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Composition operators: $\mathcal{N}_\alpha$ to the Bloch space to $\mathcal{Q}_\beta$

by

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Abstract. Let $\mathcal{N}_\alpha$, $\mathcal{B}$ and $\mathcal{Q}_\beta$ be the weighted Nevanlinna space, the Bloch space and the $\mathcal{Q}$ space, respectively. Note that $\mathcal{B}$ and $\mathcal{Q}_\beta$ are Möbius invariant, but $\mathcal{N}_\alpha$ is not. We characterize, in function-theoretic terms, when the composition operator $C_\phi f = f\circ\phi$ induced by an analytic self-map ϕ of the unit disk defines an operator $C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$, $\mathcal{B}\to\mathcal{Q}_\beta$, $\mathcal{N}_\alpha\to\mathcal{Q}_\beta$ which is bounded resp. compact.

1. Introduction. Let Δ be the unit disk $\{z\in\mathbb{C} : |z|<1\}$ in the complex plane, and let $\mathcal{H}(\Delta)$ be the space of all analytic functions on Δ. Any analytic map $\phi:\Delta\to\Delta$ gives rise to an operator $C_\phi:\mathcal{H}(\Delta)\to\mathcal{H}(\Delta)$ defined by $C_\phi f = f\circ\phi$, the *composition operator* induced by ϕ.

One of the central problems on composition operators is to know when C_ϕ maps between two subclasses of $\mathcal{H}(\Delta)$ and in fact to relate function-theoretic properties of ϕ to operator-theoretic properties of C_ϕ. This problem is addressed here for the weighted Nevanlinna, the Bloch and the $\mathcal{Q}$ spaces with respect to boundedness and compactness of the operator. The related research has recently been done by various authors (see for example [JX], [MM], [RU], [SZ], [T] and [X2]). The present paper continues their work, but also solves two problems which remained open in [SZ].

For each $\alpha\in(-1,\infty)$, let $\mathcal{N}_\alpha$ be the space of all functions $f\in\mathcal{H}(\Delta)$ satisfying

$$T_\alpha(f)=\frac{1+\alpha}{\pi}\int_\Delta[\log^+|f(z)|](1-|z|^2)^\alpha\,dm(z)<\infty.$$

Here and afterwards, dm means the usual element of the area measure on Δ, and $\log^+ x$ is $\log x$ if $x>1$ and 0 if $0\le x\le 1$.

From $\log^+ x\le\log(1+x)\le 1+\log^+ x$ for $x\ge 0$ we see that a function $f\in\mathcal{H}(\Delta)$ belongs to $\mathcal{N}_\alpha$ if and only if

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$$\|f\|_{\mathcal{N}_\alpha} = \int_\Delta [\log(1+|f(z)|)]\,(1-|z|^2)^\alpha\,dm(z) < \infty.$$

Obviously,

$$\max\{\|f+g\|_{\mathcal{N}_\alpha}, \|fg\|_{\mathcal{N}_\alpha}\} \le \|f\|_{\mathcal{N}_\alpha} + \|g\|_{\mathcal{N}_\alpha}$$

for all $f, g \in \mathcal{N}_\alpha$. Consequently, $\mathcal{N}_\alpha$ is not only a vector space but even an algebra. Further, by setting

$$d_\alpha(f,g) = \|f-g\|_{\mathcal{N}_\alpha}$$

for $f, g \in \mathcal{N}_\alpha$, we obtain a translation invariant metric on $\mathcal{N}_\alpha$. More is true: $\|\cdot\|_{\mathcal{N}_\alpha}$ is an F-norm, and under this norm, $\mathcal{N}_\alpha$ is an F-space, i.e. a complete metrizable topological vector space (cf. [J]).

The Bloch space $\mathcal{B}$ consists of all functions $f \in \mathcal{H}(\Delta)$ obeying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty.$$

$\|\cdot\|_{\mathcal{B}}$ is a norm and makes $\mathcal{B}$ a Banach space.

Given $w \in \Delta$, let

$$\varphi_w(z) = \frac{w-z}{1-\overline{w}z}$$

be a Möbius transformation which exchanges w and 0. Stroethoff's ideas in the proof of Theorems 4.1 and 4.2 in [Str] yield that $f \in \mathcal{H}(\Delta)$ lies in $\mathcal{B}$ if and only if

$$\sup_{w\in\Delta} T_\alpha(C_{\varphi_w}f - f(w)) < \infty.$$

That is to say, $\mathcal{B}$ is the Möbius bounded subspace of $\mathcal{N}_\alpha$.

For $\beta \in (-1,\infty)$, let $\mathcal{Q}_\beta$ be the class of all functions $f \in \mathcal{H}(\Delta)$ with

$$\|f\|_{\mathcal{Q}_\beta} = |f(0)| + \sup_{w\in\Delta}\Big[\int_\Delta |(C_{\varphi_w}f)'(z)|^2(1-|z|^2)^\beta\,dm(z)\Big]^{1/2} < \infty.$$

Observe that if $\beta \in (-1,0)$, $\beta = 0$, $\beta = 1$ and $\beta \in (1,\infty)$, then $\mathcal{Q}_\beta = \mathbb{C}$, $\mathcal{D}$ (the classical Dirichlet space), BMOA and $\mathcal{B}$, respectively (cf. [NX], [Ba], [AXZ], [AL], [X1]). Of course, $\mathcal{Q}_\beta$ is the Möbius bounded subspace of the weighted Dirichlet space (see also [ANZ], [ASX], [EX]). The spaces $\mathcal{N}_\alpha$, $\mathcal{B}$ and $\mathcal{Q}_\beta$ are linked by the inclusions $\mathcal{N}_\alpha \supset \mathcal{B} \supset \mathcal{Q}_\beta$. Notice that $\mathcal{B}$ and $\mathcal{Q}_\beta$ are Möbius invariant, but $\mathcal{N}_\alpha$ is not.

We are going to work with the composition operators sending "big" spaces to "small" spaces since the converse is clear. In fact, $C_\phi : \mathcal{B} \to \mathcal{N}_\alpha$ and $C_\phi : \mathcal{Q}_\beta \to \mathcal{N}_\alpha$ are always compact (cf. [X2, Proposition 4.3]), while $C_\phi : \mathcal{Q}_\beta \to \mathcal{B}$ is compact if and only if $\lim_{|\phi(z)|\to 1}(1-|z|^2)|\phi'(z)|/(1-|\phi(z)|^2) = 0$ (cf. [MM, Theorem 2] and [SZ, Theorem 6.4]).

The main results of this paper are the next three theorems. The first concerns boundedness and compactness of $C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$.

1.1. THEOREM. *Let $\alpha \in (-1,\infty)$ and let $\phi : \Delta \to \Delta$ be analytic. Then the following are equivalent*:

(i) *$C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$ exists as a bounded operator.*

(ii) *$C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$ exists as a compact operator.*

(iii) *For all $c > 0$,*

$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2}\exp\Big[\frac{c}{(1-|\phi(z)|^2)^{2+\alpha}}\Big] = 0. \tag{1.1}$$

Before giving the second assertion on boundedness and compactness of $C_\phi : \mathcal{B} \to \mathcal{Q}_p$, we explain the necessary notation.

Arcs in the unit circle $\partial\Delta$ are sets of the form $I = \{z \in \partial\Delta : \theta_1 \le \arg z < \theta_2\}$ where $\theta_1, \theta_2 \in [0, 2\pi)$ and $\theta_1 < \theta_2$. The length of an arc $I \subset \partial\Delta$ will be denoted by $|I|$. The *Carleson box* based on an arc I is the set

$$S(I) = \Big\{z \in \Delta : 1 - \frac{|I|}{2\pi} \le |z| < 1,\ \frac{z}{|z|} \in I\Big\}.$$

Also for an $r \in (0,1)$ and an analytic self-map ϕ of Δ, put $\Omega_r = \{z \in \Delta : |\phi(z)| > r\}$. The characteristic function of a set $E \subset \Delta$ is denoted by 1_E.

1.2. THEOREM. *Let $\beta \in (0,\infty)$ and let $\phi : \Delta \to \Delta$ be analytic. Then*

(i) *$C_\phi : \mathcal{B} \to \mathcal{Q}_\beta$ exists as a bounded operator if and only if*

$$\sup_{I\subset\partial\Delta} |I|^{-\beta}\int_{S(I)}\Big[\frac{(1-|z|^2)^{\beta/2}|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 dm(z) < \infty. \tag{1.2}$$

(ii) *$C_\phi : \mathcal{B} \to \mathcal{Q}_\beta$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_\beta$ and*

$$\lim_{r\to 1}\sup_{I\subset\partial\Delta} |I|^{-\beta}\int_{S(I)}\Big[\frac{(1-|z|^2)^{\beta/2}|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 1_{\Omega_r}(z)\,dm(z) = 0. \tag{1.3}$$

Note that (i) of Theorem 1.2 is essentially known (cf. [SZ, Theorem 1.5]) and is listed here only for the sake of completeness. However, (ii) is new and is just what Smith–Zhao did not figure out. Moreover, if $\beta > 1$ then (1.3) is equivalent to $\lim_{|\phi(z)|\to 1}(1-|z|^2)|\phi'(z)|/(1-|\phi(z)|^2) = 0$ (cf. [MM, Theorem 2]).

The third theorem deals with boundedness and compactness of $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$. This requires the Möbius invariant version of the generalized Nevanlinna counting function (cf. [T, Definition 2.2]). More precisely, for $\beta \in (0,\infty)$ and an analytic map $\phi : \Delta \to \Delta$, let

$$N(\beta, w, z, \phi) = \begin{cases} \sum_{\phi(v)=z}[1-|\varphi_w(v)|^2]^\beta, & z \in \phi(\Delta),\\ 0, & z \in \Delta\setminus\phi(\Delta).\end{cases}$$

1.3. THEOREM. *Let $\alpha \in (-1,\infty)$, $\beta \in (0,\infty)$ and let $\phi : \Delta \to \Delta$ be analytic. Then the following are equivalent*:

(i) $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$ *exists as a bounded operator.*
(ii) $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$ *exists as a compact operator.*
(iii) $\phi \in \mathcal{Q}_\beta$ *and for all* $c > 0$,

$$\sup_{w\in\Delta}\sup_{I\subset\partial\Delta}\frac{|I|^{-2(\alpha+3)}}{\exp(c|I|^{2+\alpha})}\int_{S(I)} N(\beta,w,z,\phi)\,dm(z)<\infty. \tag{1.4}$$

Comparing Theorem 1.3 with Theorem 1.1 we find that (1.4)⇔(1.1) when $\beta > 1$.

We devote Section 2 to the proof of Theorem 1.1 and its consequences. The proof of Theorem 1.2 and its extension are presented in Section 3. The last section is devoted to proving Theorem 1.3 and a further discussion.

Throughout this paper, we denote positive constants by M, M_0, M_1, $M_2, \ldots$ Those constants depend only on some parameters such as α and β unless a special remark is made. Also, given two families $x = (x(\omega))_{\omega\in\Omega}$ and $y = (y(\omega))_{\omega\in\Omega}$ of non-negative real numbers (or functions) on the given domain Ω, we write $x \asymp y$ if (there exist constants $M_1, M_2 > 0$ such that) $M_1 x(\omega) \le y(\omega) \le M_2 x(\omega)$ for all $\omega \in \Omega$.

2. $C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$. The space $\mathcal{H}(\Delta)$ is a Fréchet space with respect to the compact-open topology, that is, the topology of uniform convergence on compact subsets of Δ; in fact, $\mathcal{H}(\Delta)$ is even a Fréchet algebra. By Montel's theorem, bounded sets in $\mathcal{H}(\Delta)$ are relatively compact; accordingly, bounded sequences in $\mathcal{H}(\Delta)$ admit convergent subsequences. Convergence in this space will be referred to as *locally uniform* (*l.u.*) *convergence.*

Recall that $\mathcal{N}_\alpha$ is a linear subspace (even a subalgebra) of $\mathcal{H}(\Delta)$. Note that $\mathcal{N}_\alpha$ is a topological vector space with respect to the F-norm $\|\cdot\|_{\mathcal{N}_\alpha}$. This is in marked contrast to the situation for the classical Nevanlinna class which is not a topological vector space [SS]. Under $\|\cdot\|_{\mathcal{N}_\alpha}$, the topology of $\mathcal{N}_\alpha$ is stronger than that of locally uniform convergence. This is a simple consequence of the following estimate:

$$\log(1+|f(z)|)\le\frac{M_0\|f\|_{\mathcal{N}_\alpha}}{(1-|z|^2)^{2+\alpha}},\qquad f\in\mathcal{N}_\alpha, \tag{2.1}$$

where $M_0 > 0$ is a constant depending only on α.

As in [Str], $\mathcal{N}_\alpha$ has $\mathcal{B}$ as its Möbius bounded subspace.

2.1. PROPOSITION. *Let* $\alpha \in (-1,\infty)$ *and* $f \in \mathcal{H}(\Delta)$. *Then the following are equivalent*:

(i) f *belongs to* $\mathcal{B}$.
(ii) $\sup_{w\in\Delta} T_\alpha(C_{\varphi_w} f - f(w)) < \infty$.
(iii) $\sup_{w\in\Delta} \|C_{\varphi_w} f - f(w)\|_{\mathcal{N}_\alpha} < \infty$.

Proof. It suffices to show (i)⇔(iii), for (i)⇔(ii) can be verified in a similar manner to proving Theorems 4.1 and 4.2 of [Str].

Observe that if f is a Bloch function with $\|f\|_{\mathcal{B}} > 0$ then for $z \in \Delta$,

$$|C_{\varphi_w} f(z) - f(w)| \le \frac{\|f\|_{\mathcal{B}}}{2}\log\frac{1+|z|}{1-|z|}.$$

It follows that for each $t > 0$,

$$m_\alpha[t] = m_\alpha\{z\in\Delta : |C_{\varphi_w} f(z) - f(w)| > t\} \le M_1\exp\left[-\frac{2(\alpha+1)t}{\|f\|_{\mathcal{B}}}\right].$$

Let now f be a Bloch function. Without loss of generality, we may assume that $\|f\|_{\mathcal{B}} > 0$. There is a constant $M_2 > 0$ depending only on α such that for each $w \in \Delta$,

$$\|C_{\varphi_w} f - f(w)\|_{\mathcal{N}_\alpha} = \int_0^\infty \frac{m_\alpha[t]}{1+t}\,dt \le M_2\|f\|_{\mathcal{B}}, \tag{2.2}$$

which proves (iii).

Suppose conversely that (iii) is true. Let $r \in (0,1)$. If $z \in \Delta$ is such that $|\varphi_w(z)| < r$ then, by (2.1) and since φ_w is an analytic automorphism of Δ with $\varphi_w^{-1} = \varphi_w$,

$$\log(1+|f(z)-f(w)|) \le \frac{M_0\|C_{\varphi_w} f - f(w)\|_{\mathcal{N}_\alpha}}{(1-r)^{2+\alpha}}. \tag{2.3}$$

An application of (3.1) in [Str] shows that f is a Bloch function. The proof is complete.

Note that $\mathcal{B}$ has a closed subspace, the *little Bloch space* $\mathcal{B}_0$ of all functions $f \in \mathcal{B}$ obeying

$$\lim_{|z|\to1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that the polynomials are dense in $\mathcal{B}_0$ under $\|\cdot\|_{\mathcal{B}}$. Furthermore, we have

2.2. COROLLARY. *Let* $\alpha \in (-1,\infty)$ *and* $f \in \mathcal{H}(\Delta)$. *Then the following are equivalent*:

(i) f *belongs to* $\mathcal{B}_0$.
(ii) $\lim_{|w|\to1} T_\alpha(\varrho^{-1}(C_{\varphi_w} f - f(w))) = 0$ *for every* $\varrho > 0$.
(iii) $\lim_{|w|\to1} \|C_{\varphi_w} f - f(w)\|_{\mathcal{N}_\alpha} = 0$.

Proof. As in Proposition 2.1, it is enough to verify (i)⇔(iii). Suppose that f belongs to $\mathcal{B}_0$. By density, given any $\varepsilon \in (0,1)$, there is a polynomial P such that $\|f-P\|_{\mathcal{B}} < \varepsilon$. Consequently, by (2.2),

$$\|C_{\varphi_w}(f-P) - (f-P)(w)\|_{\mathcal{N}_\alpha} \le M_2\|f-P\|_{\mathcal{B}} < M_2\varepsilon.$$

This implies (iii), owing to $\lim_{|w|\to1}\|C_{\varphi_w} P - P(w)\|_{\mathcal{N}_\alpha} = 0$.

The converse follows easily from (2.3) and from Theorem 3.2 of [Str].

A subset E of $\mathcal{N}_\alpha$ is called *bounded* if it is bounded for the defining F-norm $\|\cdot\|_{\mathcal{N}_\alpha}$. Given a Banach space Y, we say that a linear map $T:\mathcal{N}_\alpha\to Y$ is *bounded* if $T(E)\subset Y$ is bounded for every bounded subset E of $\mathcal{N}_\alpha$. In addition, we say that T is *compact* if $T(E)\subset Y$ is relatively compact for every bounded set $E\subset\mathcal{N}_\alpha$. A useful tool is the following compactness criterion which follows readily from Proposition 2.3 of [JX] and Lemma 2.10 of [T].

2.3. Lemma. *Let $\alpha\in(-1,\infty)$ and Y be a Banach subspace of $\mathcal{H}(\Delta)$ with norm $\|\cdot\|_Y$. Then $C_\phi:\mathcal{N}_\alpha\to Y$ is compact if and only if for every $s>0$ and every sequence $\{f_n\}$ which satisfies $\|f_n\|_{\mathcal{N}_\alpha}\le s$ and converges to 0 l.u., $\lim_{n\to\infty}\|C_\phi f_n\|_Y=0$.*

2.4. *Proof of Theorem 1.1.* It suffices to check two implications: (i)⇒(iii) and (iii)⇒(ii).

(i)⇒(iii). Let (i) hold. For any $c>0$ and $w=\phi(z_0)$ (where $z_0\in\Delta$ is fixed), consider the test function

$$f_w(z)=\exp\left[c\left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{2+\alpha}\right]. \tag{2.4}$$

Since $\log(1+x)\le 1+\log^+x$ for $x\ge 0$,

$$\|f_w\|_{\mathcal{N}_\alpha}\le\frac{\pi}{1+\alpha}+\int_\Delta[\log^+|f_w(z)|](1-|z|^2)^\alpha\,dm(z)$$
$$\le\frac{\pi}{1+\alpha}+c\int_\Delta\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{2+\alpha}(1-|z|^2)^\alpha\,dm(z)\le M_3,$$

where $M_3>0$ does not depend on w and it comes from Lemma 4.2.2 of [Z].

Because $C_\phi:\mathcal{N}_\alpha\to\mathcal{B}$ is bounded and

$$f_w'(z)=\frac{2(2+\alpha)c\overline{w}(1-|w|^2)^{2+\alpha}}{(1-\overline{w}z)^{2(2+\alpha)+1}}\exp\left[c\left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{2+\alpha}\right],$$

there is a constant $M_4>0$ depending only on c and α such that

$$M_4\ge(1-|z|^2)|f_w'(\phi(z))|\cdot|\phi'(z)|$$
$$\ge\frac{c|w|(1-|z|^2)|\phi'(z)|(1-|w|^2)^{2+\alpha}}{|1-\overline{w}\phi(z)|^{2(2+\alpha)+1}}\exp\left[c\left(\frac{1-|w|^2}{(1-\overline{w}\phi(z))^2}\right)^{2+\alpha}\right].$$

This estimate leads to

$$\frac{(1-|z_0|^2)|\phi'(z_0)|}{1-|\phi(z_0)|^2}\exp\left[\frac{c}{(1-|\phi(z_0)|^2)^{2+\alpha}}\right]\le\frac{M_4(1-|\phi(z_0)|^2)^{2+\alpha}}{c|\phi(z_0)|}, \tag{2.5}$$

which forces (iii) to hold.

(iii)⇒(ii). Assume that (iii) is valid for all $c>0$. Note that if $f\in\mathcal{N}_\alpha$ then by (2.1) and Cauchy's formula,

$$(1-|z|^2)|f'(z)|\le\frac{2}{\pi}\int_{\partial\Delta}|f(z+2^{-1}(1-|z|)\zeta)|\,|d\zeta| \tag{2.6}$$
$$\le\exp\left[\frac{4^{2+\alpha}M_0\|f\|_{\mathcal{N}_\alpha}}{(1-|z|^2)^{2+\alpha}}\right].$$

To demonstrate that $C_\phi:\mathcal{N}_\alpha\to\mathcal{B}$ is compact, we choose, for $s>0$, any sequence $\{f_n\}$ in $\mathcal{N}_\alpha$ such that $\|f_n\|_{\mathcal{N}_\alpha}\le s$ and $f_n\to 0$ l.u. on Δ. Then for each $\delta\in(0,1)$,

$$\sup_{|\phi(z)|\le\delta}(1-|z|^2)|(C_\phi f_n)'(z)|\le\sup_{|\phi(z)|\le\delta}(1-|\phi(z)|^2)|f_n'(\phi(z))|\to 0,\quad n\to\infty.$$

On the other hand, from (2.6) and (iii) it turns out that whenever $\delta\to 1$,

$$\sup_{|\phi(z)|>\delta}(1-|z|^2)|(C_\phi f_n)'(z)|$$
$$\le\sup_{|\phi(z)|>\delta}\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|}\exp\frac{4^{2+\alpha}M_0s}{(1-|\phi(z)|^2)^{2+\alpha}}\to 0.$$

Combining the above estimates we see that $\|C_\phi f_n\|_{\mathcal{B}}\to 0$ as $n\to\infty$. Hence, (ii) follows from Lemma 2.3. The proof is complete.

There is an analogue of Theorem 1.1 for the little Bloch space $\mathcal{B}_0$:

2.5. Corollary. *Let $\alpha\in(-1,\infty)$ and let $\phi:\Delta\to\Delta$ be analytic. Then the following are equivalent:*

(i) *$C_\phi:\mathcal{N}_\alpha\to\mathcal{B}_0$ exists as a bounded operator.*
(ii) *$C_\phi:\mathcal{N}_\alpha\to\mathcal{B}_0$ exists as a compact operator.*
(iii) *For all $c>0$,*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2}\exp\left[\frac{c}{(1-|\phi(z)|^2)^{2+\alpha}}\right]=0. \tag{2.7}$$

Proof. It suffices to demonstrate (iii)⇒(ii) and (i)⇒(iii). The first implication follows easily from the proof of the corresponding case of Theorem 1.1. The second will be verified by contradiction. Suppose that $C_\phi:\mathcal{N}_\alpha\to\mathcal{B}_0$ is bounded. So $\phi\in\mathcal{B}_0$. Now, if (2.7) is not true for all $c>0$, then there are c_0, ε_0 and a sequence $\{z_n\}$ tending to $\partial\Delta$ such that

$$\frac{(1-|z_n|^2)|\phi'(z_n)|}{1-|\phi(z_n)|^2}\exp\left[\frac{c_0}{(1-|\phi(z_n)|^2)^{2+\alpha}}\right]\ge\varepsilon_0. \tag{2.8}$$

Since $\phi\in\mathcal{B}_0$, (2.8) indicates that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $|\phi(z_{n_k})|\to 1$. Also since $C_\phi:\mathcal{N}_\alpha\to\mathcal{B}$ is bounded, one has (1.1) (for all

$c > 0$), which, in particular, produces the following limit:

$$\frac{(1-|z_{n_k}|^2)|\phi'(z_{n_k})|}{1-|\phi(z_{n_k})|^2}\exp\left[\frac{c_0}{(1-|\phi(z_{n_k})|^2)^{2+\alpha}}\right]\to 0. \tag{2.9}$$

It is evident that (2.9) contradicts (2.8). We are done.

3. $C_\phi : \mathcal{B} \to \mathcal{Q}_\beta$. In this section we prove Theorem 1.2. The proof will borrow a technique from [BCM, Theorem 3.1]. Before proceeding, we need an inverse inequality for $\mathcal{B}$ due to Ramey and Ullrich [RU, Proposition 5.4].

3.1. LEMMA. *There are two functions $f_1, f_2 \in \mathcal{B}$ such that*

$$\inf_{z\in\Delta}(1-|z|^2)(|f_1'(z)|+|f_2'(z)|)\ge 1. \tag{3.1}$$

For $\beta \in (0,\infty)$ we say that a positive Borel measure $d\mu$ on Δ is a β-*Carleson measure* provided $\sup_{I\subset\partial\Delta}\mu(S(I))/|I|^\beta < \infty$. This definition was introduced by [ASX, Theorem 2.2] to characterize the $\mathcal{Q}_\beta$ space.

3.2. LEMMA. *Let $\beta \in (0,\infty)$ and let $f \in \mathcal{H}(\Delta)$ with*

$$d\mu_{f,\beta}(z) = |f'(z)|^2(1-|z|^2)^\beta dm(z).$$

Then $f \in \mathcal{Q}_\beta$ if and only if $d\mu_{f,\beta}$ is a β-Carleson measure. Moreover,

$$\|f\|_{\mathcal{Q}_\beta} \asymp |f(0)| + \left[\sup_{I\subset\partial\Delta}\frac{\mu_{f,\beta}(S(I))}{|I|^\beta}\right]^{1/2}. \tag{3.2}$$

3.3. *Proof of Theorem 1.2.* From now on, $\mathbb{B}_X$ stands for the unit ball of a given Banach space $(X, \|\cdot\|_X)$.

(i) follows obviously from Lemmas 3.1 and 3.2. The key is to infer (ii).

Sufficiency of (ii). Let $\phi \in \mathcal{Q}_\beta$ and let (1.3) hold. We have to show that if $\{f_n\} \subset \mathbb{B}_\mathcal{B}$ converges to 0 l.u. on Δ then $\{\|C_\phi f_n\|_{\mathcal{Q}_\beta}\}$ converges to 0. For each $r \in (0,1)$ set $\widetilde{\Omega}_r = \Delta \setminus \Omega_r$. So $\{f_n'(\phi)\}$ tends to 0 uniformly on $\widetilde{\Omega}_r$. Hence by Lemma 3.2, for every $\varepsilon > 0$ there is an integer $N > 1$ such that for $n \ge N$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_n)'(z)|^2(1-|z|^2)^\beta 1_{\widetilde{\Omega}_r}(z)\,dm(z) \le \varepsilon M\|\phi\|^2_{\mathcal{Q}_\beta}.$$

On the other hand, from (1.3) and the growth of the derivatives of $\mathcal{B}$-functions one derives that for every $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that for $r \in [\delta, 1)$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_n)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z) < \varepsilon.$$

Combining the previous inequalities with Lemma 3.2, we obtain $\|C_\phi f_n\|_{\mathcal{Q}_\beta} \to 0$.

Necessity of (ii). This part is more difficult. Let $C_\phi : \mathcal{B} \to \mathcal{Q}_\beta$ be compact. It is clear that $\phi \in \mathcal{Q}_\beta$. So, we must show (1.3). Since $\{z^n\}$ is norm bounded in $\mathcal{B}$ and it converges to 0 l.u. on Δ, we have $\|\phi^n\|_{\mathcal{Q}_\beta} \to 0$. Applying Lemma 3.2, we find that for every $\varepsilon > 0$, there is an integer $N > 1$ such that for $n \ge N$,

$$n^2\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|\phi(z)|^{2n-2}|\phi'(z)|^2(1-|z|^2)^\beta\,dm(z) < \varepsilon;$$

thus for each $r \in (0,1)$,

$$N^2 r^{2N-2}\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|\phi'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z) < \varepsilon.$$

Taking $r \ge N^{-1/(N-1)}$, we get

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|\phi'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z) < \varepsilon. \tag{3.3}$$

Keeping (3.3) in mind, we show that for every $f \in \mathbb{B}_\mathcal{B}$ and for every $\varepsilon > 0$, there is a $\delta = \delta(f,\varepsilon)$ such that for $r \in [\delta, 1)$,

$$\begin{aligned}(3.4)\quad & T(f,\phi,\beta,r)\\ &= \sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z) < \varepsilon.\end{aligned}$$

As a matter of fact, if we let $f_t(z) = f(tz)$ for $f \in \mathbb{B}_\mathcal{B}$ and $t \in (0,1)$, then $f_t \to f$ l.u. on Δ as $t \to 1$. Since $C_\phi : \mathcal{B} \to \mathcal{Q}_\beta$ is compact, $\|f_t\circ\phi - f\circ\phi\|_{\mathcal{Q}_\beta} \to 0$ as $t \to 1$. Furthermore, Lemma 3.2 yields that for every $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_t)'(z) - (C_\phi f)'(z)|^2(1-|z|^2)^\beta\,dm(z) < \varepsilon.$$

Accordingly, by (3.3),

$$\begin{aligned}T(f,\phi,\beta,r) &\le 2\varepsilon + 2\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_t)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z)\\ &\le 2\varepsilon + 2\|f_t'\|^2_\infty\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|\phi'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z)\\ &\le 2\varepsilon(1+\|f_t'\|^2_\infty).\end{aligned}$$

Since C_ϕ sends $\mathbb{B}_\mathcal{B}$ to a relatively compact subset of $\mathcal{Q}_\beta$, there exists, for every $\varepsilon > 0$, a finite collection of functions $f_1, \ldots, f_N$ in $\mathbb{B}_\mathcal{B}$ such that for each $f \in \mathbb{B}_\mathcal{B}$ there is a $k \in \{1,\ldots,N\}$ with

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z) - (C_\phi f_k)'(z)|^2(1-|z|^2)^\beta\,dm(z) < \varepsilon.$$

Now (3.4) is used to deduce that for $\delta = \max_{1\le k\le N}\delta(f_k,\varepsilon)$ and $r\in[\delta;1)$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_k)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z)<\varepsilon;$$

thus

$$\sup_{f\in\mathbb{B}_{\mathcal{B}}}\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z)<4\varepsilon. \tag{3.5}$$

An application of Lemma 3.1 to (3.5) implies (1.3). This concludes the proof.

The space $\mathcal{Q}_\beta$, like $\mathcal{B}$, has a closed subspace $\mathcal{Q}_{\beta,0}$ which consists of those $f\in\mathcal{Q}_\beta$ satisfying

$$\lim_{|w|\to1}\int_{\Delta}|(C_{\varphi_w}f)'(z)|^2(1-|z|^2)^\beta\,dm(z)=0.$$

It is known that $\mathcal{Q}_{\beta,0}=\mathbb{C}$, VMOA and $\mathcal{B}_0$ whenever $\beta\in(-1,0]$, $\beta=1$ and $\beta\in(1,\infty)$, respectively (cf. [NX], [AL]). Moreover, the $\mathcal{Q}_{\beta,0}$-version of Lemma 3.2 states that $f\in\mathcal{Q}_{\beta,0}$ if and only if $d\mu_{f,\beta}$ is a vanishing β-Carleson measure, i.e. $\lim_{|I|\to0}\mu_{f,\beta}(S(I))/|I|^\beta=0$ uniformly for all Carleson boxes $S(I)$ (cf. [ASX, Theorem 2.2]).

The purpose of mentioning $\mathcal{Q}_{\beta,0}$ is to solve another problem in [SZ]: "When is $C_\phi:\mathcal{B}_0\to\mathcal{Q}_\beta$ or $\mathcal{Q}_{\beta,0}$ compact?" The method of treating Theorem 1.2 can be adapted to provide an answer to this question.

For convenience, let $\Delta_r=\{z\in\Delta:|z|>r\}$ where $r\in(0,1)$. We have

3.4. COROLLARY. *Let $\beta\in(0,\infty)$ and let $\phi:\Delta\to\Delta$ be analytic. Then*

(i) *$C_\phi:\mathcal{B}_0\to\mathcal{Q}_\beta$ exists as a compact operator if and only if $\phi\in\mathcal{Q}_\beta$ and* (1.3) *holds.*

(ii) *$C_\phi:\mathcal{B}_0\to\mathcal{Q}_{\beta,0}$ exists as a compact operator if and only if $\phi\in\mathcal{Q}_\beta$ and*

$$\lim_{r\to1}\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}\left[\frac{(1-|z|^2)^{\beta/2}|\phi'(z)|}{1-|\phi(z)|^2}\right]^2 1_{\Delta_r}(z)\,dm(z)=0. \tag{3.6}$$

Proof. (i) *Sufficiency.* It follows from Theorem 1.2(ii).

Necessity. Suppose that $C_\phi:\mathcal{B}_0\to\mathcal{Q}_\beta$ is compact. Then $\phi\in\mathcal{Q}_\beta$ follows right away. Note that if $f\in\mathbb{B}_{\mathcal{B}}$ then $\|f_t\|_{\mathcal{B}}\le\|f\|_{\mathcal{B}}\le1$. Now for a fixed $t\in(0,1)$, put $\mathbb{B}^t_{\mathcal{B}}=\{f_t:f\in\mathbb{B}_{\mathcal{B}}\}$. Then $\mathbb{B}^t_{\mathcal{B}}$ is a subset of $\mathbb{B}_{\mathcal{B}_0}$. By compactness of C_ϕ, $C_\phi(\mathbb{B}_{\mathcal{B}_0})$ is a relatively compact subset of $\mathcal{Q}_\beta$. The proof of Theorem 1.2(ii) actually shows that for every $\varepsilon>0$ there is a $\delta\in(0,1)$ (independent of t) such that for $r\in[\delta,1)$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_t)'(z)|^2(1-|z|^2)^\beta 1_{\Omega_r}(z)\,dm(z)<\varepsilon.$$

This estimate and Lemma 3.1 result in

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}\left[\frac{t|\phi'(z)|(1-|z|^2)^{\beta/2}}{1-t^2|\phi(z)|^2}\right]^2 1_{\Omega_r}(z)\,dm(z)<2\varepsilon,$$

and so (3.6) follows, by Fatou's lemma.

(ii) *Sufficiency.* Let $\phi\in\mathcal{Q}_\beta$ and let ϕ satisfy (3.6). Suppose that $\{f_n\}\subset\mathbb{B}_{\mathcal{B}_0}$ is a sequence which converges to 0 l.u. on Δ. To prove that $C_\phi:\mathcal{B}_0\to\mathcal{Q}_{\beta,0}$ is compact, it suffices to verify that $\lim_{n\to\infty}\|C_\phi f_n\|_{\mathcal{Q}_\beta}=0$. For each $r\in(0,1)$ put $\widetilde{\Delta}_r=\Delta\setminus\Delta_r$. Since $\widetilde{\Delta}_r$ is a compact subset of Δ, $\{f_n'(\phi)\}$ tends to 0 uniformly on $\widetilde{\Delta}_r$. From $\phi\in\mathcal{Q}_\beta$ and Lemma 3.2 it is seen that

$$\lim_{n\to\infty}\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f_n)'(z)|^2(1-|z|^2)^\beta 1_{\widetilde{\Delta}_r}(z)\,dm(z)=0.$$

This limit, together with (3.6), gives $\lim_{n\to\infty}\|C_\phi f_n\|_{\mathcal{Q}_\beta}=0$.

Necessity. Let $C_\phi:\mathcal{B}_0\to\mathcal{Q}_{\beta,0}$ be compact. It is trivial to deduce that $\phi\in\mathcal{Q}_\beta$ and $C_\phi(\mathbb{B}_{\mathcal{B}_0})$ is a relatively compact subset of $\mathcal{Q}_{\beta,0}$. Given an $\varepsilon>0$, for every $f\in\mathbb{B}_{\mathcal{B}_0}$ there are finitely many functions $g_k\in\mathcal{Q}_{\beta,0}$ such that

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z)-g_k'(z)|^2(1-|z|^2)^\beta\,dm(z)<\varepsilon,$$

where we have used Lemma 3.2. Consequently, for all $r\in(0,1)$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z)-g_k'(z)|^2(1-|z|^2)^\beta 1_{\Delta_r}(z)\,dm(z)<\varepsilon.$$

Since $g_k\in\mathcal{Q}_{\beta,0}$, there is $\delta\in(0,1)$ such that for $r\in[\delta,1)$,

$$\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|g_k'(z)|^2(1-|z|^2)^\beta 1_{\Delta_r}(z)\,dm(z)<\varepsilon,$$

which implies

$$\sup_{f\in\mathbb{B}_{\mathcal{B}_0}}\sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}|(C_\phi f)'(z)|^2(1-|z|^2)^\beta 1_{\Delta_r}(z)\,dm(z)<2\varepsilon.$$

A careful inspection of the above argument for the necessity of (i) shows that (3.6) follows immediately from another application of Lemma 3.1 and Fatou's lemma to the last inequality. The proof is complete.

We close this section by an observation on the condition (1.3). It is clear that (1.3) holds if

$$\int_{\Delta}\left[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\right]^2 dm(z)<\infty. \tag{3.7}$$

Shapiro–Taylor [ST, Proposition 2.4] showed that (3.7) forces $C_\phi:\mathcal{D}\to\mathcal{D}$ to be a Hilbert–Schmidt operator. Tjani [T, Proposition 3.9] pointed out that (3.7) ensures that $C_\phi : \mathcal{B} \to \mathcal{D}$ is compact. Since $\mathcal{D} \subset \mathcal{Q}_\beta \subset \mathcal{B}$, our conditions (1.3) and $\phi \in \mathcal{Q}_\beta$ fill up the gap between $\mathcal{D}$ and $\mathcal{B}$ in the sense of the Hilbert–Schmidt property and compactness.

4. $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$. In this final section we show Theorem 1.3. A dyadic division of Δ, quite different from the one used for Theorem 1.1, will be involved to control Theorem 1.3.

4.1. *The dyadic division.* Following [AS] and [L], we divide Δ into dyadic boxes. Let $\mathcal{I}$ denote the family of dyadic arcs in $\partial\Delta$, that is, the family of all arcs of the form

$$\{z \in \partial\Delta : 2\pi k/2^l \le \arg z < 2\pi(1+k)/2^l\}, \quad k = 0, 1, \ldots, 2^l - 1,\ l = 0, 1, \ldots$$

Given an arc $I \subset \partial\Delta$, let $H(I)$ denote the half of $S(I)$ which is closest to the origin, namely,

$$H(I) = \{z \in S(I) : 1 - |I|/(2\pi) \le |z| < 1 - |I|/(4\pi)\}.$$

Note that the $H(I)$'s for $I \in \mathcal{I}$ are pairwise disjoint and cover Δ. Fix any enumeration $\{H_j : j = 1, 2, \ldots\}$ of these sets and select a point a_j in each H_j. Almost any point would work, but in order to simplify some parts later on let us agree that a_j is the "center" of H_j in the sense that $|a_j|$ and $\arg a_j$ bisect the interval of absolute values and the interval of arguments, respectively, of points in H_j. If $H_j = H(I)$ then $|I| \asymp 1 - |a_j|$.

4.2. *Proof of Theorem 1.3.* It is enough to verify the implications (i)⇒(iii)⇒(ii). Put $dm_{\beta,w,\phi}(z) = N(\beta, w, z, \phi)\,dm(z)$. With this choice, we establish

$$\|C_\phi f\|_{\mathcal{Q}_\beta} = |f(\phi(0))| + \sup_{w\in\Delta} \Big[\int_\Delta |f'(z)|^2\, dm_{\beta,w,\phi}(z)\Big]^{1/2}. \tag{4.1}$$

(i)⇒(iii). Suppose that $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$ is bounded. Then clearly ϕ is a member of $\mathcal{Q}_\beta$. In order to show that $dm_{\beta,w,\phi}$ satisfies (1.4), fix $\theta \in [0, 2\pi)$ and $u = [1 - (2\pi)^{-1}|I|]e^{i\theta}$. Consider, for any $c > 0$, the test function

$$g_u(z) = \exp\Big[\frac{c(1-|u|^2)^{m-2-\alpha}}{(1-\overline{u}z)^m}\Big],$$

where m is the smallest integer greater than $2+\alpha$. Then

$$g_u'(z) = \frac{cm\overline{u}(1-|u|^2)^{m-2-\alpha}}{(1-\overline{u}z)^{m+1}} \exp\Big[\frac{c(1-|u|^2)^{m-2-\alpha}}{(1-\overline{u}z)^m}\Big].$$

Since $\log(1+x) \le 1 + \log^+ x$ for $x \ge 0$,

$$\|g_u\|_{\mathcal{N}_\alpha} \le \frac{\pi}{1+\alpha} + c\int_\Delta \frac{(1-|u|^2)^{m-2-\alpha}(1-|z|^2)^\alpha}{|1-\overline{u}z|^m}\, dm(z) \le M, \tag{4.2}$$

Once again, this constant $M > 0$ is independent of u and it is determined by Lemma 4.2.2 of [Z]. Let I be the arc centered at $e^{i\theta}$. Then there is $\delta \in (0,1)$ such that for $|I| < \delta$,

$$\sup_{z\in S(I)} |1 - \overline{u}z| \le M_1|I|, \quad \inf_{z\in S(I)} \mathrm{Re}[(1-u\overline{z})^m] \ge M_2\,|I|^m,$$

and hence

$$\inf_{z\in S(I)} |g_u'(z)| \ge \frac{M_3|I|^{-(3+\alpha)}}{\exp(M_4|I|^{2+\alpha})},$$

where $M_1 > 0$ and $M_2 > 0$ rely upon δ and α only, but also give

$$M_3 = \frac{cm}{2(2\pi)^{m-2-\alpha}M_1^{m+1}}, \quad M_4 = \frac{cM_2}{(2\pi)^{m-2-\alpha}M_1^{2m}}.$$

By (4.1) and since $\log^+ x \le \log(1+x)$ on $[0,\infty)$,

$$\|C_\phi g_u\|^2_{\mathcal{Q}_\beta} \ge \frac{M_3^2 m_{\beta,w,\phi}(S(I))}{|I|^{2(3+\alpha)}\exp(2M_4|I|^{2+\alpha})}. \tag{4.3}$$

Appealing to the closed graph theorem, (4.3) and (4.2), one obtains (1.4) at once. On the other hand, if $|I| \ge \delta$, then (4.1) and $\phi \in \mathcal{Q}_\beta$ easily imply (1.4) too.

(iii)⇒(ii). Assume now that $\phi \in \mathcal{Q}_\beta$ and $dm_{\beta,w,\phi}$ is such that (1.4) is valid for all $c > 0$. For every $s > 0$ we choose a sequence $\{f_n\}$ in $\mathcal{N}_\alpha$ so that $\|f_n\|_\alpha \le s$ and $\{f_n\}$ converges to 0 l.u. on Δ. With the help of the dyadic division of Δ, for $f_n \in \mathcal{N}_\alpha$ let $a_j^* \in \overline{H}_j$ (closure of H_j) be a point where $|f_n'|$ attains its maximum on $\overline{H}_j$. If l is the integer such that H_j is contained in $A_l := \{z \in \Delta : 1 - 2^{-l} \le |z| < 1 - 2^{-(l+1)}\}$, then the set

$$S_j := \{z \in \Delta : 1 - 2^{-(l+1)} \le |z| < 1 - 2^{-(l+2)},\ |\arg z - \arg a_j^*| < 2^{-l-1}\}$$

contains a disc Δ_j with center a_j^* and radius comparable to 2^{-l}. Note that S_j intersects at most 6 of the sets H_k and that $1 - |z|^2 \asymp 2^{-l}$ whenever $z \in S_j$. Using these observations, (2.6) and the submean value property of $|f_n'|$, we find that to every $\varepsilon \in (0,1)$ there corresponds an $r \in (0,1)$ such that for all f_n and all $w \in \Delta$,

$$\begin{aligned}\int_{\Delta_r} |f_n'|^2\, dm_{\beta,w,\phi} &\le \sum_j \sup_{z\in H_j\cap\Delta_r} |f_n'(z)|^2\, m_{\beta,w,\phi}(H_j \cap \Delta_r)\\ &\le \varepsilon^{2(1+\alpha)} M_5 \sum_j |f_n'(a_j^*)|^2 (1-|a_j|^2)^4 \exp[-cM_6(1-|a_j|^2)^{2+\alpha}]\end{aligned}$$

$$\le \varepsilon^{2(1+\alpha)} M_7 \sum_j \int_{\Delta_j} |f_n'(z)|^2 (1-|z|^2)^2 \exp[-cM_8(1-|z|^2)^{2+\alpha}]\, dm(z)$$

$$\le \varepsilon^{2(1+\alpha)} M_7 \sum_j \int_{H_j} [|f_n'(z)|(1-|z|^2)]^2 \exp[-cM_8(1-|z|^2)^{2+\alpha}]\, dm(z)$$

$$\le \varepsilon^{2(1+\alpha)} M_9 \int_{\Delta} \exp[-(cM_8 - 4^{2+\alpha} M_0 s)(1-|z|^2)^{2+\alpha}]\, dm(z).$$

Since (1.4) holds for all $c > 0$, it follows from picking $c > 4^{2+\alpha} s M_0/M_8$ in the above estimates that

$$\int_{\Delta_r} |f_n'|^2\, dm_{\beta,w,\phi} < \varepsilon^{2(1+\alpha)} M_{10}. \tag{4.4}$$

Also since $\phi \in \mathcal{Q}_\beta$ and $f_n' \to 0$ uniformly on $\widetilde{\Delta}_r$, to the above ε and r there corresponds an integer $N > 0$ such that for $n \ge N$,

$$\int_{\widetilde{\Delta}_r} |f_n'|^2\, dm_{\beta,w,\phi} < \varepsilon \|\phi\|^2_{\mathcal{Q}_\beta}. \tag{4.5}$$

Putting (4.1), (4.4) and (4.5) together produces that $\|C_\phi f_n\|_{\mathcal{Q}_\beta} \to 0$ as $n \to \infty$.

To end this section, we present a $\mathcal{Q}_{\beta,0}$-version of Theorem 1.3.

4.3. COROLLARY. *Let $\alpha \in (-1,\infty)$, $\beta \in (0,\infty)$ and let $\phi : \Delta \to \Delta$ be analytic. Then the following are equivalent:*

(i) *$C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_{\beta,0}$ exists as a bounded operator.*
(ii) *$C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_{\beta,0}$ exists as a compact operator.*
(iii) *$\phi \in \mathcal{Q}_{\beta,0}$ and (1.4) holds for all $c > 0$.*

Proof. It suffices to show (iii)⇒(ii) because (ii)⇒(i) is trivial and (i)⇒(iii) follows from Theorem 1.3. So let (iii) be true. Since the polynomials are dense in $\mathcal{N}_\alpha$ and in $\mathcal{Q}_{\beta,0}$ (this is easily verified via the triangle inequality), if $f \in \mathcal{N}_\alpha$ then for every $\varepsilon > 0$ there is a polynomial P such that $\|f - P\|_{\mathcal{N}_\alpha} < \varepsilon$. Observe that (iii) asserts boundedness of $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_\beta$. So, there is a constant $M > 0$ such that $\|C_\phi f - C_\phi P\|_{\mathcal{Q}_\beta} < \varepsilon M$. Also since $\phi \in \mathcal{Q}_{\beta,0}$, it follows from the $\mathcal{Q}_{\beta,0}$-version of Lemma 3.2 that $\phi^n \in \mathcal{Q}_{\beta,0}$ for every integer $n > 0$. As a result, $C_\phi P \in \mathcal{Q}_{\beta,0}$. The triangle inequality and the density of the polynomials in $\mathcal{Q}_{\beta,0}$ yield $C_\phi f \in \mathcal{Q}_{\beta,0}$. In other words, C_ϕ maps $\mathcal{N}_\alpha$ into $\mathcal{Q}_{\beta,0}$. Furthermore, the last part of the proof of Theorem 1.3 shows that $C_\phi : \mathcal{N}_\alpha \to \mathcal{Q}_{\beta,0}$ is compact, that is, (ii) holds.

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A geometrical solution of a problem on wavelets

by

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Abstract. We prove the existence of nonseparable, orthonormal, compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity by using some basic techniques of algebraic and differential geometry. We even obtain a much stronger result: "most" of the orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$, of any regularity, are nonseparable.

1. Introduction. A *wavelet basis* for $L^2(\mathbb{R}^d)$ is an orthonormal basis of the type $\{2^{jd/2}\psi_i(2^jx-k) \mid i=1,\ldots,2^d-1,\ j\in\mathbb{Z} \text{ and } k\in\mathbb{Z}^d\}$. It can generally be obtained from a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subsets of $L^2(\mathbb{R}^d)$ called a *multiresolution analysis* because it has the following properties:

(a) $\bigcap_{j\in\mathbb{Z}} V_j=\{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j}=L^2(\mathbb{R}^2)$,

(b) $V_j\subset V_{j+1}$ for all j,

(c) there exists a function $\varphi(x)$, called the *scaling function*, that belongs to V_0 and such that $\{\varphi(x-k)\mid k\in\mathbb{Z}^d\}$ is an orthonormal basis for V_0 [Le, D, M].

The wavelets that this paper deals with are both compactly supported and generated by multiresolution analyses.

We will say that a wavelet basis is *separable* if the functions ψ_i may be written as products of monodimensional scaling functions and monodimensional wavelets.

There exists a one-to-one correspondence between the wavelet bases for $L^2(\mathbb{R}^d)$ and the filter banks that satisfy Cohen–Lawton's condition. More precisely, the Fourier transforms of the functions φ and $\psi_1,\ldots,\psi_{2^d-1}$ are given by

$$\widehat{\varphi}(\xi)=\prod_{k=1}^{\infty} M_0(2^{-k}\xi), \tag{1.1}$$

$$\widehat{\psi}_i(\xi)=M_i(2^{-1}\xi)\widehat{\varphi}(2^{-1}\xi), \tag{1.2}$$

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