2b)\lambda$, this is at most $\lambda\|f\|$, as required.

The following proposition can also be found in [4].

Proposition 1. *Let k, m, n be natural numbers, with $m < n$, let x be an element of X_m, and let γ be an element of Γ with $\operatorname{rank}(\gamma) = n$, $\operatorname{cut}(\gamma) = k$. Then*

$$|x(\gamma)| \le a\|x\restriction_{\Gamma_k}\| + b\|(I - S_k)x\| \le \|S_k x\| + b\|(I - S_k)x\|.$$

Proof. Since $x \in X_m$, x has the form $j_m f$, for some $f \in \ell^\infty(\Gamma_m)$, and so

$$\begin{aligned} x(\gamma) &= (i_{m,n} f)(\gamma) = (i_{n-1} \circ i_{m,n-1} f)(\gamma) \\ &= a(i_{m,n-1} f)(\xi) \pm b[(i_{m,n-1} f) - (i_{k,n-1}(f\restriction_{\Gamma_k})](\eta) \\ &= a x(\xi) \pm b(I - S_k)x(\eta), \end{aligned}$$

where $\xi = \operatorname{base}(\gamma)$ and $\eta = \operatorname{top}(\gamma)$ as usual. The inequality is now obvious.

Corollary. *For any m and any $x \in X_m$, either $\|x\| = \|x\restriction_{\Gamma_m}\|$ or*

$$\|x\| = \max_k [a\|x\restriction_{\Gamma_k}\| + b\|(I - S_k)x\|].$$

It is apparent from the construction that for a general $f \in \ell^\infty(\Gamma_m)$ we may need to go to Γ_n, with n significantly larger than m, in order to find a coordinate γ at which $j_m f$ comes close to attaining its norm. However, it is worth remarking that if $f \in \ell^\infty(\Gamma_m)$ and f is zero, except on Δ_m (the "last" of the sets that make up Γ_m), then $\|i_{m,n} f\| = \|f\|$ for all n. Thus in this set-up the subspaces M_n that make up the finite-dimensional decomposition of X are actually isometric to $\ell^\infty(\Delta_n)$.

Embedding ℓ^p in subspaces of $X_{a,b}$. It is implicitly shown in [4] that certain sequences in $X_{a,b}$ admit lower $U_{a,b}$-estimates (and thus, as we can now see, lower ℓ^p-estimates). These are sequences of vectors which are successive (with respect to the f.d.d. (M_n)) and which have supports sufficiently well spread out. To make this precise we choose a function $F : \mathbb{N} \to \mathbb{N}$ having the property that, for every n and every non-zero $x \in X_n$,

$$\|x\restriction_{\Gamma_{F(n)}}\| > \tfrac{1}{2}\|x\|.$$

This is possible by compactness of the unit ball of the finite-dimensional space X_n. We shall say that a (finite or infinite) sequence (y_k) in X is *F-admissible* if there are integers m_k and n_k, satisfying $m_k \le n_k$, $F(n_k) + k < m_{k+1}$, with $y_k \in X_{n_k}$, $y_k\restriction_{\Gamma_{F(m_k)}} = 0$. In terms of the f.d.d. (M_n) introduced earlier, we are saying that $y_k \in \bigoplus_{m_k < n \le n_k} M_n$ for all k, or equivalently that $\operatorname{supp}(y_k) \subseteq [m_k + 1, n_k]$. Evidently, if (y_k) is admissible then so is any sequence of successive linear combinations. The following lemma is related to Lemma 3.20 of [4].

LEMMA 2. *If (y_k) is an F-admissible sequence, then, for any l,*

$$\Big\|\sum_{k=1}^{l} y_k\Big\| > \frac{1}{6}\|(\|y_1\|, \ldots, \|y_l\|)\|_{a,b}.$$

In particular, $\|y_j\| \le 6\|\sum_{k=1}^{l} y_k\|$ for each $1 \le j \le l$.

Proof. For each k let us write p_k and q_k for the minimum and maximum, respectively, of the support of y_k. The hypothesis of F-admissibility implies that $p_{k+1} > F(q_k) + k$. We shall show that, for each subinterval $I = [j, k]$ of $[1, l]$, there exists $\gamma \in \Gamma_{F(q_k)+k-j}$ such that

$$\Big|\sum_{i=j}^{k} y_i(\gamma)\Big| > \frac{1}{6}\|(\|y_j\|, \|y_{j+1}\|, \ldots, \|y_k\|)\|_{a,b}.$$

We may suppose, by induction on the length of I and a possible re-indexing, that $I = [1, l]$ and that the result has already been proved for all proper subintervals of $[1, l]$.

When we come to calculate $\|(\|y_1\|, \ldots, \|y_l\|)\|_{a,b}$, there are two possibilities, the first case being where this norm equals $\|y_j\|$ for some j. By the defining property of the function F, there is some $\gamma \in \Gamma_{F(q_j)}$ with

$$|y_j(\gamma)| > \tfrac{1}{2}\|y_j\|.$$

For $i > j$, $y_i(\gamma) = 0$ by F-admissibility, and so

$$\Big|\sum_{i=1}^{l} y_i(\gamma)\Big| = \Big|\sum_{i=1}^{j} y_i(\gamma)\Big|.$$

Now if this quantity is at least $\frac{1}{6}\|y_j\|$, we are home. Otherwise, it must be that

$$\Big\|\sum_{i=1}^{j-1} y_i\Big\| \ge \Big|\sum_{i=1}^{j-1} y_i(\gamma)\Big| \ge |y_j(\gamma)| - \Big|\sum_{i=1}^{j} y_i(\gamma)\Big| > \Big(\frac{1}{2} - \frac{1}{6}\Big)\|y_j\| = \frac{1}{3}\|y_j\|.$$

Now we see that there exists $\delta \in \Gamma_{F(q_{j-1})}$ such that

$$\Big|\sum_{i=1}^{l} y_i(\delta)\Big| = \Big|\sum_{i=1}^{j-1} y_i(\delta)\Big| \ge \frac{1}{2}\Big\|\sum_{i=1}^{j-1} y_i\Big\| \ge \frac{1}{6}\|y_j\|.$$

We now pass to the second case, where there is some k such that

$$\|(\|y_1\|, \ldots, \|y_l\|)\|_{a,b} = a\|(\|y_1\|, \ldots, \|y_k\|)\|_{a,b} + b\|(\|y_{k+1}\|, \ldots, \|y_l\|)\|_{a,b}.$$

By our inductive hypothesis, there exist $\xi \in \Gamma_{F(q_k)+k-1}$ and $\eta \in \Gamma_{F(q_l)+l-k-1}$ such that

$$\Big|\sum_{i=1}^{k} y_i(\xi)\Big| > \frac{1}{6}\|(\|y_1\|, \ldots, \|y_k\|)\|_{a,b},$$

$$\Big|\sum_{i=k+1}^{l} y_i(\eta)\Big| > \frac{1}{6}\|(\|y_{k+1}\|, \ldots, \|y_l\|)\|_{a,b}.$$

If we now consider the element

$$\gamma = (F(q_l) + l - k, F(q_k) + k - 1, \xi, \eta, \pm 1)$$

of $\Gamma_{F(q_l)+l-k}$ (with an appropriate choice of sign), we see that

$$\begin{aligned}\Big|\sum_{i=1}^{k} y_i(\gamma)\Big| &= a\Big|\sum_{i=1}^{k} y_i(\xi)\Big| + b\Big|\sum_{i=k+1}^{l} y_i(\eta)\Big| \\ &> \tfrac{1}{6}a\|(\|y_1\|, \ldots, \|y_k\|)\|_{a,b} + \tfrac{1}{6}b\|(\|y_{k+1}\|, \ldots, \|y_l\|)\|_{a,b} \\ &= \tfrac{1}{6}\|(\|y_1\|, \ldots, \|y_l\|)\|_{a,b}.\end{aligned}$$

It is also shown in [4] that, for certain carefully chosen admissible sequences, there is an upper estimate as well. This is the way in which Bourgain and Delbaen show that $X_{a,b}$ is not isomorphic to $X_{a,b'}$ if $b \neq b'$ (and then deduce the existence of a continuum of non-isomorphic separable $\mathcal{L}^\infty$-spaces). Of course we can now see that these special sequences are ℓ^p-bases.

We shall shortly show that from every admissible sequence we can form a normalized sequence of successive *linear combinations* which is an ℓ^p-basis. Before going on to that, however, let us note that not every normalized admissible sequence is itself an ℓ^p basis. We note that the same calculation shows that $X_{a,b}$ is not an asymptotic ℓ^p-space in the sense of [10].

PROPOSITION 2. *Let $F : \mathbb{N} \to \mathbb{N}$ be any function. For all positive integers m and k there exist an F-admissible sequence $y_1, \ldots, y_{2^k-1}$ of successive vectors in $X_{a,b}$, with $\min \operatorname{supp} y_1 > m$, such that*

$$\sum_{j=1}^{2^k-1} \|y_j\|^p = 1, \qquad \Big\| \sum_{j=1}^{2^k-1} y_j \Big\| \geq k^{1/p'}.$$

Proof. We shall prove the statement by induction on k and shall show, moreover, that the construction may be carried out in such a way that the vector $\sum_{j=1}^{2^k-1} y_j$ attains a value of at least $k^{1/p'}$ at some element ξ of Γ. The construction will use some special vectors w_γ ($\gamma \in \Gamma$) which we shall now define. For each $\gamma \in \Gamma$ we set $n = \operatorname{rank}(\gamma)$ and let e_γ be the usual unit vector in $\ell^\infty(\Gamma_n)$ defined by $e_\gamma(\delta) = 1$ if $\delta = \gamma$ and 0 otherwise. We then define $w_\gamma = j_n(e_\gamma)$, noting that $\|w_\gamma\| = 1$, by the remark we made at the end of the last section.

We now pass to the inductive proof. For $k = 1$ there is of course no real problem, but in order to be sure about attainment of the norm, we might as well be specific, taking y_1 to be w_γ with $\operatorname{rank}(\gamma)$ sufficiently large.

Now suppose that the result is true for k. Given m there exist successive, F-admissible vectors $y'_1, \ldots, y'_{2^k-1}$, with $\min \operatorname{supp} y'_1 > m$, together with an element ξ' of Γ such that

$$\sum_{j=1}^{2^k-1} \|y'_j\|^p = 1, \qquad \sum_{j=1}^{2^k-1} y'_j(\xi') \geq k^{1/p'}.$$

We now use our inductive hypothesis again to obtain $y''_1, \ldots, y''_{2^k-1}$ and ξ'' satisfying the same conditions, and with

$$\min \operatorname{supp} y''_1 > \max\{\operatorname{rank}(\xi'),\ F(\max \operatorname{supp} y'_{2^k-1}) + 2^k - 1\}.$$

We choose $n > \max\{\operatorname{rank}(\xi''), F(\max \operatorname{supp} y''_{2^k-1}) + 2^{k+1} - 2\}$ and take $\xi \in \Delta_n$ to be

$$\xi = (n, \operatorname{rank}(\xi'), \xi', \xi'', 1).$$

Finally, we define $y_1, \ldots, y_{2^{k+1}-1}$ by

$$y_j = \begin{cases} \dfrac{a^{p'-1}k^{1/p}}{(k+1)^{1/p}} y'_j & (1 \leq j \leq 2^k - 1), \\[2mm] \dfrac{b^{p'-1}k^{1/p}}{(k+1)^{1/p}} y''_{j-2^k+1} & (2^k \leq j \leq 2^{k+1} - 2), \\[2mm] (k+1)^{-1/p} w_\xi & (j = 2^{k+1} - 1). \end{cases}$$

By construction, the sequence $y_1, \ldots, y_{2^{k+1}-1}$ is F-admissible, and

$$\sum_{j=1}^{2^{k+1}-1} \|y_j\|^p = \frac{ka^{p'}}{k+1} \sum_{j=1}^{2^k-1} \|y'_j\|^p + \frac{kb^{p'}}{k+1} \sum_{j=1}^{2^k-1} \|y''_j\|^p + \frac{\|w_\xi\|}{k+1} = \frac{ka^{p'} + kb^{p'} + 1}{k+1} = 1.$$

When we evaluate at ξ we obtain

$$\begin{aligned} \sum_{j=1}^{2^k-1} y_j(\xi) &= a\frac{a^{p'-1}k^{1/p}}{(k+1)^{1/p}} \sum_{j=1}^{2^k-1} y'_j(\xi') \\ &\quad + b\frac{b^{p'-1}k^{1/p}}{(k+1)^{1/p}} \sum_{j=1}^{2^k-1} y''_j(\xi'') + \frac{1}{(k+1)^{1/p}} \\ &\geq \frac{a^{p'}k^{1/p}}{(k+1)^{1/p}} k^{1/p'} + \frac{b^{p'}k^{1/p}}{(k+1)^{1/p}} k^{1/p'} + \frac{1}{(k+1)^{1/p}} \\ &\geq \frac{(a^{p'} + b^{p'})k}{(k+1)^{1/p}} + \frac{1}{(k+1)^{1/p}} = (k+1)^{1/p'}. \end{aligned}$$

COROLLARY. *There exist normalized F-admissible sequences that are not equivalent to the usual ℓ^p-basis.*

Proof. It is clear that such sequences may be constructed by normalizing and sticking together finite sequences of the kind obtained in Proposition 2.

In view of what we have just seen it is clear that we shall have to work a bit harder in order to find ℓ^p-bases in $X_{a,b}$. We shall start with an arbitrary normalized F-admissible sequence (y_n) and then form further linear combinations. As a piece of temporary terminology, we shall say that a vector x has *height* h, and write $h(x) = h$, if x is a linear combination

$$x = \sum_{l=m}^{n} \alpha_l y_l$$

with $h = \max_l |\alpha_l|$. When I is a non-empty finite interval of integers, we shall write I^* for the subinterval obtained by removing the end-points of I: thus $I^* = I \setminus \{\max I, \min I\}$.

PROPOSITION 3. *There is a constant $c > 0$, depending only on a and b, with the following property: for any normalized F-admissible sequence (y_n), any sequence (x_r) of successive linear combinations, any finite interval I and any $\gamma \in \Gamma$, we have*

$$(1)\qquad c\Big\|\sum_{i\in I} x_i\Big\| \le \Big(\|x_{\min I}\|^p + 2\sum_{i\in I^*} 6^p\|x_i\|^p + \|x_{\max I}\|^p\Big)^{1/p} + 4\sum_{i\in I^*} h(x_i) + 3h(x_{\max I});$$

moreover, for γ with $\operatorname{rank}(\gamma) > \max\operatorname{supp} x_{\max I}$, we have

$$(2)\qquad c\Big|\sum_{i\in I} x_i(\gamma)\Big| \le \Big(\|x_{\min I}\|^p + 2\sum_{i\in I^*} 6^p\|x_i\|^p + \|x_{\max I}\|^p\Big)^{1/p} + 4\sum_{i\in I^*} h(x_i) + \frac{3}{2}h(x_{\max I}).$$

In fact, the constant c may be taken to be whichever is smaller of b and $2^{-1/p'}a$.

Proof. We proceed by induction on the length of the interval I, assuming that (1) and (2) hold for all sequences of successive linear combinations of the y_j, and all intervals shorter than I. (Of course, the case of an interval containing only one natural number is trivial.) For convenience, we shall take I to be the interval $[1, l]$; let us write x for the sum $\sum_{i=1}^{l} x_i$. We consider an arbitrary $\gamma \in \Gamma$; our aim is to show that

$$(1')\qquad c|x(\gamma)| \le \Big(\|x_1\|^p + 2\sum_{i=2}^{l-1} 6^p\|x_i\|^p + \|x_l\|^p\Big)^{1/p} + 4\sum_{i=2}^{l-1} h(x_i) + 3h(x_l),$$

with

$$(2)\qquad c|x(\gamma)| \le \Big(\|x_1\|^p + 2\sum_{i=2}^{l-1} 6^p\|x_i\|^p + \|x_l\|^p\Big)^{1/p} + 4\sum_{i=2}^{l-1} h(x_i) + \frac{3}{2}h(x_l);$$

in the special case where $\operatorname{rank}(\gamma) > \max\operatorname{supp} x_l$.

We may assume that $\operatorname{rank}(\gamma) \ge \min\operatorname{supp} x_l$. Indeed, otherwise we have $x(\gamma) = \sum_{j=1}^{l-1} x_j(\gamma)$ and our inductive hypothesis may be applied. This assumption about the rank of γ will be useful since it will allow us to apply Proposition 1 to vectors like $\sum_{j=1}^{l-1} x_j$.

Let us now write $k = \operatorname{cut}(\gamma)$; we shall deal first with the two cases $k < \min\operatorname{supp} x_2$ and $k > \max\operatorname{supp} x_{l-1}$. In the first of these cases, we may estimate $|x(\gamma)|$ as follows:

$$\begin{aligned}
|x(\gamma)| &\le \|x_1\| + \Big|\sum_{i=2}^{l-1} x_i(\gamma)\Big| + \|x_l\| \\
&\le \|x_1\| + \|x_l\| + a\Big\|S_k\Big(\sum_{i=2}^{l-1} x_i\Big)\Big\| + b\Big\|(I - S_k)\Big(\sum_{i=2}^{l-1} x_i\Big)\Big\| \quad \text{(by Prop. 1)} \\
&= \|x_1\| + \|x_l\| + b\Big\|\sum_{i=2}^{l-1} x_i\Big\|.
\end{aligned}$$

Now the interval $[2, l-1]$ is one to which our inductive hypothesis is applicable, so that we obtain

$$\begin{aligned}
c|x(\gamma)| &\le c\|x_1\| + c\|x_l\| + b\Big(\|x_2\|^p + 2\sum_{i=3}^{l-2} 6^p\|x_i\|^p + \|x_{l-1}\|^p\Big)^{1/p} \\
&\quad + 4b\sum_{i=3}^{l-2} h(x_i) + 3bh(x_{l-1}) \\
&\le (2c^{p'} + b^{p'})^{1/p'}\Big(\|x_1\|^p + \|x_2\|^p + 2\sum_{i=3}^{l-2} 6^p\|x_i\|^p + \|x_{l-1}\|^p + \|x_l\|^p\Big)^{1/p} \\
&\quad + 4b\sum_{i=3}^{l-1} h(x_i),
\end{aligned}$$

by Hölder's inequality. Comparing terms and recalling that $b < 1/2$, we see that this implies inequality (2), provided that $2c^{p'} + b^{p'} \le 1$, or equivalently $c \le 2^{-1/p'}(1 - b^{p'})^{1/p'} = 2^{-1/p'}a$.

The argument in the case $k > \max\operatorname{supp} x_{l-1}$ is similar:

$$\begin{aligned}
c|x(\gamma)| &\le c\|x_l\| + c\Big|\sum_{i=1}^{l-1} x_i(\gamma)\Big| \\
&\le c\|x_l\| + ac\Big\|S_k\Big(\sum_{i=1}^{l-1} x_i\Big)\Big\| + bc\Big\|(I - S_k)\Big(\sum_{i=1}^{l-1} x_i\Big)\Big\| \quad \text{(by Prop. 1)} \\
&= c\|x_l\| + ac\Big\|\sum_{i=1}^{l-1} x_i\Big\| \\
&\le c\|x_l\| + a\Big(\|x_1\|^p + 2\sum_{i=2}^{l-2} 6^p\|x_i\|^p + \|x_{l-1}\|^p\Big)^{1/p} \\
&\quad + 4a\sum_{i=2}^{l-2} h(x_i) + 3ah(x_{l-1}),
\end{aligned}$$

$$\leq (c^{p'}+a^{p'})^{1/p'}\Big(\|x_1\|^p+2\sum_{i=2}^{l-2}6^p\|x_i\|^p+\|x_{l-1}\|^p+\|x_l\|^p\Big)^{1/p} + 4\sum_{i=2}^{l-1}h(x_i),$$

which implies inequality (2) provided $c^{p'}+a^{p'}\leq 1$ or equivalently $c\leq b$.

From now on, we shall assume that $\min\operatorname{supp}x_2\leq k=\operatorname{cut}(\gamma)\leq\max\operatorname{supp}x_{l-1}$. We consider next the case where $\operatorname{rank}(\gamma)>\max\operatorname{supp}x_l$ and need to establish inequality (2). An easy case is where the cut k lies between the supports of consecutive x_i's, say $\max\operatorname{supp}x_{i^*}<k<\min\operatorname{supp}x_{i^*+1}$ (where $2\leq i^*\leq l-2$ by what we have just proved). By our inductive hypothesis, we have inequality (1) for each of the intervals $[1,i^*]$ and $[i^*+1,l]$. Moreover, Proposition 1 is applicable, giving

$$\begin{aligned}c|x(\gamma)| &\leq ca\|S_kx\|+cb\|(I-S_k)x\| = ac\Big\|\sum_{i\leq i^*}x_i\Big\|+bc\Big\|\sum_{i>i^*}x_i\Big\|\\ &\leq a\Big[\Big(\|x_1\|^p+2\sum_{1<i<i^*}6^p\|x_i\|^p+\|x_{i^*}\|^p\Big)^{1/p}+4\sum_{1<i<i^*}h(x_i)+3h(x_{i*})\Big]\\ &\quad+b\Big[\Big(\|x_{i^*+1}\|^p+2\sum_{i^*+1<i<l}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}\\ &\quad+4\sum_{i^*+1<i<l}h(x_i)+3h(x_l)\Big]\\ &\leq\Big(\|x_1\|^p+2\sum_{1<i<i^*}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}+4\sum_{1<i<l}h(x_i)+\frac{3}{2}h(x_l),\end{aligned}$$

by Hölder's inequality and the facts that $a<1$, $b<1/2$.

A slightly more complicated case arises if $\min\operatorname{supp}x_i\leq k\leq\max\operatorname{supp}x_i$ for some $i=i^*$, say. By what we proved earlier, it must be that $1<i^*<l$. We now study the fine structure of the vector x_{i^*}, recalling that

$$x_{i^*}=\sum_{n_{i^*-1}<j\leq n_{i^*}}\alpha_jy_j.$$

We may suppose that k is somewhere between $\min\operatorname{supp}y_{j^*}$ and $\max\operatorname{supp}y_{j^*}$, for some j^*. We then set

$$x^{\rm L}_{i^*}=\sum_{n_{i^*-1}<j<j^*}\alpha_jy_j,\quad x^{\rm R}_{i^*}=\sum_{j^*<j\leq n_{i^*}}\alpha_jy_j,$$
$$x^{\rm L}=x_1+x_2+\ldots+x_{i^*-1}+x^{\rm L}_{i^*},\quad x^{\rm R}=x^{\rm R}_{i^*}+x_{i^*+1}+\ldots+x_l.$$

By minimality of l and the fact that $1<i^*<l$, inequality (1) is true for the vectors $x^{\rm R}$ and $x^{\rm L}$. Hence we have

$$\begin{aligned}c\|x^{\rm L}\|&\leq\Big(\|x_1\|^p+2\sum_{1<i<i^*}6^p\|x_i\|^p+\|x^{\rm L}_{i^*}\|^p\Big)^{1/p}+4\sum_{1<i<i^*}h(x_i)+3h(x^{\rm L}_{i^*})\\ &\leq\Big(\|x_1\|^p+2\sum_{1<i<i^*}6^p\|x_i\|^p+\|x^{\rm L}_{i^*}\|^p\Big)^{1/p}+4\sum_{1<i<i^*}h(x_i)+3h(x_{i^*}),\end{aligned}$$

since $h(x^{\rm L}_{i^*})\leq h(x_{i^*})$ by the definition of the function h, and

$$c\|x^{\rm R}\|\leq\Big(\|x^{\rm R}_{i^*}\|^p+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}+4\sum_{i^*<i<l}h(x_i)+3h(x_l).$$

If we now write $x^*=x^{\rm L}+x^{\rm R}=x-\alpha_{j^*}y_{j^*}$, and apply Proposition 1, we obtain

$$\begin{aligned}c|x(\gamma)|&\leq c|x^*(\gamma)|+c|\alpha_{j^*}|\leq ac\|S_kx^*\|+bc\|(I-S_k)x^*\|+ch(x_{i^*})\\ &=ac\|x^{\rm L}\|+bc\|x^{\rm R}\|+ch(x_{i^*})\\ &\leq a\Big[\Big(\|x_1\|^p+2\sum_{1<i<i^*}6^p\|x_i\|^p+\|x^{\rm L}_{i^*}\|^p\Big)^{1/p}+4\sum_{1<i<i^*}h(x_i)+3h(x_{i^*})\Big]\\ &\quad+b\Big[\Big(\|x^{\rm R}_{i^*}\|^p+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}\\ &\quad+4\sum_{i^*<i<l}h(x_i)+3h(x_l)\Big]+ch(x_{i^*})\\ &\leq\Big(\|x_1\|^p+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x^{\rm L}_{i^*}\|^p+\|x^{\rm R}_{i^*}\|^p\\ &\quad+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}\\ &\quad+4\sum_{1<i<i^*}h(x_i)+(3+c)h(x_{i^*})+4\sum_{i^*<i<l}h(x_i)+3bh(x_l),\end{aligned}$$

using Hölder's inequality and the values of a and b as before. Lemma 2, applied to the admissible sequence $(x^{\rm L}_{i^*},\alpha_jy_j,x^{\rm R}_{i^*})$, implies that each of $\|x^{\rm L}_{i^*}\|$ and $\|x^{\rm R}_{i^*}\|$ is at most $6\|x_{i^*}\|$ so that we can finally write

$$c|x(\gamma)|\leq\Big(\|x_1\|^p+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x_l\|^p\Big)^{1/p}+4\sum_{1<i<l}h(x_i)+\frac{3}{2}h(x_l),$$

which is inequality (2) as required. (Of course, we have also used the facts that $b<1/2$ and $3+c<4$.)

To finish the proof, we now need to look at $|x(\gamma)|$ where $\operatorname{rank}(\gamma)\leq\max\operatorname{supp}x_l$ and show that inequality (1′) holds. We do this by another induction, this time on the number n_l-n_{l-1} of non-zero coefficients in the expression for the last vector x_l as a linear combination of the y_j. We set

$$x_l^* = \sum_{n_{l-1}<j<n_l} \alpha_j y_j = x_l - \alpha_{n_l} y_{n_l}, \qquad x^* = \sum_{1\le i<l} x_i + x_l^* = x - \alpha_{n_l} y_{n_l}.$$

Our additional inductive hypothesis is applicable to x^*, and if $\operatorname{rank}(\gamma) < \min\operatorname{supp} y_{n_l}$ we have $x(\gamma) = x^*(\gamma)$, giving the result immediately. If, on the other hand, $\operatorname{rank}(\gamma) \ge \min\operatorname{supp} y_{n_l} > \max\operatorname{supp} x^*$, it is inequality (2) which holds for x^*. Thus we obtain

$$\begin{aligned} c|x(\gamma)| &\le c|x^*(\gamma)| + c|\alpha_{n_l} y_{n_l}(\gamma)| \\ &\le \Big(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l^*\|^p\Big)^{1/p} \\ &\qquad + 4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l^*) + c|\alpha_{n_l}| \\ &\le \Big(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\Big)^{1/p} + \|x_l^* - x_l\| \\ &\qquad + 4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l^*) + c|\alpha_{n_l}| \quad \text{(by Minkowski's inequality)} \\ &\le \Big(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\Big)^{1/p} \\ &\qquad + 4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l^*) + (c+1)|\alpha_{n_l}| \\ &\le \Big(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\Big)^{1/p} + 4\sum_{1<i<l} h(x_i) + 3h(x_l) \end{aligned}$$

since $c < 1/2$ and $h(x_l) = \max\{|\alpha_{n_l}|, h(x_l^*)\}$. We have thus established inequality (1′) as required.

THEOREM 2. *Let a, b be real constants satisfying $0 < b < 1/2 < a < 1$, $a + b > 1$ and let p, p' be given by $1/p + 1/p' = 1 = a^{p'} + b^{p'}$. Every closed infinite-dimensional subspace of $X_{a,b}$ has a subspace isomorphic to ℓ^p.*

Proof. By a standard approximation argument, it is enough to consider the case of a subspace Y which is the closed linear span of a normalized F-admissible sequence (y_j). Because of the lower estimates of Lemma 2 and Theorem 1, we may construct successive linear combinations z_i with $\|z_i\| = 1$ and $h(z_i)$ very small, say $\sum_{i=1}^\infty h(z_i) < 1$. Now, for arbitrary $l \in \mathbb{N}$ and arbitrary scalars β_i, we may apply the above proposition to the vectors

$x_i = \beta_i z_i$, obtaining

$$c\Big\|\sum_{i=1}^l \beta_i z_i\Big\| \le 12\Big(\sum_{i=1}^l |\beta_i|^p\Big)^{1/p} + 4\sum_{i=1}^l |\beta_i| h(z_i) \le 16\Big(\sum_{i=1}^l |\beta_i|^p\Big)^{1/p}.$$

On the other hand, from Lemma 2 and Theorem 1 again, we get the lower estimate

$$\Big\|\sum_{i=1}^l \beta z_i\Big\| \ge \frac{1}{6}\|(\beta_1, \ldots, \beta_l)\|_{a,b} \ge d\Big(\sum_{i=1}^l |\beta_i|^p\Big)^{1/p},$$

where d is a strictly positive constant.

References

[1] D. Alspach, *The dual of the Bourgain–Delbaen space*, preprint, Oklahoma State Univ., 1998.

[2] S. M. Bates, *On smooth non-linear surjections of Banach spaces*, Israel J. Math. 100 (1997), 209–220.

[3] M. Boddington, *On a certain class of recursively defined norms*, in preparation.

[4] J. Bourgain, *New Classes of $\mathcal{L}^p$-Spaces*, Lecture Notes in Math. 889, Springer, Berlin, 1981.

[5] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}^\infty$-spaces*, Acta Math. 145 (1980), 155–176.

[6] G. Godefroy, N. J. Kalton and G. Lancien, *Szlenk indices and uniform homeomorphisms*, preprint.

[7] P. Hájek, *Smooth functions on c_0*, Israel J. Math. 104 (1998), 17–28.

[8] W. B. Johnson, *A uniformly convex Banach space which contains no ℓ_p*, Compositio Math. 29 (1974), 179–190.

[9] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. 6 (1996), 430–470.

[10] B. Maurey, V. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, in: Oper. Theory Adv. Appl. 77, Birkhäuser, Basel, 1995, 149–175.

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