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School of Mathematical Sciences
Peking University
Beijing 100871, China

Institute of Analysis
TU-Braunschweig
PK 14(Forum)
D-38106 Braunschweig, Germany
E-mail: xiao@badbit.math2.nat.tu-bs.de

Received January 1, 1999
Revised version September 23, 1999 (4239)

STUDIA MATHEMATICA 139 (3) (2000)

A geometrical solution of a problem on wavelets

by

ANTOINE AYACHE (Toulouse)

Abstract. We prove the existence of nonseparable, orthonormal, compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity by using some basic techniques of algebraic and differential geometry. We even obtain a much stronger result: "most" of the orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$, of any regularity, are nonseparable.

1. Introduction. A *wavelet basis* for $L^2(\mathbb{R}^d)$ is an orthonormal basis of the type $\{2^{jd/2}\psi_i(2^jx-k) \mid i=1,\ldots,2^d-1,\ j\in\mathbb{Z} \text{ and } k\in\mathbb{Z}^d\}$. It can generally be obtained from a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subsets of $L^2(\mathbb{R}^d)$ called a *multiresolution analysis* because it has the following properties:

(a) $\bigcap_{j\in\mathbb{Z}} V_j=\{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j}=L^2(\mathbb{R}^2)$,

(b) $V_j\subset V_{j+1}$ for all j,

(c) there exists a function $\varphi(x)$, called the *scaling function*, that belongs to V_0 and such that $\{\varphi(x-k)\mid k\in\mathbb{Z}^d\}$ is an orthonormal basis for V_0 [Le, D, M].

The wavelets that this paper deals with are both compactly supported and generated by multiresolution analyses.

We will say that a wavelet basis is *separable* if the functions ψ_i may be written as products of monodimensional scaling functions and monodimensional wavelets.

There exists a one-to-one correspondence between the wavelet bases for $L^2(\mathbb{R}^d)$ and the filter banks that satisfy Cohen–Lawton's condition. More precisely, the Fourier transforms of the functions φ and $\psi_1,\ldots,\psi_{2^d-1}$ are given by

$$\widehat{\varphi}(\xi)=\prod_{k=1}^{\infty} M_0(2^{-k}\xi), \tag{1.1}$$

$$\widehat{\psi}_i(\xi)=M_i(2^{-1}\xi)\widehat{\varphi}(2^{-1}\xi), \tag{1.2}$$

2000 *Mathematics Subject Classification*: Primary 42A16.

where the trigonometric polynomials $M_0(\xi), \ldots, M_{2^d-1}(\xi)$ form a d-dimensional *filter bank*, i.e.

$$M_0(0) = 1, \tag{1.3}$$

and for all $k, l \in \{0, \ldots, 2^d - 1\}$,

$$\sum_{\nu \in \{0,1\}^d} M_k(\xi + \pi\nu)\overline{M_l(\xi + \pi\nu)} = \delta_{k,l}. \tag{1.4}$$

Thus, our problem and many other problems on wavelets can be set in the filter banks framework.

We will say that a filter bank *has* $L \geq 1$ *vanishing moments* if all the partial derivatives of order $\leq L-1$ of $M_0(\xi)$ vanish at the points of the type $\pi\nu$ where $\nu \in \{0,1\}^d$ and $\nu \neq (0, \ldots, 0)$.

From now on and unless otherwise mentioned, we will only be concerned with bidimensional filter banks. $V_{N,L}$ will be the set of filter banks $\{M_j(\xi)\}_{0 \leq j \leq 3}$ with L vanishing moments such that

$$M_j(\xi) = \sum_{k \in \{0,\ldots,2N-1\}^2} c_j(k)e^{-ik\cdot\xi}$$

where $c_j(k) \in \mathbb{R}$. *We will always identify* $\{M_j(\xi)\}_{0 \leq j \leq 3}$ *with the sequence of coefficients* $(c_j(k))$, $j \in \{0, \ldots, 3\}$ *and* $k \in \{0, \ldots, 2N-1\}^2$. Thus it results from (1.3) and (1.4) that $V_{N,L}$ is an algebraic set in $\mathbb{R}^{16N^2}$, i.e. the set of common zeros of a family of polynomials in $16N^2$ variables.

This paper is organized as follows.

In Section 2, we introduce the algebraic geometry results that we need in this paper.

In Section 3, we show the following results. The separable filter banks in $V_{N,L}$ form an algebraic subset $T_{N,L}$. The filter banks in $V_{N,L}$ that do not satisfy Cohen–Lawton's condition form an algebraic subset $W_{N,L}$, and the filter banks in $V_{N,L}$ that generate wavelet bases with exactly L vanishing moments and with critical Sobolev exponent $> \alpha$ form a Euclide open subset $S^{\alpha}_{N,L}$ of $V_{N,L}$. Thus, to establish the existence of nonseparable, compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity, it is sufficient to show that for any $\alpha > 0$, there exist N and L such that $S^{\alpha}_{N,L}$ is not contained in $T_{N,L}$.

In Section 4, $P_{M,L}$ is the set of couples $(\lambda(x), \mu(x))$ of monovariate π-periodic trigonometric polynomials $\lambda(x) = \sum_{k=0}^{M} a(k)e^{-i2kx}$ and $\mu(x) = \sum_{k=0}^{M} b(k)e^{-i2kx}$ with real coefficients that satisfy

$$\lambda(0) = 1, \quad |\lambda(x)|^2 + |\mu(x)|^2 = 1, \tag{1.5}$$

and

$$\mu'(0) = \ldots = \mu^{(L-1)}(0) = 0. \tag{1.6}$$

We will always identify the couple $(\lambda(x), \mu(x))$ *with the sequence* (a, b) *of coefficients of* $\lambda(x)$ *and of* $\mu(x)$. Thus it results from (1.5) and (1.6) that $P_{M,L}$ is an algebraic set in $\mathbb{R}^{2(M+1)}$.

First, we show that there exists a polynomial isomorphism f between $P_{M,L}$ and the algebraic subset of $V_{2M+N,L}$ consisting of the filter banks of the type

$$\begin{cases} M_0(\xi_1, \xi_2) = e^{-i2M\xi_1}[\lambda(\xi_1)S_{00}(\xi_1, \xi_2) + \mu(\xi_1)S_{11}(\xi_1, \xi_2)], \\ M_1(\xi_1, \xi_2) = e^{-i2M\xi_1}[\overline{\lambda(\xi_1)}S_{11}(\xi_1, \xi_2) - \overline{\mu(\xi_1)}S_{00}(\xi_1, \xi_2)], \\ M_2(\xi_1, \xi_2) = e^{-i2M\xi_1}[\lambda(\xi_1)S_{10}(\xi_1, \xi_2) + \mu(\xi_1)S_{01}(\xi_1, \xi_2)], \\ M_3(\xi_1, \xi_2) = e^{-i2M\xi_1}[\overline{\lambda(\xi_1)}S_{01}(\xi_1, \xi_2) - \overline{\mu(\xi_1)}S_{10}(\xi_1, \xi_2)], \end{cases} \tag{1.7}$$

where $\{S_{kl}(\xi_1, \xi_2)\}_{0 \leq k,l \leq 1}$ is an $\Omega = (\omega_{ij})$-separable ($\Omega \in \mathrm{SL}(2, \mathbb{Z})$) filter bank in $S^{\alpha}_{N,L}$; this means that it satisfies

$$S_{kl}(\xi_1, \xi_2) = a_k(\omega_{11}\xi_1 + \omega_{12}\xi_2)b_l(\omega_{21}\xi_1 + \omega_{22}\xi_2), \tag{1.8}$$

$\{a_k(x)\}_{0 \leq k \leq 1}$ and $\{b_k(x)\}_{0 \leq k \leq 1}$ being 2 monodimensional filter banks. We set

$$\sigma_0 = \{e^{-i2M\xi_1}S_{kl}(\xi_1, \xi_2)\}. \tag{1.9}$$

It is an Ω-separable filter bank of $V_{2M+N,L}$. Then we show that the couple $(\lambda_0(x), \mu_0(x))$, where $\lambda_0 \equiv 1$ and $\mu_0 \equiv 0$, is a nonsingular point of $P_{M,L}$ of dimension $M+1-L$. Since $f(\lambda_0, \mu_0) = \sigma_0$, there exists an irreducible algebraic subset $\widetilde{V}_{2M+N,L} \subset V_{2M+N,L}$ of dimension $M+1-L$ such that all the filter banks in $\widetilde{V}_{2M+N,L}$ are of the type (1.7) and σ_0 is a nonsingular point of $\widetilde{V}_{2M+N,L}$.

Finally, we show that there exists a real-analytic manifold $\widetilde{\mathcal{V}}_{2M+N,L} \subset S^{\alpha}_{2M+N,L}$ of dimension $M+1-L$ such that σ_0 is the only separable filter bank in $\widetilde{\mathcal{V}}_{2M+N,L}$. This, roughly, means that "most" of the orthonormal, compactly supported wavelet bases of arbitrary regularity are nonseparable.

2. Some preliminary results of algebraic geometry. All the results that we state in this section and their proofs can be found in [AK].

Definition 2.1. An *algebraic set* in $\mathbb{R}^n$ is any subset of $\mathbb{R}^n$ of the form

$$V(J) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) = 0, \ \forall f \in J\},$$

where J is a subset of $\mathbb{R}[X_1, \ldots, X_n]$, the ring of polynomials in n variables with real coefficients.

We obviously have:

(a) $I \subset J \Rightarrow V(I) \supset V(J)$,
(b) $V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$,

(c) $V(IJ) = V(I) \cup V(J)$ (where $IJ = \{fg \mid f \in I \text{ and } g \in J\}$).

DEFINITION 2.2. We say that an algebraic set is *irreducible* if it cannot be written as the union of two proper algebraic subsets.

PROPOSITION 2.3. *Every algebraic set V can be uniquely written as $V = \bigcup_{i=1}^m V_i$, where each V_i is an irreducible algebraic subset and no V_i is contained in another V_j. We say that the algebraic subsets V_i are the irreducible components of V.*

DEFINITION 2.4. Let $V \subset \mathbb{R}^n$ be an algebraic set. We say $x \in V$ is *nonsingular* of dimension d in V if there exists a neighborhood U of x in $\mathbb{R}^n$ and $n-d$ polynomials $f_1, \ldots, f_{n-d}$ in n variables such that:

(1) $U \cap V = U \cap \bigcap_{i=1}^{n-d} f_i^{-1}(0)$,
(2) the gradients $\nabla f_i(x)$ for $i = 1, \ldots, n-d$ are linearly independent.

DEFINITION 2.5. Let $\mathcal{V} \subset \mathbb{R}^n$. We say that $\mathcal{V}$ is a *real-analytic manifold* (resp. a *differentiable manifold*) of dimension d if for all $x \in \mathcal{V}$, there exists a neighborhood U of x in $\mathbb{R}^n$ and $n-d$ real-analytic functions (resp. $n-d$ differentiable functions) such that:

(1) $U \cap V = U \cap \bigcap_{i=1}^{n-d} f_i^{-1}(0)$,
(2) the gradients $\nabla f_i(x)$ for $i = 1, \ldots, n-d$ are linearly independent.

It is clear that any real-analytic manifold is a differentiable manifold.

The implicit function theorem allows us to locally identify every d-dimensional differentiable manifold with an open subset of $\mathbb{R}^d$.

When a point x of an algebraic set V of $\mathbb{R}^n$ is nonsingular of dimension d, there exists an open neighborhood O of x in $\mathbb{R}^n$ such that $V \cap O$ is a d-dimensional real-analytic manifold of $\mathbb{R}^n$.

DEFINITION 2.6. The *dimension* of an algebraic set V is the largest d so that V has a point which is nonsingular of dimension d. We always have $\dim V \le n$.

PROPOSITION 2.7. (i) *The dimension of an algebraic set vanishes if and only if this algebraic set is finite.*

(ii) *Let W be a proper algebraic subset of an irreducible algebraic set V. Then $\dim W < \dim V$.*

By a proper subset of a set D we mean any part of D which is not equal to D.

DEFINITION 2.8. A point of an algebraic set is said to be *nonsingular* if it is nonsingular of dimension $\dim V$.

PROPOSITION 2.9. *Let V be an algebraic set and let x be a point of V. Then x is nonsingular of dimension d if and only if x is contained in exactly one irreducible component S of V, $\dim S = d$ and x is a nonsingular point of S.*

DEFINITION 2.10. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be two algebraic sets. A function $f : V \to W$ is a *polynomial morphism* if there is a polynomial function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(V) \subset W$ and $g|_V = f$. When the function $f^{-1} : W \to V$ exists and is also a polynomial morphism, we say that f is a *polynomial isomorphism.*

PROPOSITION 2.11. *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be two algebraic sets and let $f : V \to W$ be a polynomial isomorphism. Then:*

(i) *$R \subset V$ is an irreducible algebraic subset if and only if $f(R) \subset W$ is an irreducible algebraic subset,*

(ii) *$x \in V$ is a nonsingular point of dimension d if and only if $f(x) \in W$ is a nonsingular point of dimension d.*

3. A geometrical formulation of our problem

DEFINITION 3.1. $V_{N,L}$ is the algebraic set of 4-uples $\{M_j(\xi)\}_{0\le j\le 3}$ of bivariate trigonometric polynomials of the form

$$M_j(\xi) = \sum_{k\in\{0,\ldots,2N-1\}^2} c_j(k) e^{-ik\cdot\xi}$$

with $c_j(k) \in \mathbb{R}$ that satisfy (1.3), (1.4) and such that all partial derivatives of $M_0(\xi)$ of order less than or equal to $L-1$ vanish at all points of the type $\pi\nu$ where $\nu \in \{0,1\}^2$ and $\nu \ne (0,0)$. We say that $\{M_j(\xi)\}_{0\le j\le 3}$ is a *filter bank with L vanishing moments.*

3.1. *Separable filter banks*

DEFINITION 3.2. Let $A = (a_{ij})$ be a fixed matrix in $\mathrm{SL}(2,\mathbb{Z})$. We say that a filter bank $\{M_{kl}(\xi_1,\xi_2)\}_{0\le k,l\le 1}$ is *A-separable* if there exist two monodimensional filter banks $\{r_k(x)\}_{0\le k\le 1}$ and $\{t_l(x)\}_{0\le l\le 1}$ such that

$$M_{kl}(\xi_1,\xi_2) = r_k(a_{11}\xi_1 + a_{12}\xi_2)t_l(a_{21}\xi_1 + a_{22}\xi_2). \tag{3.1}$$

We will say more generally that a filter bank is *separable* if it is A-separable for some matrix A in $\mathrm{SL}(2,\mathbb{Z})$.

It is clear that a wavelet basis is separable modulo the action of $\mathrm{SL}(2,\mathbb{Z})$ if and only if it is generated by a separable filter bank.

PROPOSITION 3.3. *Let $T_{N,L}^A \subset V_{N,L}$ be the subset of A-separable filter banks. Then $T_{N,L}^A$ is an algebraic subset of $V_{N,L}$.*

Proof. Let $\{M_{ij}(\xi_1,\xi_2)\}$ be an A-separable filter bank. It follows from (3.1) that

$$M_{kl}\left(A^{-1}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right)=r_k(\eta_1)t_l(\eta_2).$$

Since $r_k(k\pi)=\pm1$ and $t_l(l\pi)=\pm1$, taking in this last equality $\eta_1=k\pi$ and η_2 arbitrary and then $\eta_2=l\pi$ and η_1 arbitrary we obtain

$$M_{kl}\left(A^{-1}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right)=(-1)^{\varepsilon}M_{kl}\left(A^{-1}\begin{pmatrix}\eta_1\\ l\pi\end{pmatrix}\right)M_{kl}\left(A^{-1}\begin{pmatrix}k\pi\\ \eta_2\end{pmatrix}\right),$$

where $\varepsilon=\pm1$. This last equality shows that a filter bank $\{M_{ij}(\xi_1,\xi_2)\}$ is A-separable if and only if the coefficients of the trigonometric polynomials $M_{ij}(\xi_1,\xi_2)$ are the common zeros of a family of polynomials with $16N^2$ variables.

PROPOSITION 3.4. *Let A and B be two matrices in* $\mathrm{SL}(2,\mathbb{Z})$. *If $T^A_{N,L}\cap T^B_{N,L}\neq\emptyset$, then necessarily $T^A_{N,L}=T^B_{N,L}$.*

Proof. Suppose there exists a filter bank which is both $A=(a_{ij})$-separable and $B=(b_{ij})$-separable. Then there exist four monodimensional filter banks $\{r_k(x)\}$, $\{t_k(x)\}$, $\{u_k(x)\}$, $\{v_k(x)\}$ such that

$$r_0(a_{11}\xi_1+a_{12}\xi_2)t_0(a_{21}\xi_1+a_{22}\xi_2)=u_0(b_{11}\xi_1+b_{12}\xi_2)v_0(b_{21}\xi_1+b_{22}\xi_2).$$

It follows that

$$(*)\qquad r_0(\xi_1)t_0(\xi_2)=u_0(c_{11}\xi_1+c_{12}\xi_2)v_0(c_{21}\xi_1+c_{22}\xi_2),$$

where the matrix $C=(c_{ij})$ is in $\mathrm{SL}(2,\mathbb{Z})$ and $C=BA^{-1}$. Since $|\det C|=1$, at least one of the coefficients c_{11} and c_{12} is odd. If c_{11} is odd, taking in the equality $(*)$ $(\xi_1,\xi_2)=(c_{12}x+\pi,-c_{11}x)$ where x is an arbitrary real, we obtain

$$r_0(c_{12}x+\pi)t_0(-c_{11}x)=0.$$

Therefore $r_0(c_{12}x+\pi)=0$ for all x and consequently $c_{12}=0$. If c_{11} is even then c_{12} is odd and by a similar method we obtain $c_{11}=0$.

Thus c_{11} or c_{12} must vanish and we can show similarly that c_{21} or c_{22} must vanish. Since $C\in\mathrm{SL}(2,\mathbb{Z})$, we have $C=\begin{pmatrix}\varepsilon_1&0\\0&\varepsilon_2\end{pmatrix}$ or $C=\begin{pmatrix}0&\varepsilon_1\\ \varepsilon_2&0\end{pmatrix}$ where $\varepsilon_i=\pm1$ and this implies that $T^A_{N,L}=T^B_{N,L}$. ∎

THEOREM 3.5. *Let $T_{N,L}\subset V_{N,L}$ be the subset of separable filter banks. We have $T_{N,L}=\bigcup_{A\in\mathrm{SL}(2,\mathbb{Z})}T^A_{N,L}$, where only a finite number of $T^A_{N,L}$ are nonempty. Consequently, $T_{N,L}$ is an algebraic subset of $V_{N,L}$.*

Proof. If we show that $T_{N,L}$ is a finite union of $T^A_{N,L}$, then it follows immediately from (c) (see the beginning of Section 2) that $T_{N,L}$ is an algebraic set. To show that $T_{N,L}$ is a finite union of $T^A_{N,L}$ we will use the following lemma. ∎

LEMMA 3.6. *If a filter bank in $V_{N,L}$ is A-separable, then the coefficients of the matrix A satisfy $|a_{11}|,|a_{12}|,|a_{21}|,|a_{22}|\le 2N-1$.*

Proof. First, notice that the coefficients of a row (resp. of a column) of the matrix A have no common divisor. This is a consequence of the Bezout theorem, since $\det A=a_{11}a_{22}-a_{12}a_{21}=\pm1$.

Let

$$\Big\{M_{ij}(\xi_1,\xi_2)=\sum_{0\le k_1,k_2\le 2N-1}c_{ij}(k_1,k_2)e^{-i(k_1\xi_1+k_2\xi_2)}\Big\}$$

be an A-separable filter bank. We have

$$M_{00}(\xi_1,\xi_2)=r_0(a_{11}\xi_1+a_{12}\xi_2)t_0(a_{21}\xi_1+a_{22}\xi_2),$$

where $\{r_k(x)\}$ et $\{t_k(x)\}$ are two monodimensional filter banks.

If $a_{11}a_{12}\neq0$ and a_{11} is odd then since $r_0(\pi)=0$ it follows from the last equality that for all real x,

$$M_{00}(a_{12}x+\pi,-a_{11}x)=0$$

and thus

$$\sum_{0\le k_1,k_2\le 2N-1}(-1)^{k_1}c_{00}(k_1,k_2)e^{-i(k_1a_{12}-k_2a_{11})x}=0.$$

Therefore, there exist (k_1,k_2) and (k_1',k_2') in $\{0,1,\ldots,2N-1\}^2$ such that $k_1a_{12}-k_2a_{11}=k_1'a_{12}-k_2'a_{11}$ or equivalently $(k_1-k_1')a_{12}=(k_2-k_2')a_{11}$.

We then use the Gauss lemma: if a, b and c are three integers such that a and b have no common divisor and if a is a divisor of bc then a is a divisor of c.

a_{11} and a_{12} being with no common divisor, there exist two integers u_1 and u_2 such that $a_{11}u_1=k_1'-k_1$ and $a_{12}u_2=k_2'-k_2$. Thus, we have $|a_{11}|\le|k_1'-k_1|\le 2N-1$ and $|a_{12}|\le|k_2'-k_2|\le 2N-1$.

If $a_{11}a_{12}\neq0$ and a_{11} is even, then a_{12} is necessarily odd, so by a similar method we obtain the same conclusion.

We can show similarly that $|a_{21}|\le 2N-1$ and $|a_{22}|\le 2N-1$ when $a_{21}a_{22}\neq0$.

Finally, when one of the coefficients of the matrix A vanishes then the other coefficient is equal to ±1 and the conclusion of Lemma 3.6 remains true. ∎

3.2. *Filter banks that do not satisfy Cohen–Lawton's condition.* It is well known that every filter bank generates a "tight frame" for $L^2(\mathbb{R}^2)$ [L, LR]. This means that every function $f\in L^2(\mathbb{R}^2)$ may be written $f(x)=\sum_{i,j,k}\alpha(i,j,k)2^j\psi_i(2^jx-k)$ but the coefficients $\alpha(i,j,k)$ are not necessarily unique. It is worthwhile noting that there exist filter banks that do not generate wavelet bases for $L^2(\mathbb{R}^2)$.

A. Cohen has first found a necessary and sufficient condition for a filter bank to generate a wavelet basis; it involves the structure of the set of zeros of $M_0(\xi)$ [C]. W. M. Lawton, starting from a completely different view point, has then found another necessary and sufficient condition, which involves the dimension of the eigenspace of the eigenvalue 1 for the transfer operator [L, LR, RW]. This last condition may be stated as follows.

LEMMA 3.7 ([LR]). *Let $\{M_k(\xi)\}_{0\le k\le 3}$ be a filter bank, let $\mathcal{Q}_N$ be the vector space*

$$\mathcal{Q}_N = \Big\{G \,\Big|\, G(\xi) = \sum_{k\in\{-2(N-1),\ldots,2(N-1)\}^2} a(k)e^{-ik\cdot\xi},\ a(k)\in\mathbb{C}\Big\}$$

and let $T:\mathcal{Q}_N\to\mathcal{Q}_N$ be the transfer operator defined by

$$TG(\xi) = \sum_{\nu\in\{0,1\}^2} |M_0(2^{-1}\xi+\pi\nu)|^2 G(2^{-1}\xi+\pi\nu). \tag{3.2}$$

Then the filter bank $\{M_k(\xi)\}_{0\le k\le 3}$ generates a wavelet basis if and only if $\dim\ker(T-I)=1$.

THEOREM 3.8. *The filter banks in $V_{N,L}$ that do not generate wavelet bases form an algebraic subset $W_{N,L}$.*

Proof. Let T be the matrix of the transfer operator that corresponds to a filter bank that does not satisfy Cohen–Lawton's condition. It follows from Lemma 3.7 that $\dim\ker(T-I)\ge 2$. Therefore, the rank of the matrix $T-I$ is less than $(4N-3)^2-1$ and consequently, all the $(4N-3)^2-1$ minors of this matrix vanish. This shows that a filter bank $\{M_j(\xi_1,\xi_2)\}$ does not satisfy Cohen–Lawton's condition if and only if the coefficients of the trigonometric polynomials $M_j(\xi_1,\xi_2)$ are the common zeros of a family of polynomials with $16N^2$ variables. ∎

3.3. *Filter banks that generate wavelet bases with critical Sobolev exponent $>\alpha$.* Let $\alpha\ge 0$. A function $f\in L^2(\mathbb{R}^2)$ belongs to the Sobolev space $W_2^\alpha(\mathbb{R}^2)$ if

$$\int_{\mathbb{R}^2} |\widehat{f}(\xi)|^2(1+|\xi|^\alpha)^2\,d\xi<\infty.$$

For all integers k, functions in $W_2^k(\mathbb{R}^2)$ are C^{k-2} functions. One can measure the regularity of a function $f\in L^2(\mathbb{R}^2)$ by its *critical Sobolev exponent* $\alpha(f)$, defined by

$$\alpha(f)=\sup\{\alpha \mid f\in W_2^\alpha(\mathbb{R}^2)\}. \tag{3.3}$$

LEMMA 3.9 ([CGV, J]). *Let T be the transfer operator of a filter bank that generates a wavelet basis with exactly L vanishing moments. Suppose that T is defined on the space of all bivariate trigonometric polynomials. Then the critical Sobolev exponent of these wavelets is given by*

$$-\log_4(\varrho(T|_{\tau_{2L}})), \tag{3.4}$$

where τ_{2L} is the space of bivariate trigonometric polynomials that have a zero of order $2L$ at the origin and $\varrho(T|_{\tau_{2L}})$ is the spectral radius of the restriction of T to τ_{2L}.

THEOREM 3.10. *Let $S_{N,L}^\alpha\subset V_{N,L}$ be the subset of filter banks that generate wavelet bases both with exactly L vanishing moments and with critical Sobolev exponents $>\alpha$. Then $S_{N,L}^\alpha$ is a Euclide open subset of $V_{N,L}$.*

Proof. This theorem is a consequence of the continuity of the spectral radius. ∎

It follows from this section that the problem of establishing the existence of nonseparable compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrary regularity is equivalent to the geometrical problem of showing that for all $\alpha\ge 0$ one can find N and L such that $S_{N,L}^\alpha$ is not included in $T_{N,L}$.

4. A geometrical solution of our problem

DEFINITION 4.1. Let $P_{M,L}$ be the set of couples $(\lambda(x),\mu(x))$ of π-periodic, monovariate, trigonometric polynomials with real coefficients that satisfy (1.5) and (1.6) and such that

$$\lambda(x)=\sum_{k=0}^{M} a(k)e^{-i2kx}\quad\text{and}\quad \mu(x)=\sum_{k=0}^{M} b(k)e^{-i2kx}.$$

$P_{M,L}$ may be identified with an algebraic subset of $\mathbb{R}^{2(M+1)}$.

The following result gives some insight on $P_{M,L}$.

PROPOSITION 4.2. *Let f be the function from $P_{M,L}$ to $V_{2M+N,L}$* (see Definition 3.1) *such that the image of a couple of trigonometric polynomials $(\lambda(x),\mu(x))$ is the filter bank $\{M_k(\xi_1,\xi_2)\}$ defined in* (1.7). *Then $f(P_{M,L})$ is an algebraic subset of $V_{2M+N,L}$ and $f:P_{M,L}\to f(P_{M,L})$ is a polynomial isomorphism.*

Proof. It is clear that the function f is injective: indeed, we have for all ξ_1, ξ_2,

$$\begin{pmatrix} M_0(\xi_1,\xi_2)\\ M_0(\xi_1+\pi,\xi_2)\\ M_0(\xi_1,\xi_2+\pi)\\ M_0(\xi_1+\pi,\xi_2+\pi)\end{pmatrix} = e^{-2iM\xi_1}\lambda(\xi_1)\begin{pmatrix} S_{00}(\xi_1,\xi_2)\\ S_{00}(\xi_1+\pi,\xi_2)\\ S_{00}(\xi_1,\xi_2+\pi)\\ S_{00}(\xi_1+\pi,\xi_2+\pi)\end{pmatrix} + e^{-2iM\xi_1}\mu(\xi_1)\begin{pmatrix} S_{11}(\xi_1,\xi_2)\\ S_{11}(\xi_1+\pi,\xi_2)\\ S_{11}(\xi_1,\xi_2+\pi)\\ S_{11}(\xi_1+\pi,\xi_2+\pi)\end{pmatrix}$$

and it follows that

$$(4.1)\quad \begin{cases} \lambda(\xi_1)e^{-2iM\xi_1} \\ \quad = \sum_{(\nu_1,\nu_2)\in\{0,1\}^2} M_0(\xi_1+\pi\nu_1,\xi_2+\pi\nu_2)\overline{S_{00}(\xi_1+\pi\nu_1,\xi_2+\pi\nu_2)}, \\ \mu(\xi_1)e^{-2iM\xi_1} \\ \quad = \sum_{(\nu_1,\nu_2)\in\{0,1\}^2} M_0(\xi_1+\pi\nu_1,\xi_2+\pi\nu_2)\overline{S_{11}(\xi_1+\pi\nu_1,\xi_2+\pi\nu_2)}, \end{cases}$$

since the vectors

$$\begin{pmatrix} S_{00}(\xi_1,\xi_2) \\ S_{00}(\xi_1+\pi,\xi_2) \\ S_{00}(\xi_1,\xi_2+\pi) \\ S_{00}(\xi_1+\pi,\xi_2+\pi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_{11}(\xi_1,\xi_2) \\ S_{11}(\xi_1+\pi,\xi_2) \\ S_{11}(\xi_1,\xi_2+\pi) \\ S_{11}(\xi_1+\pi,\xi_2+\pi) \end{pmatrix}$$

in $\mathbb{C}^4$ are orthonormal. It results from the definition of f and from (4.1) that the coefficients of $M_k(\xi_1,\xi_2)$ are polynomial functions of the coefficients of $\lambda(x)$ and of $\mu(x)$ and conversely. As a consequence, $f(P_{M,L})$ is an algebraic subset of $V_{2M+N,L}$ and f is a polynomial isomorphism from $P_{M,L}$ to $f(P_{M,L})$. ■

LEMMA 4.3. *The couple $(\lambda_0(x),\mu_0(x))$ where $\lambda_0\equiv 1$ and $\mu_0\equiv 0$ is a nonsingular point of dimension $M+1-L$ of $P_{M,L}$* (see Definition 2.4).

Proof. For all $m\in\{0,\dots,M\}$ and all $l\in\{0,\dots,L-1\}$ consider the following polynomials that depend on the variables $(a(k))_{0\le k\le M}$ and $(b(k))_{0\le k\le M}$:

$$P_m(a,b)=\sum_{k=0}^{M-m}(a(k)a(k+m)+b(k)b(k+m))-\delta_0(m)$$

where $\delta_p(q)=1$ when $p=q$ and 0 otherwise, and

$$H_l(a,b)=\sum_{k=0}^{M}k^l b(k).$$

Let U be the open subset of $\mathbb{R}^{2(M+1)}$ defined by $\sum_{k=0}^{M}a(k)\ne -1$. We have

$$U\cap P_{M,L}=P_{M,L}=\Big(\bigcap_{m=0}^{M}P_m^{-1}(0)\Big)\cap\Big(\bigcap_{l=0}^{L-1}H_l^{-1}(0)\Big).$$

Let (a_0,b_0) be the sequences of the coefficients of $\lambda_0(x)$ and $\mu_0(x)$. It is clear that the gradients $\nabla P_m(a_0,b_0)$ and $\nabla H_l(a_0,b_0)$, $m\in\{0,\dots,M\}$ and $l\in\{0,\dots,L-1\}$, are linearly independent vectors of $\mathbb{R}^{2(M+1)}$. Indeed, we have

$$\frac{\partial P_0}{\partial a(k)}(a_0,b_0)=2\delta_0(k)\quad\text{and}\quad\frac{\partial P_0}{\partial b(k)}(a_0,b_0)=0,$$

for all $m\in\{1,\dots,M\}$ we have

$$\frac{\partial P_m}{\partial a(k)}(a_0,b_0)=\delta_m(k)\quad\text{and}\quad\frac{\partial P_m}{\partial b(k)}(a_0,b_0)=0,$$

and for all $l\in\{0,\dots,L-1\}$,

$$\frac{\partial H_l}{\partial a(k)}(a_0,b_0)=0\quad\text{and}\quad\frac{\partial H_l}{\partial b(k)}(a_0,b_0)=k^l. \ ■$$

LEMMA 4.4. *There exists an irreducible algebraic subset $\widetilde{V}_{2M+N,L}\subset V_{2M+N,L}$ of dimension $M+1-L$ such that all the filter banks in $\widetilde{V}_{2M+N,L}$ are of the type* (1.7) *and the Ω-separable filter bank σ_0* (see (1.9)) *is a nonsingular point of $\widetilde{V}_{2M+N,L}$.*

Proof. Let $f:P_{M,L}\to f(P_{M,L})$ be the polynomial isomorphism defined in Proposition 4.2 and let $(\lambda_0,\mu_0)\in P_{M,L}$ be as in Lemma 4.3. We have $f(\lambda_0,\mu_0)=\sigma_0$ and it follows from Proposition 2.11 that σ_0 is a nonsingular point of dimension $M+1-L$ of $f(P_{M,L})$. Thus by using Proposition 2.9 we obtain the assertion. ■

The following lemma means that there exist "very few" Ω-separable filter banks in $\widetilde{V}_{(M+N),L}$.

LEMMA 4.5. *Let $\widetilde{T}^{\Omega}_{2M+N,L}=\widetilde{V}_{2M+N,L}\cap T^{\Omega}_{2M+N,L}$ be the proper algebraic subset of Ω-separable filter banks in $\widetilde{V}_{2M+N,L}$* (see Proposition 3.3). *Then $\widetilde{T}^{\Omega}_{2M+N,L}$ is a finite set.*

Proof. Let $\{M_k(\xi_1,\xi_2)\}$ be an Ω-separable filter bank of the type (1.7). We have

$$e^{-i2M(\gamma_{11}\xi_1+\gamma_{12}\xi_2)}[\lambda(\gamma_{11}\xi_1+\gamma_{12}\xi_2)a_0(\xi_1)b_0(\xi_2) \\ +\mu(\gamma_{11}\xi_1+\gamma_{12}\xi_2)a_1(\xi_1)b_1(\xi_2)]=r_0(\xi_1)t_0(\xi_2),$$

where $\{a_k(x)\}$, $\{b_k(x)\}$, $\{r_k(x)\}$ and $\{t_k(x)\}$ are four monodimensional filter banks and where $\Gamma=(\gamma_{ij})=\Omega^{-1}$. Since $\Gamma\in \mathrm{SL}(2,\mathbb{Z})$, at least one of the two integers γ_{11} and γ_{12} is nonzero; so we can suppose that $\gamma_{11}\ne 0$. As $b_0(0)=t_0(0)=1$ and $b_1(0)=0$, we obtain

$$e^{-i2M\gamma_{11}\xi_1}\lambda(\gamma_{11}\xi_1)a_0(\xi_1)=r_0(\xi_1).$$

Since $\lambda(\gamma_{11}\xi_1)$ is π-periodic and

$$|r_0(\xi_1)|^2+|r_0(\xi_1+\pi)|^2=|a_0(\xi_1)|^2+|a_0(\xi_1+\pi)|^2=1,$$

it follows that

$$(*)\qquad |\lambda(\gamma_{11}\xi_1)|^2=1.$$

Thus it is clear that there are only a finite number of elements $(\lambda(x), \mu(x))$ of $P_{M,L}$ that satisfy $(*)$; these elements are of the form $\lambda(x) = e^{-i2kx}$ and $\mu(x) = 0$, where $k \in \{0, \ldots, M\}$. ■

The following proposition means that "most" of the filter banks in $\widetilde{V}_{2M+N,L}$ are both nonseparable and satisfy Cohen–Lawton's condition.

PROPOSITION 4.6. *Let $\widetilde{W}_{2M+N,L} = W_{2M+N,L} \cap \widetilde{V}_{2M+N,L} \subset \widetilde{V}_{2M+N,L}$ be the algebraic subset of filter banks that do not satisfy Cohen–Lawton's condition and let $\widetilde{T}_{2M+N,L} = T_{2M+N,L} \cap \widetilde{V}_{2M+N,L}$ be the algebraic subset of separable filter banks. Then*

$$\dim(\widetilde{W}_{2M+N,L} \cup \widetilde{T}_{2M+N,L}) < \dim(\widetilde{V}_{2M+N,L}).$$

Proof. As $\sigma_0 \notin \widetilde{W}_{2M+N,L}$, $\widetilde{W}_{2M+N,L}$ is a proper algebraic subset of $\widetilde{V}_{2M+N,L}$.

Moreover, for all $A \in \mathrm{SL}(2, \mathbb{Z})$, $\widetilde{T}^{A}_{2M+N,L}$ is a proper algebraic subset of $\widetilde{V}_{2M+N,L}$. Indeed, suppose that for some A we have $\widetilde{T}^{A}_{2M+N,L} = \widetilde{V}_{2M+N,L}$. Proposition 3.4 will then imply that $\widetilde{T}^{A}_{2M+N,L} = \widetilde{T}^{\Omega}_{2M+N,L}$. This contradicts Lemma 4.5 since this lemma entails that $\widetilde{T}^{\Omega}_{2M+N,L}$ is a proper algebraic subset of $\widetilde{V}_{2M+N,L}$.

All this and Theorem 3.5 imply that $\widetilde{W}_{2M+N,L} \cup \widetilde{T}_{2M+N,L}$ is a finite union of proper algebraic subsets of $\widetilde{V}_{2M+N,L}$, therefore, $\widetilde{W}_{2M+N,L} \cup \widetilde{T}_{2M+N,L}$ is a proper algebraic subset of $\widetilde{V}_{2M+N,L}$; then the assertion follows from Proposition 2.7. ■

Let us now state the main result of this paper.

THEOREM 4.7. *For every integer $M \geq L - 1$, one can construct a real-analytic manifold $\widetilde{\mathcal{V}}_{2M+N,L} \subset V_{2M+N,L}$ of dimension $M + 1 - L$ with the following properties.*

(i) *All filter banks in $\widetilde{\mathcal{V}}_{2M+N,L}$ generate wavelet bases with critical Sobolev exponent $> \alpha$ and with exactly L vanishing moments.*

(ii) σ_0 (see (1.9)) *is the only separable filter bank in $\widetilde{\mathcal{V}}_{2M+N,L}$.*

Proof. Since σ_0 is a nonsingular point of $\widetilde{V}_{2M+N,L}$ and $\widetilde{S}^{\alpha}_{2M+N,L}$ is an open subset of $\widetilde{V}_{2M+N,L}$ (see Theorem 3.10 for the definition of $S^{\alpha}_{2M+N,L}$), there exists a real-analytic manifold $\widetilde{\mathcal{V}}'_{2M+N,L} \subset \widetilde{S}^{\alpha}_{2M+N,L} = S^{\alpha}_{2M+N,L} \cap \widetilde{V}_{2M+N,L}$ of dimension $M + 1 - L$ that contains σ_0.

As $T^{\Omega}_{2M+N,L} - \{\sigma_0\}$ is a finite set (see Lemma 4.5), it is a closed subset of $V_{2M+N,L}$, and consequently $\widetilde{\mathcal{V}}_{2M+N,L} = (\widetilde{\mathcal{V}}'_{2M+N,L} - T^{A}_{2M+N,L}) \cup \{\sigma_0\}$ is a nonempty open subset of $\widetilde{\mathcal{V}}_{2M+N,L}$ (see Proposition 3.4 and Theorem 3.5).

Finally, $\widetilde{\mathcal{V}}_{2M+N,L}$ is an analytic manifold that satisfies (i) and (ii). ■

Theorem 4.7 roughly shows that "most" of compactly supported wavelet bases with critical Sobolev exponent $> \alpha$ are nonseparable.

Acknowledgements. I would like to thank Professor Yves Meyer for fruitful discussions.

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Laboratoire de Statistique et Probabilités
Université Paul Sabatier
118, Route de Narbonne
31062 Toulouse, France
E-mail: ayache@cict.fr

Received February 9, 1999
Revised version November 19, 1999 (4258)