Fractional Sobolev norms and structure of Carnot–Carathéodory balls for Hörmander vector fields

by

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Abstract. We study the notion of fractional $L^p$-differentiability of order $s \in (0, 1)$ along vector fields satisfying the Hörmander condition on $\mathbb{R}^n$. We prove a modified version of the celebrated structure theorem for the Carnot–Carathéodory balls originally due to Nagel, Stein and Wainger. This result enables us to demonstrate that different $W^{s,p}$-norms are equivalent. We also prove a local embedding $W^{1,p} \subset W^{q,q}$, where $q$ is a suitable exponent greater than $p$.

1. Introduction. It is well known that the classical theory of Sobolev spaces plays an important role in many problems concerning partial differential equations. It has also been realized in the last years that an essential tool in the study of second order differential operators arising from degenerate vector fields on $\mathbb{R}^n$ is the construction of generalized Sobolev spaces suitably related to the fields.

To motivate our discussion we recall some simple features of first order Sobolev spaces. Given a family $X_1, \ldots, X_m$ of (at least Lipschitz continuous) vector fields on $\mathbb{R}^n$, $X_j = \sum_{k=1}^n a_{j,k}(x) \partial / \partial x_k$, a natural generalization of the usual $W^{1,p}$ space can be defined by means of the norm

$$\|u\|_{W^{1,p} (\Omega)} = \|u\|_{L^p (\Omega)} + \|Xu\|_{L^p (\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $Xu = (X_1u, \ldots, X_mu)$ denotes the “degenerate gradient”, $X_j u = \sum a_{j,k} \partial_k u$. If we assume that the fields are smooth and satisfy the Hörmander condition (see (5)), then a Sobolev-type embedding holds for the space $W^{1,p}_X$. Namely, representing a function $u$ as a “convolution” by means of the fundamental solution $\Gamma$ of $\sum X_j^2$, using the estimates of $\Gamma$ and $\nabla \Gamma$ (see Nagel, Stein and Wainger [47] and Sánchez-Calle [52]), together with the continuity of some “fractional integration op-

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operators’ (see Capogna, Danielli and Garofalo [7]), one can show that, if $p$ is greater than 1 and $\Omega$ is a bounded set, then
\[ \|u\|_{L^p(\Omega)} \leq c \|Xu\|_{L^p(\Omega)}, \quad u \in C^0_0(\Omega), \]
for suitable $q = q(\Omega, X, p) > p$.

Several recent papers are devoted to the study of geometric and embedding properties of first order Sobolev spaces in various degenerate situations. We refer to Rothschild and Stein [49], Franchi and Lanconelli [22, 23], Jerison [35], Saloff-Coste [51], Varopoulos, Saloff-Coste and Coulhon [54], Capogna, Danielli and Garofalo [7, 8], Biolo and Mosco [4], Franchi, Lu and Wheeden [24, 25], Hajlasz and Koskelo [29, 30], Maheux and Saloff-Coste [46], Garofalo and Nieuw [27, 28], Franchi, Serapioni and Serra Cassano [26], Berhanu and Pesenson [3] and to the references of those papers.

The aim of this paper is to give some properties of a family of spaces which are “intermediate” between $L^p$ and $W^{1,p}_X$. It seems natural to define the fractional (semi)norm of order $s$, $0 < s < 1$, as a sum of fractional derivatives along the fields, setting
\[ [u]_{W^{s,p}(\Omega)} = \left( \sum_{j=1}^m \int_{\Omega} \int_{(e^{X_j}(x) \in \Omega)} \frac{dt}{|x|^{1+sp/d_j}} \left| u(e^{X_j}(x)) - u(x) \right|^p \right)^{1/p}, \]
where $\Omega$ is a bounded set, $1 \leq p < \infty$, and $t \mapsto e^{X_j}(x)$ denotes the integral curve of the field $X_j$, starting from $x$ at $t = 0$.

One of the results of this paper (Section 4) is that if Höldermand’s condition is satisfied, then the norm (1) is locally equivalent to
\[ [u]_{W^{s,p}(\Omega)} := \left( \int_{\Omega \times \partial \Omega} \frac{|u(x) - u(y)|^p}{d(x, y)^p|B(x, d(x, y))|} \, dx \, dy \right)^{1/p}. \]
In (2), $d$ denotes the $C^n$–Carnot–Carathéodory distance associated with the fields (see Section 2) and $|B|$ is the Lebesgue measure of the $d$-ball $B$ (we let $B(x, r) := \{ y : d(x, y) < r \}$). This equivalence result shows that the norm (1) is determined only by the distance $d$. A similar phenomenon occurs for first order Sobolev spaces (see Hajlasz and Koskelo [30, Theorem 11.11]).

The main tool in the proof of the equivalence between (1) and (2) (Section 3) consists of a new structure theorem for $d$-balls. Our result is a modified version of a “classical” theorem by Nagel, Stein and Wainger [47, Theorem 7]. Roughly speaking, we prove that for any $d$-ball $B = B(x, r)$, there exists a $C^1$ diffeomorphism $E$ defined on a neighborhood of the origin in $\mathbb{R}^n$ such that
\[ E(c_1 Q) \subset B \subset E(c_2 Q), \]
where $Q$ is a suitable box in $\mathbb{R}^n$, $c_1$ and $c_2$ are positive constants and $c_j Q := \{ c_j x : x \in Q \}$, $j = 1, 2$, is the homothetic box. The precise statement of our result is given in Theorem 3.1. We only remark that the difference between this theorem and Nagel–Stein–Wainger’s original result is an alternative choice of the “exponential” maps $E$. The new feature of our maps is that they can be easily factorized as a composition of a finite number of elementary translations along integral curves of the vector fields $X_1, \ldots, X_m$. These maps are already known in the literature (they appeared in Nagel, Stein and Wainger [47], Lanconelli [41], Varopoulos, Saloff-Coste and Coulhon [54] and Danielli [16]). However their properties have not been completely exploited. The results of Section 3 give a contribution in this direction. We also remark that our Theorem 3.1 has been used in the new proof of Jerison’s Poincaré inequality by E. Lanconelli and the author in [42].

We actually consider more general “anisotropic” norms, of the form
\[ [u]_{W^{s,p}(\Omega)} = \left( \sum_{j=1}^m \int_{\Omega} \left[ \int_{(e^{X_j}(x) \in \Omega)} \frac{dt}{|x|^{1+sp/d_j}} \left| u(e^{X_j}(x)) - u(x) \right|^p \right]^{1/p} \right)^{1/p}, \]
where the integer $d_j = d(X_j)$ is the formal degree (in the sense of [47]) of the field $X_j$ and $0 < s < 1$. In this weighted situation we prove the equivalence between (3) and (2) provided the distance $d$ is suitably defined taking account of the degrees of the fields. The interest of this generalization stems from the fact that (3) is related to the “parabolic” operator $X_0 + \sum_{j=m}^m X_j x_2$ if we let $d(X_0) = 2$ and $d(X_j) = \ldots = d(X_m) = 1$.

In Section 5 we prove an embedding result of the form
\[ [u]_{W^{s,p}(\Omega)} \leq c \|Xu\|_{L^p(\Omega)}, \quad u \in C^0_0(\Omega), \]
where $p > 1$ and $q > p$ is suitable. We also give a “parabolic version” of (4). The proofs of these results rely on some properties of the fundamental solutions of Höldermand operators, essentially established by Sánchez-Calle [52] and Nagel, Stein and Wainger [47].

Before closing this introduction we quote some papers partially related to ours. Bakry, Coulhon, Ledoux and Saloff-Coste [1, Section 8] prove, as an application of their results, that, in a general situation, the space $W^{s,p}$ defined by (2) embeds in $L^q$ for a suitable $q > p$. A similar embedding, for $p = 2$ and for Höldermand fields of type 2, is proved by Chemin and Xu [11]. Their spaces $W^{s,2}$, $s > 0$, are constructed by means of pseudodifferential techniques.

We finally remark that several results concerning fractional Sobolev spaces, in the particular situation of Carnot groups (all the fields have degree one and are left invariant on a nilpotent stratified Lie group) are given in Folland [19] and Saka [50]. The cited papers develop a quite rich theory. Here we obtain only partial results, but we work in a more general setting.

The paper is organized as follows. In Section 2 we recall some known results about Höldermand vector fields. In Section 3 we prove the structure
In Section 4 we study the equivalence between different $W^{s,p}$-norms. Section 5 is devoted to some embedding results.

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2. Notations and known results. In this section we recall some known properties of Hörmander's vector fields and we introduce the notations used in what follows.

The Hörmander condition. Consider a family of $m$ vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^n$, where $X_j = \sum_{k=1}^n a_{j,k}(x) \partial / \partial x_k$ and the functions $a_{j,k} = a_{j,k}(x)$ are smooth on $\mathbb{R}^n$. Denote by $[X, Y]$ the commutator of the fields $X$ and $Y$. Setting $\text{ad}(X)(Y) = [X, Y]$, we can write the commutators of higher order by means of the following standard notation: if $I = (i_1, \ldots, i_p)$ is a multi-index ($p \in \mathbb{N}$ and $1 \leq i_j \leq m$), we set

$$X_{[I]} = \text{ad}(X_{i_1})\text{ad}(X_{i_2})\ldots \text{ad}(X_{i_p})(X_{i_p}) = [X_{i_1}, [X_{i_2}, \ldots, X_{i_p}, \ldots]].$$

We say that the commutator $X_{[I]}$ has length $p$ and we write $|I| = p$. The original fields $X_j$ are commutators of length 1.

In what follows we assume that the fields satisfy the following Hörmander condition ([33]): for any $x \in \mathbb{R}^n$ there exists an integer $r$ such that

$$\text{span}\{X_{[I]}(x) : |I| \leq r\} = \mathbb{R}^n. \tag{5}$$

The properties of vector fields satisfying (5) have been widely studied in the last years. Many papers cited in the introduction deal with Hörmander's vector fields. Some more references related to this topic are Bony [5], Hörmander and Melin [34], Jerison and Sánchez-Calle [36], Kusuoka and Strook [39, 40], Lu [43, 44, 45], Xu [56], Citti, Garofalo and Lanconelli [15], Franci, Gallot and Wheeden [20], Citti and Di Fazio [14], Buckley, Koskela and Lu [6], Vodop'yanov and Markina [55], Capogna, Danielli and Garofalo [9, 10], Chernikov and Vodop'yanov [12], Krylov [38], Ben Arous and Gradinariu [2], Hajlasz and Strzelecki [31] and the references of those papers. (We have not mentioned here papers dealing with analysis on Carnot groups.)

Carnot-Carathéodory distances. We introduce some noneuclidean distances associated with a family of vector fields (cf. [17], [21] and [47]; see also [37]). Let $X_1, \ldots, X_m$ be Hörmander vector fields. Attach to any field $X_j$ a degree $d(X_j) \in \mathbb{N}$ (see [47]). Assign to the commutator $X_{[I]}$ the degree $d(X_{[I]}) = \sum d(X_{i_j})$. Denote by $Y_1, \ldots, Y_q$ an enumeration of all the commutators of length at most $r$, where $r$ is an integer large enough to ensure that $Y_1, \ldots, Y_q$ span $\mathbb{R}^n$ at each point of a fixed bounded set $\Omega_0 \subset \mathbb{R}^n$. Denote also by $\Gamma_{x,y}$ the space of absolutely continuous paths $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define

$$d(x, y) = \inf \{r > 0 : \exists \gamma \in \Gamma_{x,y} \text{ such that } \dot{\gamma}(t) = \sum_{j=1}^m a_{j}(t)X_j(\gamma(t)) \text{ and } |a_{j}(t)| < r^{-d(X_j)} \text{ a.e. in } [0, 1] \}. \tag{6}$$

Set also

$$\rho(x, y) = \inf \{r > 0 : \exists \gamma \in \Gamma_{x,y} \text{ such that } \dot{\gamma}(t) = \sum_{j=1}^n a_{j}(t)Y_j(\gamma(t)) \text{ and } |a_{j}(t)| < r^{-d(Y_j)} \text{ a.e. in } [0, 1] \}, \tag{7}$$

$$\rho_2(x, y) = \inf \{r > 0 : \exists \gamma \in \Gamma_{x,y} \text{ such that } \dot{\gamma}(t) = \sum_{j=1}^n a_{j}(t)Y_j(\gamma(t)) \text{ and } |a_{j}| < r^{-d(Y_j)} \text{ a.e. in } [0, 1] \}. \tag{8}$$

It is not elementary to prove that $d < \infty$, that is, given two points $x$ and $y$ there exists at least a path which connects $x$ and $y$ and whose tangent vector lies in $\text{span}\{X_j\}$. The existence of such a path (under Hörmander's condition) is a classical reachability result due to Chow [13]. Various "quantitative versions" of this result are contained in [41], [47] and [54]. Also our results of Section 3 give a proof of Chow's Theorem.

The functions $d$, $\rho$ and $\rho_2$ just introduced are trivially symmetric. Moreover $d$ and $\rho$ clearly satisfy the triangle inequality. The distance $\rho$ satisfies locally $\rho(x, y) \leq c(\rho_2(x, z) + \rho_2(y, z))$, where $c$ can be greater than 1. This inequality is a consequence of the local equivalence between $\rho$ and $\rho_2$ [47, Theorem 7].

A remarkable property of the distance arising from a family of Hörmander vector fields is a local estimate of the form

$$|x - y| \leq c_1 d(x, y) \leq c_2 |x - y|^{\varepsilon}. \tag{6}$$

Here $\varepsilon < 1$ depends on the geometric properties of the vector fields. The proof of (6) is easy for the distance $\rho$ (see [47, Proposition 1.1]). The fact that (6) holds for $d$ (in particular the second inequality) is a consequence of Theorem 3.1 (or of the results of the papers [47], [41] and [54], if all the fields have degree one).

We now introduce some notations in order to recall the results of [47]. Given a multi-index $I = (i_1, \ldots, i_p)$, $i_j \leq q$, we set
\[ \phi_{t,x}(h) = \exp \left( \sum_{j=1}^{n} h_j Y_j \right)(x), \quad h \in \mathbb{R}^n, \text{ small,} \]

\[ \lambda_f(x) = \det\{Y_{j_1}(x), \ldots, Y_{j_m}(x)\}, \quad \|h\|_f = \max_{j=1,\ldots,m} |h_j|^{1/d(Y_{j})}, \]
\[ d(I) = d(Y_{i_1}) + \ldots + d(Y_{i_n}). \]

Nagel, Stein and Wainger proved the following theorem.

**Theorem 2.1.** Let \( X_1, \ldots, X_m \) be Hörmander vector fields of degrees \( d_1, \ldots, d_m \) and let \( K \subset \mathbb{R}^n \) be a compact set. Then there exist \( r_0 > 0 \) and \( \eta > 0 \) such that, if \( x \in K \) and \( r < r_0 \), it is possible to find a multi-index \( I \) so that:

(i) \( |\lambda_f(x)|^{d(I)} \geq \frac{1}{2} \max_j |\lambda_{j}(x)|^{d(I)} \), where the maximum is taken over the set \( \{I = (j_1, \ldots, j_m) : f_k \leq g_k, k = 1, \ldots, m\} \);

(ii) if \( \|h\|_f < \eta r \), then \( \frac{1}{2} \lambda_f(x) \leq \det \phi_{f,x}(h)/\det h \leq 4 \lambda_f(x) \);

(iii) \( B_e(x, \eta r) \subset \phi_{f,x}(\{\|h\|_f < \eta r\}) \subset B_e(x, \eta r) \);

(iv) the function \( \phi_{f,x} \) is one-to-one on \( \{\|h\|_f < \eta r\} \).

An easy consequence of Theorem 2.1 is the polynomial behavior of the measure of the ball, i.e. \( |B(x, r)| \sim \sum_j |\lambda_j(x)|^{d(I)} \). This equivalence is uniform in \( x \) in each compact set \( K \) and \( r \leq r_0(K) \). Moreover, there exists \( c > 0 \) such that the following doubling property holds:

\[ |B(x, 2r)| \leq c |B(x, r)|, \quad x \in K, r \leq r_0. \]

**Fundamental solutions.** It has been proved by Sánchez-Calle [52] (see also [47]) that, given a family \( X_1, \ldots, X_m \) of Hörmander vector fields on \( \mathbb{R}^n, n \geq 3, \) and a bounded set \( \Omega \), there exists a kernel \( \Gamma(x, y) \) smooth off the diagonal of \( \Omega \times \Omega \) which is a fundamental solution of the differential operator \( \Delta : = \sum_{j=1}^{m} X_j^2 \) (1), i.e. the equation \( \Delta \phi = \delta, \phi \in C_0^\infty(\Omega) \), is solved by \( f(x) = \int \Gamma(x, y) \delta(y) \, dy \). The kernel also satisfies the estimates

\[ \frac{1}{c} \frac{d(x, y)^2}{|B(x, d(x, y))|} \leq \Gamma(x, y) \leq c \frac{d(x, y)^2}{|B(x, d(x, y))|}, \]

\[ |X_1 \Gamma(x, y)| \leq c \frac{d(x, y)}{|B(x, d(x, y))|} \quad \text{and} \quad |X_1 X_2 \Gamma(x, y)| \leq \frac{c}{|B(x, d(x, y))|}, \]

\[ x, y \in \Omega, d(x, y) \leq r_0 = r_0(\Omega). \]

In (9) each derivative can act both on the first and on the second argument.

Multiplying the equation \( \Delta \phi = \delta \) by a function \( u \in C_0^\infty(\Omega) \) and integrating by parts we obtain the representation formula

\[ u(x) = \int \Gamma(x, y) \cdot X_j u(y) \, dy. \]

In the “parabolic” case (see [47] and [18]), given a family \( X_0, X_1, \ldots, X_m \) of Hörmander vector fields, the representation formula (10) becomes \( u(x) = \int H(y, x) L u(y) \, dy \), where \( L = X_0 + \sum_j X_j X_j \), while \( H \) is a kernel satisfying the growth estimates

\[ |H(x, y)| \leq c \frac{d(x, y)^2}{|B(x, d(x, y))|}, \quad |X_j H(x, y)| \leq c \frac{d(x, y)^2 - d(x_j)}{|B(x, d(x, y))|}, \]

where \( j = 0, 1, \ldots, m, d(X_0) = 2, \) while \( d(X_1) = \ldots = d(X_m) = 1. \)

**Riesz potentials.** We give the generalization to our context of the classical continuity result concerning “fractional integration operators”. For any compact set \( K \) the “pointwise homogeneous dimension” defined by

\[ D(x) = \min\{d(Y_{i_1}) + \ldots + d(Y_{i_m}) : \lambda_f(x) \neq 0\}. \]

Recall that \( |B(x, r)| \sim \sum_j |\lambda_j(x)|^{d(I)} \). Thus \( |B(x, r)| \) behaves as \( r^{D(x)} \) as \( r \to 0 \). We will need the following result (see [7] for a proof).

**Theorem 2.2.** Let \( K \) be a compact set. Then there exists \( r_0 > 0 \) so that, for every ball \( B = B(x, r), x \in K, r \leq r_0 \), if we set \( \bar{D} = \max_{x \in B} D(x), \) and, for a fixed \( a \in ]0, \bar{D}[ \),

\[ I_a f(x) := \int_B f(y) \frac{d(x, y)^a}{|B(x, d(x, y))|} \, dy, \]

then for any \( p \in ]1, \bar{D}/a[ \), there exists \( c > 0 \) such that

\[ \|I_a f\|_{L^p(B)} \leq c \|f\|_{L^p(B)}, \quad f \in C_0^\infty(B), \quad q = \frac{\bar{D}}{\bar{D} - ap}. \]

The Campbell–Hausdorff formula. Given a smooth vector field \( X \), we denote by \( t \mapsto e^{tx}(X) \) the integral curve of \( X \) starting from \( x \) at \( t = 0 \). The map \( x \mapsto e^{tx}(X) \) is a diffeomorphism between suitable subsets of \( \mathbb{R}^n \). We denote this function by \( e^{tx}(X) \equiv \exp(tX) \). We will get useful information on the composition of \( e^{tx}(X) \) and \( e^{ty}(Y) \) by means of this version of the Campbell–Hausdorff formula.

**Proposition 2.3.** Let \( X \) and \( Y \) be smooth vector fields on the open set \( \Omega \subset \mathbb{R}^n \). Then the following formal equality holds:

\[ \exp(sX) \exp(tY) = \exp \left( sX + tyX - \frac{st}{2} [X, Y] + \sum_{k+t+1} s^k t^j C_{k,j} \right). \]

Here \( C_{k,j} \) denotes a finite linear combination of commutators. Any commutator contains \( k \) times the field \( X \) and \( t \) times the field \( Y \). The meaning of (12) is the following: for any fixed compact set \( K \subset \Omega \), given two integers
there exists \( r_0 > 0 \) such that, if \( |s_t| < r_0 \), then

\[
e^{sX}e^{tY}(x) = \exp \left( sX + tY - \frac{st}{2}[X,Y] + \sum_{k \leq k_0, j \leq j_0} s^k t^j C_{k,j} \right)(x) + O(s^k t^{k+1}) + O(t^{j+1}),
\]

where \( |O(t^n)| \leq \epsilon t^n \), uniformly in \( x \in K \).

Classical references on the Campbell–Hausdorff formula are Hochschild [32] and Serre [53]. The applications of this tool to our context are discussed in Hörmander [33], Rothschild and Stein [49], Nagel, Stein and Wainger [47] and Varopoulos, Saloff-Coste and Coulhon [54].

3. Structure of balls. In this section we construct some modified versions of the exponential maps used by Nagel, Stein and Wainger in the proof of their representation result [47, Theorem 7]. We consider a class of “almost exponential” maps, defined in (16), which can be factorized as a composition of a finite number of elementary translations along integral curves of the fields \( X_1, \ldots, X_n \). We prove in Theorem 3.1 that our maps give a good representation of the Carnot–Carathéodory balls. Our result also gives (not surprisingly) a proof of the equivalence between the distances \( d, \varrho \) and \( \varphi_2 \), for any choice of the degrees of the fields.

Let now \( S_1, \ldots, S_l \) be fields belonging to the family \( X_1, \ldots, X_n \). Set \( d_j = d(S_j) \), \( j = 1, \ldots, l \). Keeping the notations of (47), we define for \( a \in \mathbb{R} \),

\[
C_1(a, S_1) = \exp(a_{d_1} S_1),
\]

\[
C_2(a; S_1, S_2) = \exp(-a_{d_2} S_2) \exp(-a_{d_1} S_1) \exp(a_{d_2} S_2) \exp(a_{d_1} S_1),
\]

\[ \vdots \]

\[
C_l(a; S_1, \ldots, S_l) = C_{l-1}(a; S_2, \ldots, S_l) \exp(-a_{d_1} S_1) C_{l-1}(a; S_2, \ldots, S_l) \exp(a_{d_1} S_1).
\]

By the Campbell–Hausdorff formula and the Jacobi identity (a commutator of commutators is a sum of commutators), we get the following equality of formal series:

\[
C_2(a, S_1, S_2) = \exp \left( a_{d_1+d_2} [S_1, S_2] + \sum_{d(I) > d_1+d_2} c_I a^{d(I)} S_{[I]} \right),
\]

where \( c_I \) is a suitable number, \( I = (i_1, \ldots, i_p) \) is a multi-index, \( i_j \in \{1, 2\} \), \( p \in \mathbb{N} \), \( S_{[I]} \) is the commutator \([S_{i_1}, \ldots, [S_{i_{p-1}}, S_{i_p}] \ldots] \) and \( d(I) = d(i_1) + \ldots + d(i_p) \) is the degree of \( S_{[I]} \). Iterating and using again the Campbell–

Hausdorff formula and the Jacobi identity we have

\[
C_1(a; S_1, \ldots, S_l) = \exp \left( a_{d_1} + \ldots + a_{d(I)} S_{[I]} + \sum_{d(I) > d_1, \ldots, d_l} c_I a^{d(I)} S_{[I]} \right),
\]

where \( i_j \in \{1, \ldots, l\} \) and \( p \in \mathbb{N} \). If \( d_j = 1 \) for each \( j \), then (13) is contained in [47, Lemma 2.21].

Set now \( d = d(S_1) + \ldots + d(S_l) \). We can define, for \( \sigma \in \mathbb{R} \), small,

\[
\exp^*(\sigma S_{[\ldots, [S_1, \ldots] \ldots]} \right) = \begin{cases} C_1(\sigma/4; S_1, \ldots, S_l), & \sigma > 0, \\ C_1(\sigma/4; S_1, \ldots, S_l)^{-1}, & \sigma < 0. \\ \end{cases}
\]

Then, by means of (13), we discover that

\[
\exp^*(\sigma S_{[\ldots, [S_1, \ldots] \ldots]} \right) = \exp(\sigma S_{[\ldots, [S_1, \ldots] \ldots]} \right) + \text{sgn}(\sigma) \sum_{d(I) > d} c_I \sigma^{d(I)} S_{[I]}.
\]

Roughly speaking, the map \( \exp^*(S) \) can be thought of as an “approximate exponential” of a commutator \( S \). It is also factorizable in paths which are piecewise integral curves of the original fields. A computation of the derivative of the function \( \sigma \mapsto \exp^*(\sigma S_{[\ldots, [S_1, \ldots] \ldots]}(x)) \) is contained in Lemma 3.2 and it is the fundamental step in the proof of Theorem 3.1.

From now on we fix an open bounded set \( \Omega_0 \) and we denote by \( Y_1, \ldots, Y_q \) a fixed enumeration of the commutators of length at most \( r \), where \( r \) is so large that span\( \{Y_1(x), \ldots, Y_q(x)\} = \mathbb{R}^q \) at every \( x \in \Omega_0 \). Consider also a multi-index \( I = (i_1, \ldots, i_q) \), \( i_j \leq q \), and denote by \( U_j = Y_{i_1}, \ldots, U_n = Y_{i_n} \) the associated commutators. Set, for \( h \in \mathbb{R}^q \) small enough,

\[
E_I(x,h) = E_{I,2}(h) = \exp^*(h_1 U_1) \ldots \exp^*(h_n U_n)(x).
\]

We are now ready to state the main result of this section.

**Theorem 3.1.** Let \( K \subset \Omega_0 \) be a compact set. Then there exists \( \delta_0 > 0 \) and positive numbers \( a \) and \( b \), \( b < a < 1 \), so that, given any \( n \)-tuple \( I \) of commutators such that

\[
|\lambda_I(x)|^\delta(I) \geq \frac{1}{2} \max_j |\lambda_j(x)|^\delta(I),
\]

for \( x \in K \) and \( \delta < \delta_0 \), we have

(i) if \( |h_j| < \delta^a \) and \( J_h E_I(x, h) \) is the jacobian determinant of \( E_I(x, \cdot) \), then

\[
\frac{1}{2} |\lambda_I(x)| \leq |J_h E_I(x, h)| \leq 4|\lambda_I(x)|;
\]

(ii) if \( B_{\delta} \) and \( B_{\delta^a} \) are the balls with respect to the metrics \( \varrho \) and \( d \), then \( B_{\delta^a}(x, \delta^a) \subset E_{I,a}(\{ |h_j| < \delta^a \}) \subset B_{\delta}(x, \delta) \);

(iii) the function \( E_{I,a} \equiv E_I(x, \cdot) \) is one-to-one on \( \{ |h_j| < \delta^a \} \).

The proof of the theorem is organized as follows: (i) is an easy consequence of Lemmas 3.2–3.4. The proof of (ii) is contained in Lemma 3.5, while assertion (iii) is proved in Lemma 3.6.

Lemmas 3.2 and 3.3 below give the analogues of [47, Lemma 2.12].
Lemma 3.2. Let \( S_1, \ldots, S_l \) be vector fields of degrees \( d_1, \ldots, d_l \). Set \( U = [S_1, [S_2, [\ldots, [S_l, \ldots]]] \). If \( K \subset \subset \Omega_0 \) is a compact set and \( M \) is a fixed integer, then there exists \( \delta_0 > 0 \) such that, if \( 0 < \lambda < \delta_0 \) and \( x \in K \), then
\[
\frac{\partial}{\partial \lambda} \exp^*(\lambda U)(x) = U(\exp^*(\lambda U)(x)) + \sum_{d < h \leq M} \lambda^{h/d} Z_k(\exp^*(\lambda U)(x)) + R_M(\lambda, x),
\]
where \( d = d(U) \) is the degree of \( U \), \( Z_k \) is a linear combination of commutators of degree \( k \) of the fields \( S_1, \ldots, S_l \), while \( |R_M(\lambda, x)| \leq c|\lambda|^{(M+1)/d(U)-1} \), \( x \in K \), \( 0 < \lambda < \delta_0 \). An analogous result holds for \( -\delta_0 < \lambda < 0 \). In that case both \( Z_k \) and \( R_M \) have to be replaced by different expressions \( Z'_k \) and \( R'_M \), with the same properties.

Proof. Set \( u^d = \lambda \) and \( \exp^*(\lambda U)(x) = \xi \). Then
\[
\exp^*((u + \sigma)^d U)(x) - \exp^*(u^d U)(x) = \exp^*((u + \sigma)^d U) \exp^*(-u^d U)(\xi) - \xi.
\]

From now on, \( R \) denotes any formal series of the form \( \sum_{j,k \geq 0} \sigma^j u^k V_{j,k} \), where \( V_{j,k} \) is a vector field, and by \( Z_k \) any finite linear combination of commutators of degree \( k \). The \( R \) and \( Z_k \) may not be the same at each occurrence. We can also assume that \( 0 < \sigma < u/2 \). By means of (15) and the Campbell–Hausdorff formula, we get
\[
\exp^*((u + \sigma)^d U) \exp^*(-u^d U) = \exp\left((u + \sigma)^d U + \sum_{k > d} (u + \sigma)^k Z_k\right) \exp(-u^d U - \sum_{k > d} u^k Z_k)
= \exp\left(u^d U + \sum_{k > d} u^k Z_k + \sigma du^{d-1} U + \sigma \sum_{k > d} ku^{k-1} Z_k + \sigma^2 R\right)
\cdot \exp(-u^d U - \sum_{k > d} u^k Z_k)
= \exp\left(\sigma (du^{d-1} U + \sum_{k > d} ku^{k-1} Z_k)\right)
+ \sigma_1 \exp\left((u^d U + \sum_{k > d} u^k Z_k) (du^{d-1} U + \sum_{k > d} ku^{k-1} Z_k) + \ldots\right)
+ \sigma_{M-1} \exp\left((u^d U + \sum_{k > d} u^k Z_k)^{M-1} (du^{d-1} U + \sum_{k > d} ku^{k-1} Z_k)\right)
+ \sigma^2 R + \sigma u^{M} R,
\]
where the \( \sigma_j \)'s are constants coming from the Campbell–Hausdorff formula.

Each term of the finite sum can be written as
\[
\exp(\sigma \left( u^d U + \sum_{k > d} u^k Z_k \right)) \exp(\sigma \left( du^{d-1} U + \sum_{k > d} ku^{k-1} Z_k \right))
= \sigma \exp\left( \sum_{d < k \leq M} c_{j,p} u^p Z_p + \sigma u^M R \right),
\]
where \( Z_p \) is a linear combination of commutators of degree \( p \), while \( c_{j,p} \) is a constant. Then
\[
(18) \quad \exp^*(u^d U) \exp^*(-u^d U) = \exp\left(\sigma du^{d-1} U + \sigma \sum_{d < k \leq M} u^k Z_k + \sigma^2 R + \sigma u^M R\right).
\]

In other words
\[
\frac{1}{\sigma} \{ \exp^*((u + \sigma)^d U)(x) - \exp^*(u^d U)(x) \}
= \frac{1}{\sigma} \left\{ \exp\left(\sigma du^{d-1} U + \sigma \sum_{d < k \leq M} u^k Z_k\right)(\xi) - \xi \right\}
+ O(\sigma^2) + O(u^M)
\rightarrow du^{d-1} U(\xi) + \sum_{d < k \leq M} u^k Z_k(\xi) + O(u^M)
\]
as \( \sigma \to 0 \). Finally keeping in mind that we set \( u^d = \lambda \),
\[
\frac{\partial}{\partial \lambda} \exp^*(\lambda U)(x) = \frac{1}{\sigma} \frac{\partial}{\partial u} \exp^*(u^d U)(x)
= U(\exp^*(\lambda U)(x)) + \sum_{d < k \leq M} \lambda^{k/d} Z_k(\exp^*(\lambda U)(x))
+ O(|\lambda|^{(M+1)/d(u)-1})
\]
which ends the proof if \( \lambda > 0 \). In the case \( \lambda < 0 \) the proof is analogous. \( \blacksquare \)

The following lemma is an adapted version of [47, Lemma 2.12].

Lemma 3.3. Let \( E_l(x, h) \) be the map defined in (16). Let also \( K \) be a compact set. Then there exists \( \delta_0 > 0 \) such that, if \( x \in K \) and \( |h| < \delta_0 \), then
\[
(19) \quad \frac{\partial}{\partial h_j} E_l(x, h) = U_j(E_l(x, h)) + \sum_{\substack{k_1 + \ldots + k_l + j \leq d_j \leq M \leq d_j \leq M \ldots \leq M}} \frac{|h_1|^{k_1/d_1} \ldots |h_{j-1}|^{k_{j-1}/d_{j-1}} |h_j|^{k_j/d_j} Z_k(E_l(x, h))}{|h|^M \cdot |h|^{M-d_j} + O(|h|^{M+1} \cdot \ldots \cdot \ldots \cdot |h|^M)}.
\]
where $Z_k = Z_0^{(k)}$ denotes a finite linear combination (with constant coefficients) of commutators of degree $k_1 + \ldots + k_j$, which may possibly change if the coordinates of $h$ change their sign.

**Proof.** The case $j = 1$ is a consequence of Lemma 3.2. If $j = 2$ we need a computation of the derivative $\frac{\partial}{\partial u_2} \exp^*(h_1 U_1) \exp^*(h_2 U_2)(x)$. We can assume $h_1 \geq 0$ and $h_2 \geq 0$. Set $h_i = u_i^{d_i}$, where $d_i$ and $d_2$ are the degrees of $U_1$ and $U_2$. Write $\exp^*(u_1^{d_1} U_1) \exp^*(u_2^{d_2} U_2)(x) = \xi$. Then

\begin{equation}
\frac{\partial}{\partial u_2} \exp^*(u_1^{d_1} U_1) \exp^*(u_2^{d_2} U_2)(x) = \lim_{\sigma \to 0} \frac{1}{\sigma} (\exp^*(u_1^{d_1} U_1) \exp^*((u_2 + \sigma) u_2^{d_2} U_2) \exp^*(-u_2^{d_2} U_2) \exp^*(-u_1^{d_1} U_1)(\xi) - \xi).
\end{equation}

Let us now introduce the following notation:

\begin{align}
H &= u_1^{d_1} U_1 + \sum_{k > d_1} u_k^{d_1} Z_k, \\
W &= d_2 u_2^{d_2-1} U_2 + \sum_{d_2 < k_2 \leq M} u_2^{k_2-1} Z_{k_2}.
\end{align}

In view of (15), we can assert that $\exp^*(u_1^{d_1} U_1) = e^{\sigma H}$. Then, by means of (18) (choosing $U = U_2$) and (21),

\begin{equation}
\frac{\partial}{\partial h_2} \exp^*(h_1 U_1) \exp^*(h_2 U_2)(x) = e^{\sigma W + \sigma^2 R + \sigma u_2^M R} e^{-H} \exp\left\{H + \sigma W + \sum_{k \geq 1} \alpha_k \text{ad}(H)^k \left(\sigma W + \sigma^2 R + \sigma u_2^M R\right)\right\} e^{-H}
\end{equation}

\begin{align}
\text{where } \alpha_k, \beta_k, \gamma_k \text{ are suitable numbers. But}
\ad(H)(W) &= \left[ u_1^{d_1} U_1 + \sum_{k_1 > d_1} u_1^{k_1} Z_{k_1}, d_2 u_2^{d_2-1} U_2 + \sum_{d_2 < k_2 \leq M} u_2^{k_2-1} Z_{k_2}\right] \\
&= \sum_{d_2 < k_2 \leq M} u_1^{k_1} u_2^{k_2-1} Z_{k_1 + k_2} + u_1^{M+1} R,
\end{align}

where $Z_{k_1 + k_2}$ is a linear combination of commutators of degree $k_1 + k_2$. An analogous argument shows that all the other terms of (22) (those containing $\ad(H)(W)$) can be written in the same form. The explicit form (21) of $W$ enables us to rewrite (22) as follows:

\begin{align}
\exp^*(u_1^{d_1} U_1) \exp^*((u_2 + \sigma) u_2^{d_2} U_2) \exp^*(-u_2^{d_2} U_2) \exp^*(-u_1^{d_1} U_1) = \exp \left\{ \sigma W + \sigma \sum_{d_2 < k_2 \leq M} u_2^{k_1} u_2^{d_2-1} Z_{k_1 + k_2} + \sigma^2 R + \sigma u_2^M R + \sigma u_1^{M+1} R \right\}
\end{align}

\begin{align}
\exp \left\{ \sigma d_2 u_2^{d_2-1} U_2 + \sigma \sum_{k_1 \leq M, k_2 \leq M} u_2^{k_1} u_2^{k_2-1} Z_{k_1 + k_2} + \sigma^2 R + \sigma u_2^M R + \sigma u_1^{M+1} R \right\}
\end{align}

It is now easy to compute the derivative (20). Keeping in mind that we have written $\xi = \exp^*(u_1^{d_1} U_1) \exp^*(u_2^{d_2} U_2)(x)$ and $u_2^{d_2} = h_2$, we get

\begin{align}
\frac{\partial}{\partial h_2} \exp^*(h_1 U_1) \exp^*(h_2 U_2)(x) = \frac{\partial}{\partial h_2} \frac{\partial}{\partial u_2} \exp^*(u_1^{d_1} U_1) \exp^*(u_2^{d_2} U_2)(x)
\end{align}

\begin{align}
&= \frac{\partial}{\partial h_2} \frac{\partial}{\partial u_2} \exp^*(u_1^{d_1} U_1)(u_2^{d_2} U_2)(x) \\
&= \frac{h_2^{1/d_2-1}}{d_2} \left\{d_2 u_2^{d_2-1} U_2(\xi) + \sum_{k_1 \leq M, k_2 \leq M} u_2^{k_1} u_2^{k_2-1} Z_{k_1 + k_2}(\xi) + O(u_2^M) + O(u_1^{M+1})\right\}
\end{align}

\begin{align}
&= U_2(\xi) + \sum_{k_1 \leq M, k_2 \leq M} h_2^{1/d_2} h_2^{k_2/d_2-1} Z_{k_1 + k_2}(\xi)
\end{align}

\begin{align}
&= h_2^{1/d_2-1} U_2(\xi) + O(h_2^{1/d_2-1}) + O(h_2^{1/d_2-1} h_1^{1/d_1-1})
\end{align}

That proves the lemma if $j = 2$ (as $h_1^{1/d_1} \leq \|H_1\|$).

Finally, in order to compute the derivative $\frac{\partial}{\partial z_j} E(x, \cdot)$ in the case $j > 2$ it is enough to remark that, by means of the Campbell–Hausdorff formula,
we have
\[
\exp^* (u_1^{d_1} U_1) \exp^* (u_2^{d_2} U_2) \cdots \exp^* (u_{j-1}^{d_{j-1}} U_{j-1}) = \exp \left( \sum_{k_1 \geq d_1, \ldots, k_{j-1} \geq d_{j-1}} u_1^{k_1} u_2^{k_2} \cdots u_{j-1}^{k_{j-1}} Z_{k_1 + \cdots + k_{j-1}} \right) =: \exp (\tilde{H}),
\]
where \( Z_{k_1 + \cdots + k_{j-1}} \) is a linear combination of commutators of degree \( k_1 + \cdots + k_{j-1} \). The proof follows by the same argument, on setting \( \tilde{H} \) instead of \( H \).

**Lemma 3.4.** Let \( \chi > 0 \), let \( K \) be a compact set and \( t > 0 \). Then there exists \( \varepsilon = \varepsilon (t) > 0 \), depending also on \( \chi \) and \( K \), such that, if
\[
|\lambda_I (x)| \delta^{d(I)} \geq t \max_j |\lambda_I (x)| \delta^{d(j)}
\]
for some \( x \in K \), \( \delta > 0 \) and an \( n \)-tuple \( I \), then, as soon as \( ||h||_f < \varepsilon (t) \delta \), we have
\[
\partial_{h_j} E_I (x, h) = U_j (E_I (x, h)) + \sum_{s=1}^n b_{j,s} (E_I (x, h)) U_s (E_I (x, h)),
\]
where \( |b_{j,s} (E_I (x, h))| \leq \chi \delta^{d(U_s) - d(U_j)} \).

**Proof.** Let \( I \) be an \( n \)-tuple of commutators \( Y_i = U_1, \ldots, Y_n = U_n \), of degrees \( d_1, \ldots, d_n \) and such that (23) holds for suitable \( t > 0 \), \( x \in K \) and \( \delta > 0 \). By the triangle inequality, if \( \varepsilon (t) \) is small and \( ||h||_f \leq \max_j |\lambda_j (x)| \delta^{d(U_j)} \leq \varepsilon (t) \delta \), it is easy to see that the point \( E_I (x, h) \) belongs to the ball \( B_2 (x, c||h||_f) \), where \( c \) is an absolute constant. Then, by [47, Lemma 2.10] we get
\[
\frac{1}{2} |\lambda_I (x)| \leq |\lambda_I (E_I (x, h))| \leq 2 |\lambda_I (x)|.
\]
Moreover,
\[
|\lambda_I (x)| \delta^{d(I)} \geq t \max_j |\lambda_I (x)| \delta^{d(j)} \geq t (\max_j |\lambda_I (x)|) \delta^{d_{\max}(L)} \geq t c \delta^{d_{\max}(L)},
\]
by Hörmander's condition. Thus \( |\lambda_I (x)| \geq c t \delta^{N_0} \) provided \( N_0 \) is large enough (say, \( N_0 = \max_J (d(J) - d(K)) \)). We remark that the vectors \( U_j (E_I (x, h)) \), \( j = 1, \ldots, n \), are independent, by (25). Therefore the remainder in (19) can be written as follows:
\[
O(||h||_f^{M+1-d_j}) = \sum_{s=1}^n \mu_{j,s} (E_I (x, h)) U_s (E_I (x, h)),
\]
where \( \mu_{j,s} \) are suitable functions. Taking (25) into account and solving for the \( \mu_{j,s} \)'s, we have
\[
|\mu_{j,s} (E_I (x, h))| \leq \frac{c}{|\lambda_I (E_I (x, h))|} O(||h||_f^{M+1-d_j})
\]
\[
\leq \frac{c}{|\lambda_I (x)|} O(||h||_f^{M+1-d_j}) \leq \frac{c ||h||_f^{M+1-d_j}}{t \delta^{N_0}}
\]
\[
\leq \frac{c (c \delta)^{d+1-d_j}}{t} \frac{\delta - N_0}{2} \leq \frac{X \delta^{N_0}}{2}
\]
provided \( M + 1 = 2 N_0 + \max_j d (J) \) and \( \varepsilon = \varepsilon (t) \) is small enough.

Again by the independence of the \( U_k \)'s, we can write each term of the finite sum in (19) as \( Z_k = \sum_{s=1}^n a_k (U_s) \). But \( Z_k \) is a finite sum of commutators of degree \( k_1 + \cdots + k_{j-1} \). Therefore by [47, Theorem 6], we have \( |a_k (E_I (x, h))| \leq (\varepsilon / t) \delta^{d_{\max} (k_1 + \cdots + k_{j-1})} \) provided \( ||h||_f < \varepsilon (t) \delta \). Then the finite sum in (19) is estimated as follows:
\[
\sum_{k_1 \leq M, \ldots, k_{j-1} \leq M} \frac{||h||_f^{k_1 + \cdots + k_{j-1} - d}}{\delta^{d_{\max} (k_1 + \cdots + k_{j-1})}} \leq \frac{c}{t} \sum_{k_1 \leq M, \ldots, k_{j-1} \leq M} \frac{||h||_f^{k_1 + \cdots + k_{j-1} - d}}{\delta^{d_{\max} (k_1 + \cdots + k_{j-1})}}
\]
\[
\leq \frac{c \delta^{d_{\max} - d} \frac{\delta - N_0}{2}}{2} \leq \frac{X \delta^{N_0}}{2}
\]
as soon as \( \varepsilon (t) \) is small depending on the compact set \( K \), \( t \) and \( \chi \). That proves the lemma.

It is now easy to prove part (i) of Theorem 3.1 in the following way. By the properties of the determinant, if \( \delta_{jk} \) denotes the Kronecker delta, then
\[
\det (\delta_{js} + b_{js} (E_I (x, h))) = \det (\delta_{js} + b_{js} (E_I (x, h)) \delta^{d_{\max} - d}) \in [1/2, 2],
\]
provided we apply Lemma 3.4 for a suitably small \( \chi \) (depending on the dimension \( n \)). Computing the determinant in (24) and using (25) we get (i).

We can now prove assertion (ii) of Theorem 3.1.

**Lemma 3.5.** Let \( K \) be a compact set and \( t > 0 \). Then there exist positive numbers \( m \), \( \varepsilon = \varepsilon (t) \), \( \eta = \eta (t) \) and \( \delta_0 \) (also depending on \( K \)) such that, if \( |\lambda_I (x)| \delta^{d(I)} \geq t \max_j |\lambda_I (x)| \delta^{d(j)} \), then
\[
B_{\eta} (x, \eta (t) \varepsilon (t) \delta) \subset E_I (x, ||h||_f < \varepsilon (t) \delta) \subset B_{\delta_0} (x, m \varepsilon (t) \delta).
\]
**Proof.** The second inclusion is a trivial consequence of the definition of the distance \( d \). The other one has been proved in [47, Lemma 2.16]. For the convenience of the reader we give a proof. The argument is very similar (but not identical) to the one of the cited authors.

Let \( y \in B_{\eta} (x, \eta (t) \varepsilon (t) \delta) \) (\( \eta \) and \( \varepsilon \) will be made precise later). Then there exists an absolutely continuous path \( \varphi \) (we can assume that \( \varphi \) is one-to-one) such that \( \varphi (s) = \sum_{j=1}^n b_j (x) Y_j (\varphi) \) if \( 0 < s < 1 \), \( \varphi (0) = x \), \( \varphi (1) = y \) and
In order to prove the existence of a path $\theta$ satisfying (26), we can set
$$\Sigma = \{ s_0 \in [0, 1] : \text{there exists } \theta \text{ a.c. such that } \theta(0) = x$$
$$\text{and } E_I(x, \theta) \equiv \varphi \text{ on } [0, s_0] \}. $$

We remark that, if $s_0 \in \Sigma$, then the function $\theta$ is unique (if we had $\theta_1$ and $\theta_2$ the set $\{ s \in [0, s_0] : \theta_1(s) = \theta_2(s) \}$ would be open, closed and nonempty).

We now prove that $\Sigma = [0, 1]$. $\Sigma$ is open: if $s_0 \in \Sigma$, then since $E_I(x, \cdot)$ is a diffeomorphism near $\theta(s_0)$, we can extend the map $\theta$ on an interval $[0, s_0 + \sigma]$ for a small $\sigma > 0$. To prove that $\Sigma$ is closed, we take a sequence $s_j \in \Sigma$, $s_j \to s_0$. By uniqueness we have a map $\theta' : [0, s_0] \to \{ ||h||_1 < \varepsilon(t)\delta/2 \}$. Then $\theta([0, s_0])$ is contained in a set where $E_I$ is “strictly” nonsingular. Then $\theta(s_j)$ is a Cauchy sequence. More precisely, for any $s < s_0$, we denote again by $\Psi^s$ a local inverse of $E_I|_s$ defined near $\varphi(s)$ (which sends $\varphi(s)$ to $\theta(s)$). We have
$$|\theta(s_j) - \theta(s_k)| = \left| \int_{s_k}^{s_j} \frac{d}{ds} \Psi^s(\varphi(s + \sigma)) \right| ds$$
$$\leq c \sup_{||h||_1 < \varepsilon(t)\delta/2} |h(s)| s_j - s_k.$$

Here we used the fact that the path $\theta$ lies in the set $||h||_1 < \varepsilon(t)\delta/2$; on that set (25) holds, and gives easily the estimate $||dE_I(\theta(s_0))^{-1}|| \leq c_1/|\lambda_I(x)|$. Then $\theta(s_j) \to h_0$ and $||h_0||_1 < \varepsilon(t)\delta/2$. Thus $\Sigma = [0, 1]$ and finally $E_I(x, \theta(1)) = y = \theta(1))_1 < \varepsilon(t)\delta/2$. 

We now prove the injectivity of the map $E$ (part III) of Theorem 3.1.

**Lemma 3.6.** For any compact set $K$, there exist $\delta$ and $\alpha$ depending on $K$ such that, for $x \in K$, $\delta < \bar{\delta}$ and $I$ such that
$$|\lambda_I(x)| d^{(I)} \geq \frac{1}{2} \max_J |\lambda_J(x)| d^{(J)},$$
then the function $E_I(x, \cdot)$ is one-to-one on the set $||h||_1 < \alpha \delta$.

**Proof.** The argument here is the same as that of [47, pp. 123–133]. We give some details for the reader who is not familiar with problems concerning global inversion of functions.

For any $x \in K$, among the $n$-tuples such that $|\lambda_I(x)| > 0$ we first select all the $n$-tuples of minimal degree. Then we choose from this family an $n$-tuple $I$ such that $|\lambda_I(x)|$ is maximal. Under this choice, for suitable $\delta > 0$, we have $|\lambda_I(x)| d^{(I)} \geq \max_J |\lambda_J(x)| d^{(J)}$ for any $\delta < \delta_0$.

Then by a compactness argument, we cover $K$ by a finite family of open sets $\cup U_k \supset \{ x_k \}$, $k = 1, \ldots, p$, such that there exist $\delta_k > 0$ and an $n$-tuple
one, we can deform the closed path \( E_{I_1}(r(s)) := \gamma(s) \) to a point, letting \( q(\lambda, \gamma) = E_{I_1}(\lambda E_{I_1}^{-1}(y) + (1 - \lambda) E_{I_1}^{-1}(\gamma(s))) \), \((\lambda, \gamma) \in [0,1] \times [0,1]\). Again by (32) we have \( q(\lambda, \gamma) \in \{ \| h \|_{I_2} < \gamma \delta_1 \} \). We can now use the same argument used in the proof of Lemma 3.5—essentially the nonsingularity of the map \( E_{I_1} \) to construct, for any fixed \( s \in [0,1] \), a (unique) lifted path \( \lambda \mapsto p(\lambda, s) \), globally defined on \([0,1], \) such that \( p(0, s) = r(s) \) and \( E_{I_1}(p(\lambda, s)) = q(\lambda, s) \). A standard argument (see for example [48]) shows that the function \( p \) is actually continuous on \([0,1] \times [0,1]\). Moreover, \( q(\lambda, 0) = y, \lambda \in [0,1] \). Thus \( E_{I_1}(p(\lambda, 0)) = y, \lambda \in [0,1] \). But \( E_{I_1} \) is a local diffeomorphism. Therefore \( p(\lambda, 0) = \text{constant} = p(0, 0) = h \). Analogously, \( p(\lambda, 1) = \text{constant} = p(0, 1) = h' \). Finally, \( E_{I_1}(p(1, s)) = q(1, s) = y \) if \( s \in [0,1] \). Thus \( p(1, s) \) is constant in \( s \) and takes the values \( h \) and \( h' \), respectively, for \( s = 0 \) and \( 1 \). Consequently, \( h = h' \).

Now we know that \( E_{I_1} \) is one-to-one on \( \{ \| h \|_{I_2} < \alpha_3 \delta_1 \} \). Moreover (31) holds with \( \delta_2 \) instead of \( \delta_1 \), and \( I_2 \) instead of \( I_1 \). Thus the first inclusion in (32) becomes \( E_{I_1}(\{ \| h \|_{I_2} < \alpha_2 \delta_2 \}) \subset E_{I_2}(\{ \| h \|_{I_3} < \beta \delta_2 \}) \). Choose \( \zeta \) such that \( \zeta(t) < \alpha_1 \). By Lemma 3.5 we have

\[
E_{I_1}(\| h \|_{I_2} < \alpha_1 \delta_2) \supset E_{I_2}(\| h \|_{I_3} < \zeta(t) \delta_2) \supset E_{I_2}(\| h \|_{I_3} < \zeta(t) \delta_2)
\]

\[
\supset E_{I_2}(\| h \|_{I_3} < \zeta(t) \delta_2)/m = E_{I_2}(\| h \|_{I_3} < \alpha_2 \delta_2).
\]

\( \alpha_2 \) is defined by the last equality and depends only on the compact set \( K \). Thus, by the same argument as before it is easy to prove that \( E_{I_2} \) is one-to-one on \( \{ \| h \|_{I_3} < \alpha_2 \delta_2 \} \).

Iterating (at most) \( N \) times, where \( N \) is the number of \( n \)-tuples available, and setting \( \alpha_N = \alpha \) we complete the proof. 

4. A characterization of the spaces. In this section (Proposition 4.1 and 4.2) we prove the local equivalence between the fractional norms (3) and (2). 

PROPOSITION 4.1. Let \( X_1, \ldots, X_m \) be a family of Hörmander vector fields on \( \mathbb{R}^n \) and denote by \( d_1, \ldots, d_m \) their degrees. Let also \( \tilde{\Omega} \subset \mathbb{R}^n \) be an open bounded set and fix an open set \( \Omega \subset \subset \tilde{\Omega} \). Then there exist \( r_0 > 0 \) and \( c > 0 \) such that, for any \( u \in C^1(\tilde{\Omega}) \),

\[
\int_{\Omega \times \tilde{\Omega}} \frac{|u(x) - u(y)|^p}{d(x, y)^{p \rho}} |B(x, d(x, y))| dx dy
\]

\[
\leq c \sum_{j=1}^m \int_{x \in \tilde{\Omega}} \int_{t \in \mathbb{R}} \frac{dt}{|t|^{1 + p \rho/2}} |u(x) - u(e^{it} x_j(x))|^p.
\]

\( e^{it} \) is the exponential map associated with the vector field \( X_j \).
Proof. The proof relies on the results of Section 3. Roughly speaking, by means of the maps constructed there we can connect two points \( z \) and \( y \) by suitable integral curves of the fields.

We begin by fixing \( r_0 = b \delta_0 \), where \( \delta_0 \) and \( b \) are the constants appearing in Theorem 3.1, applied to the compact set \( K = \Omega \). Let \( z, y \in \Omega \). For any \( n \)-tuple \( I \) we define

\[
M_{I,z} = \left\{ y \in \mathbb{R}^n : d(z,y) < r_0 \right\}
\]

\[
\left| \lambda_I(z) \right| \left( \frac{d(z,y)}{b} \right)^d \geq \frac{1}{2} \max \left\{ \lambda_I(x) \right\} \left( \frac{d(x,y)}{b} \right)^d
\]

It is easy to see that the set \( M_{I,z} \) is an annulus (possibly empty), i.e. for some \( 0 \leq R_{I,z} \leq R_{I,x} \) we have \( M_{I,z} = \{ y \in \mathbb{R}^n : r_{I,z} < d(x,y) < R_{I,z} \} \), where \( \alpha \wedge \beta = \min\{\alpha, \beta\} \). Thus, for any \( x \in \Omega \) we also have \( \bigcup_I M_{I,z} = B(x, r_0) \).

\[
\int_{\{ \alpha \wedge \beta \}} \frac{|u(x) - u(y)|^p}{d(x,y)^{ns}} \|B(x, d(x,y))\| dx \ dy
\]

\[
\text{where } \sigma_j = \pm 1. \text{ This last expression is actually incorrect; the definition of the map } E_I \text{ depends on the sign of the } h_j \text{'s (cf. (14)). This difficulty can be easily overcome by splitting the last integral of (37) in } 2^a \text{ integrals such that in each integral the sign of the } h_j \text{'s does not change.}

Thus the right hand side of (37) can be estimated by means of a finite sum of terms of the form

\[
\int_{\|h\|_I < (a/b)/2} \|h\|_I \|I\|^{(d+ps)(I)}^{1/2} \times \int_{\|h\|_I < (a/b)/2} \|h\|_I \|I\|^{(d+ps)(I)}^{1/2} \times \int_{\|h\|_I < (a/b)/2} \|h\|_I \|I\|^{(d+ps)(I)}^{1/2}
\]

Belongs to a fixed open set \( \Omega^* \) such that \( \Omega \subset \subset \Omega^* \subset \tilde{\Omega} \). For any fixed \( h \), \( x \mapsto z \) is a change of variable whose Jacobian is bounded by geometric constants (depending on the fields and on the sets \( \Omega \) and \( \tilde{\Omega} \)).

Our next step is to split the variables in the integral (38) and to apply Fubini's Theorem. For fixed \( j = 1, \ldots, n \), let \( h = (h_j, \tilde{h}_j) \), \( d_j = (d(u_1), \ldots, d(u_n)) \), and \( d(I) = \sum_k d_k \). Then \( \|h\|_I = \max \|h\|_I^{1/d_k} \sum_k h_k^{1/d_k} \sim h_k^{1/d_k} + \tilde{h}_j \).
and for any measurable function $\psi \geq 0$ of a real variable, we have

\begin{equation}
\sum_{j=1}^{m} \int \frac{\psi(h_j)}{||h_j||^{d(I)+ps}} \, dh_j \, d\hat{h}_j \\
\sim \sum_{||h_j|| < \delta_0} \frac{\psi(h_j)}{(h_j)^{\frac{d(I)+ps}{d_I}}} \, dh_j \, d\hat{h}_j \\
= \int \frac{\psi(h_j)}{||h_j||^{d(I)+ps}} \, dh_j \int_{||\hat{h}_j|| < \delta_0} \frac{d\hat{h}_j}{(1 + ||\hat{h}_j||)^{d(-I)+ps}}.
\end{equation}

Set now $h_k = |h_j|^{d_k/d_j} u_k$, $k \neq j$. Then $\hat{h}_j = |h_j|^{d(I)/d_j} - 1 \, d\hat{u}_j$ and moreover (40) takes the form

\begin{equation}
\sum_{j=1}^{m} \int \frac{dh_j}{h_j^{(d(I)+ps)/d_j}} \int_{||h_j|| < \delta_0} \frac{d\hat{h}_j}{(1 + ||\hat{h}_j||)^{d(I)+ps}} \\
= c \int \frac{\psi(h_j)}{|h_j|^{1+ps/d_j}} \, dh_j.
\end{equation}

We have also used the fact that $\int_{\mathbb{R}^{d-I}} (1 + ||\hat{h}_j||)^{-(d(I)+ps)} \, d\hat{h}_j < \infty$.

In order to estimate (38) we perform the change of variable $x \mapsto z$ (see (39)) and integrate in $d\hat{h}_k$. Then (38) is less than

\begin{equation}
\int \frac{dh}{||h||^{d(I)+ps}} \int \frac{dx |u(\exp(h_{\hat{r}}) \sigma_{X_{\hat{r}}}) - u(x)|^p}{d_{\hat{r}}}
\leq c \int \frac{dx}{d_{\hat{r}}^{1+ps/d_{\hat{r}}}} |u(\exp(h_{\hat{r}}) \sigma_{X_{\hat{r}}}) - u(x)|^p
\leq c \int \frac{dx dt}{|t|^{1+ps/d(X_t)}} |u(\exp(tX_t)) - u(x)|^p.
\end{equation}

In the last inequality we have again changed variable, letting $h_{\hat{r}} \sigma_{X_{\hat{r}}} = t$.

We have also assumed that $r_0$ is small enough such that $e^{r_0} X_t \subset \Omega$, for any field $X$ of the family and for each $|t| < r_0 |d(X)|$.

The counterpart of Proposition 4.1 is:

**Proposition 4.2.** Let $X_1, \ldots, X_m$ be a family of Hörmander vector fields on $\mathbb{R}^n$. Fix $x_0 \in \mathbb{R}^n$. Then there exists a neighborhood $U$ of $x_0$ so that

for any $O \subset \subset \tilde{O} \subset \subset U$ there exist positive constants $\alpha$, $\delta_0$ and $c$ such that

$$
\int \frac{dt}{|t|^{1+ps/d(X_t)}} |u(e^{\alpha t} X_t(x)) - u(x)|^p
\leq c \int_{\delta_0 < d(x, y) \leq \beta} \frac{|u(x) - u(y)|^p}{d(x, y)^{ps/d(X_t)}}
$$

The proof of Proposition 4.2 relies on the so-called lifting method, introduced by Rothschild and Stein. In [49, Theorem 4] the following is proved. Assume that $X_1, \ldots, X_m$ are smooth vector fields whose commutators of length at most $r$ span $\mathbb{R}^n$ at a point $x_0 \in \mathbb{R}^n$. Then there exists an open set $U$ containing $x_0$ so that it is possible to introduce new coordinates $\tau \in V \subset \mathbb{R}^d$ and new fields $\hat{X}_j = X_j + \sum_{l=1}^{d} a_{jl}(x, \tau) \partial / \partial \tau_l$, on $U \times V \subset \mathbb{R}^{n+d}$, which are free up to $r$. That means that the only linear relations (at any point of $U \times V$) between the commutators of length at most $r$ of the fields $\hat{X}_j$ are given by antisymmetry and the Jacobi identity.

Following [49, p. 272], we can select $q$ commutators

\begin{equation}
\hat{Y}_1 = \hat{X}_1, \ldots, \hat{Y}_q = \hat{X}_m, \hat{Y}_{m+1}, \ldots, \hat{Y}_d
\end{equation}

linearly independent at any point of $U \times V$. The remaining commutators can be expressed as linear combinations (with constant coefficients, given by antisymmetry and the Jacobi identity) of $\hat{Y}_1, \ldots, \hat{Y}_q$.

The proof of Proposition 4.2 relies on the following lemmas.

**Lemma 4.3.** Let $E \subset \subset U$ and $H \subset \subset V$. Then there exist $\delta_0 > 0$, $b < 1$ and $c > 0$ such that, if $x, y \in E$ and $d(x, y) < \delta_0$, we have

$$
\int \frac{d\tau d\sigma}{d((x, \tau), (y, \sigma))^{Q+ps}} \leq c \frac{1}{d(x, y)^{ps/d(X_t)}}
$$

Here $d$ denotes the distance defined by the $\hat{Y}_j$'s.

**Proof.** It is proved in [47, Lemma 3.2] (see also [35, Lemma 4.1]) that, for $Q = d(\hat{Y}_1) + \ldots + d(\hat{Y}_d)$, if $x \in E$ and $\delta < \delta_0$, then

\begin{equation}
|(x, \tau), (y, \sigma)| \leq c \frac{B(x, \tau, \delta)}{B(x, \delta)} \sim \frac{\delta Q}{B(x, \delta)}.
\end{equation}

with constants depending on the choice of the sets $E, H, U, V$.

Now $d((x, \tau), (y, \sigma)) \equiv d(x, y) \delta$. Take $x, y \in E$ with $d(x, y) = \delta_0$. From (43) we get
Here $\psi \geq 0$ is any measurable function. Then

\begin{align*}
\int_{G} d\xi \int_{\exp(\alpha \tilde{Y}_j)(\xi) \in G} & \frac{dh_j}{|h_j|^{1+ps/d_j}} |u(e^{h_j \alpha Y_j}(\xi)) - u(\xi)|^p \\
\leq c \int_{\tilde{G}} d\xi \int_{\frac{1}{|h|} < \delta_0} \frac{dh}{|h|^{Q+ps}} |u(\tilde{\theta}(\xi, h) - u(\xi)|^p \\
\leq c \left\{ \int_{\tilde{G}} d\xi \left( \int_{\frac{1}{|h|} < \delta_0} \frac{dh}{|h|^{Q+ps}} |u(\tilde{\theta}(\xi, h))| \right)^p \\
& + \int_{\tilde{G}} d\xi \int_{\frac{1}{|h|} < \delta_0} \frac{dh}{|h|^{Q+ps}} |u(e^{h \alpha Y_j}(\xi)) - u(\tilde{\theta}(\xi, h))|^p \right\}.
\end{align*}

The first term can be handled by performing the change of variable $h \mapsto \eta = \tilde{\theta}(\xi, h)$, whose Jacobian is strictly nonsingular as soon as $|h| < \delta_0$ and $\delta_0$ is small enough. Moreover by the properties of the distance, we have $d(\xi, \eta) \leq |\eta|$. Thus the integral is under control by means of

$$c \int_{G^* \times \tilde{G}} d\xi d\eta \frac{|u(\xi) - u(\eta)|^p}{d(\xi, \eta)^{Q+ps}}.$$ 

Concerning the second integral we write $\exp(h_j \alpha \tilde{Y}_j)(\xi) = \eta$ and we consider the function $\tilde{\theta}_0(\eta, h) = \exp(h \tilde{Y}_1 + \ldots + h \tilde{Y}_q)(\xi)$. The function just introduced has continuous first derivatives in all the variables, $h, \eta$ and $\alpha$. By means of the Inverse Function Theorem, there exist $\epsilon > 0$ and $\delta_0 > 0$ such that, as soon as $0 < \alpha < \epsilon$, $\eta \in G^*$, the function $h \mapsto \tilde{\theta}_0(\eta, h)$ restricted to $|h| < \delta_0$ has an inverse. On the other hand,

$$d(\eta, \tilde{\theta}_0(\eta, h)) \leq \text{const} \cdot |h|$$

provided $\alpha$ is small enough. Then the last line of (44) becomes

$$\int_{G^* \times \{ |h| < \delta_0 \}} \frac{d\xi dh}{|h|^{Q+ps}} |u(e^{h \alpha Y_j}(\xi)) - u(\tilde{\theta}_0(\xi, h))|^p.$$ 

\textbf{Proof of Proposition 4.2.} Let $x_0 \in \mathbb{R}^n$. Choose an open set $U$ containing $x_0$ and small enough such that the results of Rothschild and Stein hold. We
can denote by \((x, \tau)\) \(\in U \times V\) the new coordinates. Let now \(O \subset \subset \tilde{O} \subset \subset U\).

We fix an open set \(H_1 \subset \subset H \subset \subset \tilde{H} \subset \subset V\). If \(\delta_0\) is small enough (depending on the choice of the sets), as soon as \((x, \tau) \in O \times H_1, |\tau| < \delta_0\) and \(e^{itX_j}(x) \in O, \) then \(e^{itX_j}(x, \tau) \in O \times H\). Thus

\[
\int_{O} dx \int_{|\tau| < \delta_0} \frac{dt}{|1 + px/d_j|} |u(e^{itX_j}(x, \tau) - u(x))|^p \leq c \int_{O \times H_1} dx \int_{|\tau| < \delta_0} \frac{dt}{|1 + px/d_j|} |u(e^{itX_j}(x, \tau) - u(x))|^p.
\]

This last expression can be handled by using first Lemma 4.4 (with \(G = O \times H\) and \(\tilde{G} = \tilde{O} \times \tilde{H}\)) and secondly Lemma 4.3 (with \(E = \tilde{O}\)). The last term of (45) can be finally estimated by

\[
\int_{\partial \times \tilde{H}} dx \int_{\partial \times \tilde{H}} dy \frac{|u(x) - u(y)|^p}{d((x, s), (y, \sigma))^{Q + ps}} \leq \int_{\partial \times \tilde{H}} dx \int_{\partial \times \tilde{H}} dy \frac{|u(x) - u(y)|^p}{d((x, s), (y, \sigma))^{Q + ps}}.
\]

5. Some embedding results. In this section we give some embedding results (Theorems 5.1 and 5.2) concerning the spaces introduced before. We use the representation of smooth functions by means of fundamental solutions. The embedding results hold either in the case \(d(X_j) = 1\) (we then use the fundamental solution of \(\sum X_j^2\)) or in the case where only one among the fields (say \(X_0\)) has degree two, while \(X_1, \ldots, X_m\) have degree one (in that case we use the fundamental solution of \(X_0 + \sum X_j^2\)).

**Theorem 5.1.** Let \(X_1, \ldots, X_m\) be a family of Hörmander vector fields on \(\mathbb{R}^n\). Denote by \(d\) the distance associated with \(\sum X_j^2\). Fix a compact set \(K \subset \mathbb{R}^n\). There exists \(r_0 = r_0(K)\) such that for any ball \(B = B(x_0, r)\), \(x_0 \in K, r \leq r_0\), the following holds:

\[
[u]_{W^{s,q}(B)} \leq c \frac{m}{j=1} \|X_j u\|_{L^p(B)}, \quad u \in C_0^\infty(B).
\]

Here \(0 < s < 1, p > 1, q = Dp/(D - (1 - s)p)\) and \(D = \max\{D(x) : x \in \bar{B}\}\) (see (11)) denotes the homogeneous dimension of \(B\).
\[
\left(48\right) \int_{B \times B} \frac{dx \, dy}{d(x, y)^{q \alpha} |B(x, d(x, y))|} |u(x) - u(y)|^q \\
= \int_{B \times B} \frac{dx \, dy}{d(x, y)^{q \alpha} |B(x, d(x, y))|} \left( \int_B \{X \Gamma(\xi, x) - X \Gamma(\xi, x)\} X u(\xi) \, d\xi \right)^q,
\]
where the derivatives act on the variable \(\xi\). In view of the equivalence \(|B(x, d(x, y))| \sim |B(y, d(x, y))|\), we can also assume that we are integrating on the set
\[
\{ |X \Gamma(\xi, x)| \geq |X \Gamma(\xi, y)| \}.
\]
Taking (34) and (37) into account we can write, for any \(\psi(x, y) \geq 0\),
\[
\int_{B \times B} \frac{\psi(x, y)}{d(x, y)^{q \alpha} |B(x, d(x, y))|} \, dx \, dy \leq \sum_{I} \int_{E_{r,\lambda}(M, s, \lambda \in B)} \frac{\psi(x, E_{r,\lambda}(h))}{\|h\|^{d(\ell) + q \alpha}} \, dh.
\]
Then (48) is estimated by
\[
\sum_{I} \int_{E_{r,\lambda}(M, s, \lambda \in B)} \frac{dh}{\|h\|^{d(\ell) + q \alpha}} \left( \int_B \{X \Gamma(\xi, E_{r,\lambda}(h)) - X \Gamma(\xi, x)\} X u(\xi) \, d\xi \right)^q.
\]
Letting now \(\{X \Gamma(\xi, E_{r,\lambda}(h)) - X \Gamma(\xi, x)\} =: \Delta(\xi, E_{r,\lambda}(h))\), we get, by (47),
\[
\|\Delta(\xi, E_{r,\lambda}(h))\| \leq \frac{d(x, E_{r,\lambda}(h))}{|B(x, d(x, \xi))|} \frac{d(x, E_{r,\lambda}(h))}{d(x, \xi)},
\]
while the estimate
\[
\|\Delta(\xi, x, h)\| \leq 2 |X \Gamma(\xi, x)| \leq \frac{d(x, \xi)}{|B(x, d(x, \xi))|}
\]
always holds, in view of (49).

By the Minkowski inequality, we have
\[
\sum_{I} \int_{E_{r,\lambda}(M, s, \lambda \in B)} \frac{dh}{\|h\|^{d(\ell) + q \alpha}} \left( \int_B \frac{d(x, \xi, h) X u(\xi)}{|B(x, d(x, \xi))|^q} \right)^q
\]
\[
\leq \left( \sum_{I} \int_{E_{r,\lambda}(M, s, \lambda \in B)} \frac{dh}{\|h\|^{d(\ell) + q \alpha}} \right) \left( \int_B \frac{d(x, \xi, h)}{|B(x, d(x, \xi))|} \right)^{1/q} \left( \int_B \frac{d(x, \xi, h)}{|B(x, d(x, \xi))|^q} \frac{d(x, \xi, h)}{|B(x, d(x, \xi))|} \right)^{1/q} \left( \int_B \frac{d(x, \xi, h)}{|B(x, d(x, \xi))|} \right)^{1/q}.
\]
In order to estimate (51) we remark that \(E_{r,\lambda}(M, s, \lambda \in B) \subset \{\|h\| \leq cr\}\), where \(r\) is the radius of the ball \(B\) and \(c\) is a suitable constant. Moreover, \(\|h\| \sim d(x, E_{r,\lambda}(h))\) on \(E_{r,\lambda}(M, s, \lambda \in B)\). Thus, if \(\lambda\) is a positive number, small enough, but only depending on the compact set, we can assert that \(\|h\| \leq \lambda d(x, \xi) \Rightarrow d(x, E_{r,\lambda}(h)) \leq \frac{1}{\lambda} d(x, \xi)\). Then the square bracket in (51) can be estimated as follows:

References


Composition operators: $N_\alpha$ to the Bloch space to $Q_\beta$

by

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Abstract. Let $N_\alpha$, $B$ and $Q_\beta$ be the weighted Nevanlinna space, the Bloch space and the $Q$ space, respectively. Note that $B$ and $Q_\beta$ are Möbius invariant, but $N_\alpha$ is not. We characterize, in function-theoretic terms, when the composition operator $C_\phi f = f \circ \phi$ induced by an analytic self-map $\phi$ of the unit disk defines an operator $C_\phi : N_\alpha \to B$, $B \to Q_\beta$, $N_\alpha \to Q_\beta$ which is bounded resp. compact.

1. Introduction. Let $\Delta$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, and let $\mathcal{H}(\Delta)$ be the space of all analytic functions on $\Delta$. Any analytic map $\phi : \Delta \to \Delta$ gives rise to an operator $C_\phi : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ defined by $C_\phi f = f \circ \phi$, the composition operator induced by $\phi$.

One of the central problems on composition operators is to know when $C_\phi$ maps between two subclasses of $\mathcal{H}(\Delta)$ and in fact to relate function-theoretic properties of $\phi$ to operator-theoretic properties of $C_\phi$. This problem is addressed here for the weighted Nevanlinna, the Bloch and the $Q$ spaces with respect to boundedness and compactness of the operator. The related research has recently been done by various authors (see for example [JX], [MM], [RU], [SZ], [T] and [X2]). The present paper continues their work, but also solves two problems which remained open in [SZ].

For each $\alpha \in (-1, 1)$, let $N_\alpha$ be the space of all functions $f \in \mathcal{H}(\Delta)$ satisfying

$$T_\alpha(f) = \frac{1 + \alpha}{\pi} \int_{\Delta} \frac{1}{|f(z)||1 - |z|^2|^\alpha} \, dm(z) < \infty.$$ 

Here and afterwards, $dm$ means the usual element of the area measure on $\Delta$, and $\log^+ x$ is $\log x$ if $x > 1$ and 0 if $0 \leq x \leq 1$. From $\log^+ x \leq \log(1+x) \leq 1 + \log^+ x$ for $x \geq 0$ we see that a function $f \in \mathcal{H}(\Delta)$ belongs to $N_\alpha$ if and only if

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