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$\mathcal{J}$-subspace lattices and subspace M-bases

by

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Abstract. The class of $\mathcal{J}$-lattices was defined in the second author's thesis. A subspace lattice on a Banach space X which is also a $\mathcal{J}$-lattice is called a *$\mathcal{J}$-subspace lattice*, abbreviated JSL. Every atomic Boolean subspace lattice, abbreviated ABSL, is a JSL. Any commutative JSL on Hilbert space, as well as any JSL on finite-dimensional space, is an ABSL. For any JSL $\mathcal{L}$ both $\operatorname{Lat}\operatorname{Alg}\mathcal{L}$ and $\mathcal{L}^\perp$ (on reflexive space) are JSL's. Those families of subspaces which arise as the set of atoms of some JSL on X are characterised in a way similar to that previously found for ABSL's. This leads to a definition of a *subspace M-basis* of X which extends that of a vector M-basis. New subspace M-bases arise from old ones in several ways. In particular, if $\{M_\gamma\}_{\gamma\in\Gamma}$ is a subspace M-basis of X, then (i) $\{(M'_\gamma)^\perp\}_{\gamma\in\Gamma}$ is a subspace M-basis of $\bigvee_{\gamma\in\Gamma}(M'_\gamma)^\perp$, (ii) $\{K_\gamma\}_{\gamma\in\Gamma}$ is a subspace M-basis of $\bigvee_{\gamma\in\Gamma}K_\gamma$ for every family $\{K_\gamma\}_{\gamma\in\Gamma}$ of subspaces satisfying $(0)\neq K_\gamma\subseteq M_\gamma$ $(\gamma\in\Gamma)$ and (iii) if X is reflexive, then $\{\bigcap_{\beta\neq\gamma}M'_\beta\}_{\gamma\in\Gamma}$ is a subspace M-basis of X. (Here M'_γ is given by $M'_\gamma=\bigvee_{\beta\neq\gamma}M_\beta$.)

1. Introduction. The class of $\mathcal{J}$-lattices was defined in [16] and subsequently discussed in [14], although some results concerning such lattices were given earlier in [12, 13]. In any complete lattice L the operation "$\_$" is defined by $a_- = \bigvee\{b\in L : a\not\leq b\}$ $(a\in L)$. The set of elements $a\in L$ satisfying $a\neq 0$ and $a_-\neq 1$ is called the set of *$\mathcal{J}$-elements* of L and is denoted by $\mathcal{J}(L)$ (this notation was first used in [11]). In the study of subspace lattices on Banach spaces and the operator algebras that are usually associated with them, $\mathcal{J}$-elements are of special interest primarily due to the following result.

Lemma 1.1 ([11], see also [8]). *If $\mathcal{L}$ is a subspace lattice on a Banach space, the rank one operator $e^*\otimes f$ belongs to* $\operatorname{Alg}\mathcal{L}$ *if and only if $f\in M$ and $e^*\in(M_-)^\perp$ for some $M\in\mathcal{J}(\mathcal{L})$.*

Consequently, those subspace lattices $\mathcal{L}$ for which $\mathcal{J}(\mathcal{L})$ is "large" have Alg's that are rich in rank one operators and deserve some attention. All those subspace lattices which are also $\mathcal{J}$-lattices fall into this category. (The

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definition of a $\mathcal{J}$-lattice is recalled in the next section.) Such a subspace lattice is called a *$\mathcal{J}$-subspace lattice*, abbreviated JSL. For every JSL $\mathcal{L}$, $\mathcal{J}(\mathcal{L})$ is just the set of atoms of $\mathcal{L}$. Every atomic Boolean subspace lattice, abbreviated ABSL, is a JSL. The simplest example of a JSL which is not an ABSL is any pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, X\}$. Here K, L and M are subspaces of the Banach space X satisfying $K \vee L = X$, $K \cap M = (0)$ and $L \subset M$. (For a discussion of pentagon subspace lattices see [7, 10].) It turns out (see later) that, on a Hilbert space, every commutative JSL is an ABSL. However, most JSL's on Hilbert space are non-commutative. It also turns out that every JSL on a finite-dimensional space is an ABSL so most of their particular interest lies in infinite-dimensional spaces. Also, although ABSL's are completely determined by their atoms, JSL's are not; different JSL's can have the same sets of atoms.

In the memoir [1] a connection was made between ABSL's and strong M-bases. Indeed, the set of atoms of an ABSL seems a most reasonable interpretation of "a strong M-basis consisting of subspaces". A similar connection is made in the present note, this time between JSL's and M-bases. We begin by attempting to show that the set of atoms of a JSL is a reasonable interpretation of "an M-basis consisting of subspaces". Next we show that, if $\mathcal{L}$ is a JSL, so is $\operatorname{LatAlg}\mathcal{L}$ and $\mathcal{J}(\mathcal{L}) = \mathcal{J}(\operatorname{LatAlg}\mathcal{L})$. If the underlying space is reflexive and $\mathcal{L}$ is a JSL we show that $\mathcal{L}^\perp$ is also a JSL and $\mathcal{J}(\mathcal{L}^\perp) = \{(M_-)^\perp : M \in \mathcal{J}(\mathcal{L})\}$. We conclude by discussing some ways by which new subspace M-bases arise from old ones, in some cases parallelling similar results for strong M-bases given in [1]. Briefly, amongst other things, it is shown that if $\{M_\gamma\}_{\gamma\in\Gamma}$ is a subspace M-basis of a Banach space X then

(1) $\{K_\gamma\}_{\gamma\in\Gamma}$ is a subspace M-basis of $\bigvee_{\gamma\in\Gamma} K_\gamma$ for any subspaces $\{K_\gamma\}_{\gamma\in\Gamma}$ satisfying $(0) \neq K_\gamma \subseteq M_\gamma$ for every $\gamma \in \Gamma$,

(2) $\{(M'_\gamma)^\perp\}_{\gamma\in\Gamma}$ is a subspace M-basis of $\bigvee_{\gamma\in\Gamma}(M'_\gamma)^\perp$, where $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta$ for every $\gamma \in \Gamma$,

(3) if X is reflexive, then $\{\bigcap_{\beta\neq\gamma} M'_\beta\}_{\gamma\in\Gamma}$ is a subspace M-basis of X (with M'_β as in (2)).

2. Definitions and preliminaries. Throughout, the terms "Banach space", "subspace", and "operator" will mean "real or complex Banach space", "closed linear subspace", and "bounded linear operator", respectively. The symbol X will denote a Banach space and H will denote a separable complex infinite-dimensional Hilbert space with inner product denoted by $(\cdot \mid \cdot)$. The symbol "$\subset$" will be reserved for *strict* set-inclusion. The dual of X is denoted by X^*. The set of operators acting on X is denoted by $\mathcal{B}(X)$. If $T \in \mathcal{B}(X)$ the range of T is denoted by $\operatorname{Ran}(T)$. For any family $\mathcal{E}$ of vectors of X, $\operatorname{span}\mathcal{E}$ denotes the linear span of $\mathcal{E}$ and, if $\mathcal{E} = \{f\}$, $\langle f\rangle = \operatorname{span}\mathcal{E}$.

If $\{L_\gamma\}_{\gamma\in\Gamma}$ is a family of subspaces of X, $\bigvee_{\gamma\in\Gamma} L_\gamma$ denotes the closed linear span of $\bigcup_{\gamma\in\Gamma} L_\gamma$. For any family $\{f_\gamma\}_{\gamma\in\Gamma}$ of elements of X, $\bigvee_{\gamma\in\Gamma}\langle f_\gamma\rangle$ is denoted more simply as $\bigvee_{\gamma\in\Gamma} f_\gamma$. For any subset $\mathcal{E} \subseteq X$ the annihilator $\mathcal{E}^\perp$ of $\mathcal{E}$ is given by $\mathcal{E}^\perp = \{f^* \in X^* : f^*(x) = 0 \text{ for every } x \in \mathcal{E}\}$. For every family $\{L_\gamma\}_{\gamma\in\Gamma}$ of subspaces of X we have $(\bigvee_{\gamma\in\Gamma} L_\gamma)^\perp = \bigcap_{\gamma\in\Gamma} L_\gamma^\perp$ and $(\bigcap_{\gamma\in\Gamma} L_\gamma)^\perp \supseteq \bigvee_{\gamma\in\Gamma} L_\gamma^\perp$. If $f \in X$ and $e^* \in X^*$ the operator $e^* \otimes f$ acting on X is defined by $e^* \otimes f(x) = e^*(x)f$ $(x \in X)$.

A family $\mathcal{L}$ of subspaces of X is a *subspace lattice on* X if $(0), X \in \mathcal{L}$ and both $\bigvee_{\gamma\in\Gamma} L_\gamma$ and $\bigcap_{\gamma\in\Gamma} L_\gamma$ belong to $\mathcal{L}$ for every family $\{L_\gamma\}_{\gamma\in\Gamma}$ of elements of $\mathcal{L}$. We use $\mathcal{C}(X)$ to denote the subspace lattice on X consisting of all subspaces of X. For any subset $\mathcal{F} \subseteq \mathcal{C}(X)$ the intersection of all those subspace lattices containing $\mathcal{F}$ is called the *subspace lattice generated by* $\mathcal{F}$. If X is reflexive $\mathcal{L}^\perp$ defined by $\mathcal{L}^\perp = \{M^\perp : M \in \mathcal{L}\}$ is a subspace lattice on X^*, if $\mathcal{L}$ is a subspace lattice on X. On Hilbert space, a subspace lattice is called *commutative* if the orthogonal projections onto each of its members pairwise commute. For any subset $\mathcal{F} \subseteq \mathcal{C}(X)$, $\operatorname{Alg}\mathcal{F}$ denotes the set of operators acting on X that leave every member of $\mathcal{F}$ invariant, that is,

$$\operatorname{Alg}\mathcal{F} = \{T \in \mathcal{B}(X) : T(M) \subseteq M \text{ for every } M \in \mathcal{F}\}.$$

For any subset $\mathcal{F} \subseteq \mathcal{C}(X)$, the Alg of the subspace lattice generated by $\mathcal{F}$ is equal to $\operatorname{Alg}\mathcal{F}$. For any subset $\mathcal{G} \subseteq \mathcal{B}(X)$, $\operatorname{Lat}\mathcal{G}$ denotes the set of subspaces of X that are invariant under each member of $\mathcal{G}$, that is,

$$\operatorname{Lat}\mathcal{G} = \{M \in \mathcal{C}(X) : T(M) \subseteq M \text{ for every } T \in \mathcal{G}\}.$$

We have $\mathcal{F} \subseteq \operatorname{LatAlg}\mathcal{F}$ for every subset $\mathcal{F} \subseteq \mathcal{C}(X)$.

A family $\{f_\gamma\}_{\gamma\in\Gamma}$ of elements of X is *complete* if $\bigvee_{\gamma\in\Gamma} f_\gamma = X$. A family $\{f^*_\gamma\}_{\gamma\in\Gamma}$ of elements of X^* is *total* if $\bigcap_{\gamma\in\Gamma}\ker f^*_\gamma = (0)$. A family $\{f_\gamma\}_{\gamma\in\Gamma}$ of elements of X is *minimal* if $f_\gamma \notin \bigvee_{\beta\neq\gamma} f_\beta$, for every $\gamma \in \Gamma$. If $\{f_\gamma\}_{\gamma\in\Gamma}$ is complete and minimal, there exists a unique family $\{f^*_\gamma\}_{\gamma\in\Gamma} \subseteq X^*$ biorthogonal to it, that is, satisfying $f^*_\alpha(f_\beta) = \delta_{\alpha,\beta}$ $(\alpha, \beta \in \Gamma)$. A complete minimal family $\{f_\gamma\}_{\gamma\in\Gamma}$ with a total biorthogonal family $\{f^*_\gamma\}_{\gamma\in\Gamma}$ is called an *M-basis of* X. An M-basis $\{f_\gamma\}_{\gamma\in\Gamma}$ of X is called a *strong M-basis* if $x \in \bigvee\{f_\gamma : f^*_\gamma(x) \neq 0\}$ for every $x \in X$. (It is interesting to note that it has been shown only relatively recently that every separable Banach space has a strong M-basis [19].)

Next we briefly summarise, for the reader's convenience, some of the lattice-theoretic results proved in [14, 16]. Some of these will be needed later. First, some terminology from lattice theory (mostly taken from [17]). A lattice L is called

(i) *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$,
(ii) *modular* if $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $a \geq c$,

(iii) *semi-modular* if, whenever $a, b, c \in L$ with a and c incomparable and $a \wedge c < b < a$, there exists $d \in L$ such that $a \wedge c < d \leq c$ and $a \wedge (b \vee d) = b$.

Additionally, if L is complete, with least element 0 and greatest element 1, then L is called

(iv) *atomic* if every non-zero element of L contains an atom and is the join of all the atoms it contains (an element $a \in L$ is an *atom* of L if $0 \leq b \leq a$ and $b \in L$ implies that $b = 0$ or a),

(v) (*uniquely*) *complemented* if each element of L has a (respectively, unique) complement (an element $b \in L$ is a *complement* of $a \in L$ if $a \vee b = 1$ and $a \wedge b = 0$),

(vi) *algebraic* if every element of L is a join of compact elements (an element $a \in L$ is *compact* if $a \leq \bigvee\{b : b \in \mathcal{G}\}$ implies that $a \leq \bigvee\{b : b \in \mathcal{F}\}$ for some finite subset $\mathcal{F} \subseteq \mathcal{G}$),

(vii) *Boolean* if it is complemented and distributive.

As usual, we observe the conventions that $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$.

DEFINITION 2.1 ([16]). An abstract complete lattice L is called a *$\mathcal{J}$-lattice* if

(1) $\bigvee\{a : a \in \mathcal{J}(L)\} = 1$,
(2) $\bigwedge\{a_- : a \in \mathcal{J}(L)\} = 0$,
(3) $a \vee a_- = 1$ for every $a \in \mathcal{J}(L)$,
(4) $a \wedge a_- = 0$ for every $a \in \mathcal{J}(L)$.

If B is a complete atomic Boolean lattice, then $\mathcal{J}(B)$ is precisely the set of atoms of B, and $a_- = a'$ (the Boolean complement of a) for every atom a of B [11]. Consequently, every complete atomic Boolean lattice is a $\mathcal{J}$-lattice.

In the present note we are more interested in representations of $\mathcal{J}$-lattices as subspaces lattices on Banach spaces.

DEFINITION 2.2. A subspace lattice on a Banach space X which is also a $\mathcal{J}$- lattice is called a *$\mathcal{J}$-subspace lattice* (abbreviated JSL) on X.

LEMMA 2.1 ([14, 16]). *Let L be a $\mathcal{J}$-lattice. Then*

(i) *$\mathcal{J}(L)$ is the set of atoms of L,*
(ii) *$\mathcal{J}(L) = \{a \in L : a \not\leq a_-\}$,*
(iii) *every non-zero element of L contains an atom,*
(iv) *L is complemented.*

THEOREM 2.1 ([14, 16]). *Let L be a complete lattice. The following are equivalent:*

(i) *L is atomic and Boolean,*
(ii) *L is distributive,*

(iii) *L is modular,*
(iv) *L is atomic,*
(v) *L is uniquely complemented,*
(vi) *L is semi-modular and algebraic.*

The implications (ii)⇒(i) and (iii)⇒(i) of the preceding theorem show, respectively, that every commutative JSL on Hilbert space is an ABSL, and that every JSL on finite-dimensional space is an ABSL (respectively, because commutative subspace lattices are distributive, and $\mathcal{C}(X)$ is modular if X is finite-dimensional).

It seems appropriate to include here the following observation. Recall that the *dual* $L^{\rm d}$ of a lattice L is the lattice obtained from L by "reversing the order" of L. Precisely, the dual of L is the lattice whose elements are those of L with partial order defined by $a \preceq b$ if $b \leq a$.

PROPOSITION 2.1. *The dual $L^{\rm d}$ of any $\mathcal{J}$-lattice L is a $\mathcal{J}$-lattice. Moreover, $\mathcal{J}(L^{\rm d}) = \{a_- : a \in \mathcal{J}(L)\}$.*

Proof. Let L be a $\mathcal{J}$-lattice with least element 0 and greatest element 1. Denote the "$_-$" operation in $L^{\rm d}$ by "$_\ominus$". Let $a \in \mathcal{J}(L)$. Then $(a_-)_\ominus = \bigwedge\{b \in L : b \not\leq a_-\}$ (where the inf is taken in L). Since $a \not\leq a_-$, we get $(a_-)_\ominus \leq a$. But clearly $b \not\leq a_-$ implies that $a \leq b$. Thus $a \leq (a_-)_\ominus$ so $(a_-)_\ominus = a$. Since $a_- \neq 1$ and $(a_-)_\ominus \neq 0$, we have $a_- \in \mathcal{J}(L^{\rm d})$.

Next, let $b \in \mathcal{J}(L^d)$. Since $b \neq 1$ and $\bigvee\{a : a \in \mathcal{J}(L)\} = 1$, there exists $a \in \mathcal{J}(L)$ such that $a \not\leq b$. Thus $b \leq a_-$. Also, since $b_\ominus = \bigwedge\{c \in L : c \not\leq b\} \neq 0$, it follows that $a_- \leq b$. For, if $a_- \not\leq b$ then $b_\ominus \leq a_- \wedge a = 0$. This shows that $b = a_-$ for some $a \in \mathcal{J}(L)$.

It follows that $\mathcal{J}(L^{\rm d}) = \{a_- : a \in \mathcal{J}(L)\}$ and this, together with the fact that $(a_-)_\ominus = a$ for every $a \in \mathcal{J}(L)$, shows that $L^{\rm d}$ is a $\mathcal{J}$-lattice. ■

Most of our examples involve operator ranges on Hilbert space and their properties. All of the properties needed here follow from results of [4], especially the fact (which follows from a stronger result of J. von Neumann) that if $A \in \mathcal{B}(H)$ is a positive injective non-invertible operator, then there exists a positive injective non-invertible operator $B \in \mathcal{B}(H)$ such that $\operatorname{Ran}(A) \cap \operatorname{Ran}(B) = (0)$.

3. Main results.

Let X be a real or complex Banach space. We begin by characterising those families of non-zero subspaces of X which arise as precisely the set of atoms of a JSL on X. Those families of non-zero subspaces which arise as precisely the set of atoms of an ABSL on X are, by [1, Theorem 2.4], those which are subspace strong M-bases in the sense of the following definition.

DEFINITION 3.1. A family $\{M_\gamma\}_{\gamma\in\Gamma}$ of non-zero subspaces of X is a *subspace strong M-basis* of X if

(i) $\bigvee_{\gamma\in\Gamma} M_\gamma = X$,

(ii) $(\bigvee_{\gamma\in I} M_\gamma)\cap(\bigvee_{\gamma\in J} M_\gamma) = \bigvee_{\gamma\in I\cap J} M_\gamma$ for every pair I, J of subsets of Γ.

The use of the terminology "strong M-basis" in the preceding definition is justified by the fact that a family $\{f_\gamma\}_{\gamma\in\Gamma}$ of non-zero vectors of X is a strong M-basis of X (in the usual sense) if and only if the family of one-dimensional subspaces $\{\langle f_\gamma\rangle\}_{\gamma\in\Gamma}$ is a subspace strong M-basis of X [1, Theorem 5.1]. This raises the question: What should be meant by a subspace M-basis of X? We feel that the following is a reasonable answer and we will offer some justification for this view.

DEFINITION 3.2. A family $\{M_\gamma\}_{\gamma\in\Gamma}$ of non-zero subspaces of X is a *subspace M-basis* of X if

(i) $\bigvee_{\gamma\in\Gamma} M_\gamma = X$,

(ii) $(\bigvee_{\gamma\in I} M_\gamma)\cap(\bigvee_{\gamma\in J} M_\gamma) = (0)$ for every pair I, J of disjoint subsets of Γ,

(iii) $\bigcap_{\gamma\in\Gamma} M'_\gamma = (0)$, where $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta$ for every $\gamma\in\Gamma$.

REMARKS. 1. A family $\{f_\gamma\}_{\gamma\in\Gamma}$ of non-zero vectors of X is an M-basis of X (in the usual sense) if and only if the family of one-dimensional subspaces $\{\langle f_\gamma\rangle\}_{\gamma\in\Gamma}$ is a subspace M-basis of X. Indeed, as remarked in [1, p. 45], $\{f_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of X if and only if conditions (i) and (ii) of the preceding definition are satisfied by $\{\langle f_\gamma\rangle\}_{\gamma\in\Gamma}$. Also, if $\{f_\gamma\}_{\gamma\in\Gamma}$ is an M-basis then $\langle f_\gamma\rangle' = \bigvee_{\beta\neq\gamma}\langle f_\beta\rangle = \ker f^*_\gamma$ for every $\gamma\in\Gamma$, so condition (iii) is just the totality condition on the biorthogonal family $\{f^*_\gamma\}_{\gamma\in\Gamma}$. However, condition (iii) need not follow automatically from conditions (i) and (ii) in general (see Example 3.1 below).

2. Throughout the following, the terms "subspace strong M-basis" and "subspace M-basis" will be abbreviated to "strong M-basis" and "M-basis", respectively, wherever no confusion can arise.

3. If $\{M_\gamma\}_{\gamma\in\Gamma}$ is a strong M-basis of X, then, for every $\gamma\in\Gamma$, the Boolean complement of M_γ in the (unique) ABSL having $\{M_\gamma\}_{\gamma\in\Gamma}$ as its set of atoms is $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta$. Thus $\bigcap_{\gamma\in\Gamma} M'_\gamma = (\bigvee_{\gamma\in\Gamma} M_\gamma)' = X' = (0)$, by De Morgan's Laws. It is now clear that every strong M-basis of X is an M-basis of X. (That conditions (i) and (ii) of Definition 3.1 together imply condition (iii) of Definition 3.2 is proved more directly in [1, Theorem 2.1] where it is shown that $\bigcap_{\lambda\in\Lambda}(\bigvee_{\gamma\in\Gamma_\lambda} M_\gamma) = \bigvee\{M_\gamma : \gamma\in\bigcap_{\lambda\in\Lambda}\Gamma_\lambda\}$ for every family $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ of subsets of Γ.)

4. Unlike strong M-bases, M-bases are stable under "slicing" and "selecting". Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a family of non-zero subspaces of X and let $\{K_\gamma\}_{\gamma\in\Gamma}$ be a family of subspaces satisfying $(0)\neq K_\gamma\subseteq M_\gamma$ for every $\gamma\in\Gamma$. If $\{M_\gamma\}_{\gamma\in\Gamma}$ is a strong M-basis of X, then $\{K_\gamma\}_{\gamma\in\Gamma}$ need not be a strong M-basis of $\bigvee_{\gamma\in\Gamma} K_\gamma$. This is shown by the example given in [1, Theorem 6.4]. In it Γ has minimal cardinality, namely 3. However, if $\{M_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of X, then the "slices" $\{K_\gamma\}_{\gamma\in\Gamma}$ are an M-basis of $\bigvee_{\gamma\in\Gamma} K_\gamma$. This follows easily by noting that $\bigvee_{\gamma\in I} K_\gamma\subseteq\bigvee_{\gamma\in I} M_\gamma$ for every subset $I\subseteq\Gamma$, and that $K'_\gamma = \bigvee_{\beta\neq\gamma} K_\beta\subseteq M'_\gamma$ for every $\gamma\in\Gamma$. In particular, if $\{M_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of X and vectors $\{f_\gamma\}_{\gamma\in\Gamma}$ are selected as follows: $0\neq f_\gamma\in M_\gamma$ for every $\gamma\in\Gamma$, then $\{f_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of $\bigvee_{\gamma\in\Gamma} f_\gamma$. (Other ways by which new M-bases arise from old ones will be discussed later.) Selecting vectors like this from the elements of a subspace strong M-basis need not produce vectors which are a strong M-basis of their closed linear span. An example showing this is given in [18] (see also [1, p. 71]). A better example, on separable Hilbert space, is given in [2] (see also [3]). In the latter, each element of the subspace strong M-basis, from which the vectors are selected, is two-dimensional.

EXAMPLE 3.1. Let $A, B, C\in\mathcal{B}(H)$ be positive injective operators whose ranges pairwise intersect in (0). (Such operators exist [4]. Indeed, given A, first choose B such that $\mathrm{Ran}(A)\cap\mathrm{Ran}(B) = (0)$. Then choose C such that $(\mathrm{Ran}(A)+\mathrm{Ran}(B))\cap\mathrm{Ran}(C) = (0)$.) Define subspaces of $H^{(4)}$ by

$$M_1 = \{(Ax, 0, 0, x) : x\in H\},$$
$$M_2 = \{(0, Bx, 0, x) : x\in H\},$$
$$M_3 = \{(0, 0, Cx, x) : x\in H\}.$$

Then $M'_1 = M_2\vee M_3 = (0)\oplus H\oplus H\oplus H$, $M'_2 = M_1\vee M_3 = H\oplus(0)\oplus H\oplus H$ and $M'_3 = M_1\vee M_2 = H\oplus H\oplus(0)\oplus H$. Clearly $M_1\vee M_2\vee M_3 = H^{(4)}$ and $M_n\cap M'_n = (0)$ $(n = 1, 2, 3)$. However, $M'_1\cap M'_2\cap M'_3 = (0)\oplus(0)\oplus(0)\oplus H\neq(0)$.

PROPOSITION 3.1. *The set of atoms of a JSL on X is a subspace M-basis of X.*

Proof. Let $\mathcal{L}$ be a JSL on X and let $\{M_\gamma\}_{\gamma\in\Gamma}$ be its set of atoms. Then $\mathcal{J}(\mathcal{L}) = \{M_\gamma\}_{\gamma\in\Gamma}$ and $\bigvee_{\gamma\in\Gamma} M_\gamma = X$. Let I and J be disjoint subsets of Γ. Suppose that

$$\Big(\bigvee_{\gamma\in I} M_\gamma\Big)\cap\Big(\bigvee_{\gamma\in J} M_\gamma\Big)\neq(0).$$

Then there exists an atom, M_α say, such that

$$M_\alpha\subseteq\Big(\bigvee_{\gamma\in I} M_\gamma\Big)\cap\Big(\bigvee_{\gamma\in J} M_\gamma\Big).$$

Since $I \cap J = \emptyset$, we may suppose that $\alpha \notin I$. Since distinct atoms of $\mathcal{L}$ are incomparable, $M_\gamma \subseteq M_{\alpha-}$ for every $\gamma \in I$. Thus $M_\alpha \subseteq \bigvee_{\gamma \in I} M_\gamma \subseteq M_{\alpha-}$ and this contradicts $M_\alpha \not\subseteq M_{\alpha-}$. Hence

$$\Big(\bigvee_{\gamma\in I} M_\gamma\Big) \cap \Big(\bigvee_{\gamma\in J} M_\gamma\Big) = (0).$$

Finally, if $\beta \neq \gamma$, then $M_\beta \subseteq M_{\gamma-}$ so $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta \subseteq M_{\gamma-}$. Since $\bigcap_{\gamma\in\Gamma} M_{\gamma-} = (0)$, it follows that $\bigcap_{\gamma\in\Gamma} M'_\gamma = (0)$. This completes the proof. ■

The example immediately below shows that it is not true that a subspace lattice is a JSL if its atoms are an M-basis of X.

EXAMPLE 3.2. Let $A, B \in \mathcal{B}(H)$ be positive injective operators satisfying $\operatorname{Ran}(A) \cap \operatorname{Ran}(B) = (0)$ [4]. Define subspaces of $H^{(3)}$ by $M_1 = (0)\oplus(0)\oplus H$, $M_2 = \{(Ax, Bx, x) : x \in H\}$, $K_1 = H \oplus (0) \oplus H$ and $K_2 = (0) \oplus H \oplus H$. These subspaces are the non-trivial elements of a subspace lattice $\mathcal{L}$ on $H^{(3)}$ whose partial order diagram is given in Figure 1. Since $M_1 \vee M_2 = H^{(3)}$ and $M_1 \cap M_2 = (0)$, $\{M_1, M_2\}$ is an M-basis of $H^{(3)}$. However, $\mathcal{J}(\mathcal{L}) = \{M_1\}$ so $\mathcal{L}$ is not a JSL.

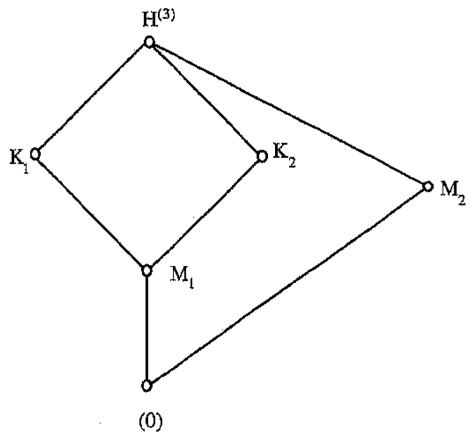

Fig. 1

PROPOSITION 3.2. *The subspace lattice generated by a subspace M-basis of X is a JSL on X. If the subspace M-basis $\{M_\gamma\}_{\gamma\in\Gamma}$ generates the subspace lattice $\mathcal{L}$, then $\{M_\gamma\}_{\gamma\in\Gamma}$ is the set of atoms of $\mathcal{L}$ and $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta = M_{\gamma-}$ for every $\gamma \in \Gamma$ (where $M_{\gamma-}$ is calculated in $\mathcal{L}$).*

Proof. Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a subspace M-basis of X and let $\mathcal{L}$ be the subspace lattice it generates. Recall that M'_γ is defined by $M'_\gamma = \bigvee_{\beta\neq\gamma} M_\beta$.

First we show that each M_γ is an atom of $\mathcal{L}$. Suppose that $N \in \mathcal{L}$ and $(0) \subseteq N \subset M_\gamma$. Choose a vector $f \in M_\gamma$, $f \notin N$. Let $e^* \in (M'_\gamma)^\perp$. Then $e^* \otimes f \in \operatorname{Alg}\{M_\beta : \beta \in \Gamma\} = \operatorname{Alg}\mathcal{L}$, so $(e^*\otimes f)(N) \subseteq N$. Since $f \notin N$, we have $e^* \in N^\perp$. This shows that $(M'_\gamma)^\perp \subseteq N^\perp$ so $N \subseteq M'_\gamma$. Since $N \subseteq M'_\gamma \cap M_\gamma = (0)$, we get $N = (0)$.

Next, let $L \in \mathcal{J}(\mathcal{L})$. Since $\bigvee_{\gamma\in\Gamma} M_\gamma = X$ and $L_- \neq X$, we have $L \subseteq M_\gamma$ for some $\gamma \in \Gamma$. But M_γ is an atom of $\mathcal{L}$ so $L = M_\gamma$. Hence $\mathcal{J}(\mathcal{L}) \subseteq \{M_\gamma : \gamma \in \Gamma\}$.

Finally, let $\gamma \in \Gamma$ and let $h \in M_\gamma$ and $g^* \in (M'_\gamma)^\perp$ be non-zero vectors. Then $g^* \otimes h \in \operatorname{Alg}\mathcal{L}$ so, by Lemma 1.1, there is an element, M_β say, of $\mathcal{J}(\mathcal{L})$ such that $h \in M_\beta$ and $g^* \in (M_{\beta-})^\perp$. Since M_γ and M_β are atoms of $\mathcal{L}$ and $M_\gamma \cap M_\beta \neq (0)$, we have $M_\gamma = M_\beta$. Thus $\mathcal{J}(\mathcal{L}) = \{M_\gamma : \gamma \in \Gamma\}$. The above also shows that $(M'_\gamma)^\perp \subseteq (M_{\gamma-})^\perp$, and so $M_{\gamma-} \subseteq M'_\gamma$ for every $\gamma \in \Gamma$. Clearly $M'_\gamma \subseteq M_{\gamma-}$ (since $M_\gamma \not\subseteq M_\beta$ if $\beta \neq \gamma$) so $M'_\gamma = M_{\gamma-}$. It is now clear that $\mathcal{L}$ is a JSL having $\{M_\gamma\}_{\gamma\in\Gamma}$ as its set of atoms. ■

Combining the preceding two propositions gives the following characterisation of the families of non-zero subspaces of X which arise as the set of atoms of a JSL on X.

THEOREM 3.1. *Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a family of non-zero subspaces of a Banach space X. There exists a JSL on X having $\{M_\gamma\}_{\gamma\in\Gamma}$ as precisely its set of atoms if and only if $\{M_\gamma\}_{\gamma\in\Gamma}$ is a subspace M-basis of X.*

REMARK. If $\mathcal{L}$ is a JSL on X having $\{M_\gamma\}_{\gamma\in\Gamma}$ as its set of atoms, then, since distinct atoms are incomparable, $M'_\gamma \subseteq M_{\gamma-}$ for every $\gamma \in \Gamma$. Propositions 3.1 and 3.2 show that we have equality here if $\mathcal{L}$ is generated by $\{M_\gamma\}_{\gamma\in\Gamma}$. We need not have equality in general, as consideration of any pentagon subspace lattice soon shows.

Next we show that $\operatorname{Lat}\operatorname{Alg}\mathcal{L}$ and $\mathcal{L}^\perp$ (on reflexive space) are JSL's if $\mathcal{L}$ is. Actually, we prove a little more than this. In the following proposition, to avoid confusion, the "$_-$" operation in the subspace lattice $\widetilde{\mathcal{L}}$ is denoted by "$_\ominus$" so that $M_\ominus = \bigvee\{K \in \widetilde{\mathcal{L}} : M \not\subseteq K\}$ for every $M \in \widetilde{\mathcal{L}}$.

PROPOSITION 3.3. *Let $\mathcal{L}$ be a subspace lattice on X and let $\mathcal{R}$ denote the set of rank one operators of $\operatorname{Alg}\mathcal{L}$. Let $\widetilde{\mathcal{L}}$ be any subspace lattice on X satisfying $\mathcal{L} \subseteq \widetilde{\mathcal{L}} \subseteq \operatorname{Lat}\mathcal{R}$ (where we take $\operatorname{Lat}\mathcal{R} = \mathcal{C}(X)$ if $\mathcal{R} = \emptyset$). Then*

(i) *$M_- = M_\ominus$, for every $M \in \mathcal{L}$,*

(ii) *if $M \in \mathcal{J}(\mathcal{L})$, every element $N \in \widetilde{\mathcal{L}}$ satisfying $(0) \neq N \subseteq M$ belongs to $\mathcal{J}(\widetilde{\mathcal{L}})$,*

(iii) *if $N \in \mathcal{J}(\widetilde{\mathcal{L}})$, then $N \subseteq M$ for some element $M \in \mathcal{J}(\mathcal{L})$,*

(iv) *if $\mathcal{L}$ is a JSL on X so is $\widetilde{\mathcal{L}}$, and then $\mathcal{J}(\mathcal{L}) = \mathcal{J}(\widetilde{\mathcal{L}})$.*

Proof. (i) Let $M \in \mathcal{L}$. Since $(0)_- = (0)_\ominus = (0)$, we may suppose that $M \neq (0)$. From the definitions it is clear that $M_- \subseteq M_\ominus$. Let $K \in \widetilde{\mathcal{L}}$ satisfy $M \not\subseteq K$. Choose a vector $f \in M$, $f \notin K$. Let $e^* \in (M_-)^\perp$. Then $e^* \otimes f \in \operatorname{Alg} \mathcal{L}$ so $e^* \otimes f(K) \subseteq K$. Since $f \notin K$, we have $e^* \in K^\perp$. Hence $(M_-)^\perp \subseteq K^\perp$ so $K \subseteq M_-$. It follows that $M_\ominus \subseteq M_-$ so $M_- = M_\ominus$.

(ii) Let $M \in \mathcal{J}(\mathcal{L})$ and let $N \in \widetilde{\mathcal{L}}$ satisfy $(0) \neq N \subseteq M$. Then $N_\ominus \subseteq M_\ominus = M_- \neq X$, so $N_\ominus \neq X$ and $N \in \mathcal{J}(\widetilde{\mathcal{L}})$.

(iii) Let $N \in \mathcal{J}(\widetilde{\mathcal{L}})$. Put $M = \bigcap\{L \in \mathcal{L} : N \subseteq L\}$. Then $M \in \mathcal{L}$ and $N \subseteq M$. Since $N \neq (0)$, we have $M \neq (0)$. Also, if $J \in \mathcal{L}$ and $M \not\subseteq J$, then $N \not\subseteq J$ so $J \subseteq N_\ominus$. Thus $M_- \subseteq N_\ominus$. Since $N_\ominus \neq X$, it follows that $M_- \neq X$. Hence $M \in \mathcal{J}(\mathcal{L})$.

(iv) Suppose that $\mathcal{L}$ is a JSL on X. By (i) above, $\mathcal{J}(\mathcal{L}) \subseteq \mathcal{J}(\widetilde{\mathcal{L}})$. Let $N \in \mathcal{J}(\widetilde{\mathcal{L}})$. By (iii) above, $N \subseteq M_1$ for some $M_1 \in \mathcal{J}(\mathcal{L})$. Since $\bigcap\{M_- : M \in \mathcal{J}(\mathcal{L})\} = \bigcap\{M_\ominus : M \in \mathcal{J}(\mathcal{L})\} = (0)$, and $N \neq (0)$, we have $M_2 \subseteq N$ for some $M_2 \in \mathcal{J}(\mathcal{L})$. Hence $M_2 \subseteq N \subseteq M_1$. But M_1 is an atom of $\mathcal{L}$ so $M_2 = M_1$. Hence $N = M_1 \in \mathcal{J}(\mathcal{L})$. It follows that $\mathcal{J}(\mathcal{L}) = \mathcal{J}(\widetilde{\mathcal{L}})$ and this, together with (i), in turn shows that $\widetilde{\mathcal{L}}$ is a JSL on X. ■

The preceding proposition, of course, applies to the cases where $\widetilde{\mathcal{L}} = \operatorname{Lat}\operatorname{Alg} \mathcal{L}$ or $\operatorname{Lat} \mathcal{R}$. A purely lattice-theoretic proof for the latter case is given in [14].

PROPOSITION 3.4. *If $\mathcal{L}$ is a JSL on a reflexive Banach space X, then $\mathcal{L}^\perp$ is a JSL on X^* and $\mathcal{J}(\mathcal{L}^\perp) = \{(M_-)^\perp : M \in \mathcal{J}(\mathcal{L})\}$.*

Proof. Let $\mathcal{L}$ be a JSL on the reflexive Banach space X. The map $L \mapsto L^\perp$ is a (lattice) isomorphism from the dual of $\mathcal{L}$ onto $\mathcal{L}^\perp$. By Proposition 2.1 the dual of a $\mathcal{J}$-lattice is a $\mathcal{J}$-lattice, so $\mathcal{L}^\perp$ is a JSL on X^*. Again by Proposition 2.1, the set of $\mathcal{J}$-elements of the dual of $\mathcal{L}$ is $\{M_- : M \in \mathcal{J}(\mathcal{L})\}$ so $\mathcal{J}(\mathcal{L}^\perp) = \{(M_-)^\perp : M \in \mathcal{J}(\mathcal{L})\}$. ■

In [6] the authors give an example of an M-basis of one-dimensional subspaces of H for which the subspace lattice $\mathcal{L}$ it generates (which is a JSL by Proposition 3.2) does not equal $\operatorname{Lat} \mathcal{R}$, where $\mathcal{R}$ denotes the set of rank one operators of $\operatorname{Alg} \mathcal{L}$ (unlike the situation for strong M-bases [11]). For their example it can be shown that $\mathcal{L} = \operatorname{Lat}\operatorname{Alg} \mathcal{L}$. Now pentagon subspace lattices are JSL's and there exist pentagons $\mathcal{P}$ satisfying $\mathcal{P} \neq \operatorname{Lat}\operatorname{Alg} \mathcal{P}$ [10, 15]. But pentagons are not generated by their atoms. In Example 3.3 below we exhibit a JSL $\mathcal{L}$ generated by its atoms and satisfying $\mathcal{L} \neq \operatorname{Lat}\operatorname{Alg} \mathcal{L}$ (so satisfying $\mathcal{L} \neq \operatorname{Lat} \mathcal{R}$). Its set of atoms, of minimal cardinality 3 for this property, is a simpler example than that given by [1, Theorem 6.4] of an M-basis which is not a strong M-basis.

EXAMPLE 3.3. Let $A \in \mathcal{B}(H)$ be a positive injective non-invertible operator. Let N be a non-zero subspace of H satisfying $\operatorname{Ran}(A) \cap N = (0)$. Define subspaces of $H^{(3)}$ by

$$M_1 = \{(0, x, Ax) : x \in H\},$$
$$M_2 = (0) \oplus N^\perp \oplus (0),$$
$$M_3 = \{(Ax, x, 0) : x \in H\}.$$

Then $\{M_1, M_2, M_3\}$ is an M-basis of $H^{(3)}$. For, $M_1^\perp \cap M_2^\perp = H \oplus (0) \oplus (0)$ so $M_1 \vee M_2 = (0) \oplus H \oplus H$. Similarly, $M_2 \vee M_3 = H \oplus H \oplus (0)$, so $M_1 \vee M_2 \vee M_3 = H^{(3)}$. Also, $M_1 \vee M_3 = \{(x, y, z) : x + z = Ay\}$. It follows that

$$M_1 \cap (M_2 \vee M_3) = M_2 \cap (M_1 \vee M_3) = M_3 \cap (M_1 \vee M_2) = (0),$$

and

$$(M_1 \vee M_2) \cap (M_2 \vee M_3) \cap (M_1 \vee M_3) = (0).$$

However, $\{M_1, M_2, M_3\}$ is not a strong M-basis of $H^{(3)}$ since $(M_1 \vee M_2) \cap (M_2 \vee M_3) = (0) \oplus H \oplus (0) \neq M_2$.

Let $\mathcal{L}$ be the JSL generated by $\{M_1, M_2, M_3\}$. A partial order diagram of $\mathcal{L}$ is given in Figure 2 (in which $L = (0) \oplus H \oplus (0)$).

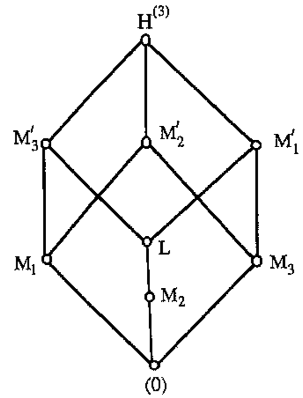

Fig. 2

Now $\mathcal{L} \neq \operatorname{Lat}\operatorname{Alg} \mathcal{L}$ if and only if $\mathcal{L}^\perp \neq \operatorname{Lat}\operatorname{Alg} \mathcal{L}^\perp$. We show that the latter is true if N is chosen appropriately.

Let N_0 be a non-zero subspace of H satisfying $\operatorname{Ran}(A^{1/2}) \cap N_0 = (0)$, and let $y_0 \in \operatorname{Ran}(A^{1/2})$ with $y_0 \notin \operatorname{Ran}(A)$. Let $N = N_0 + \langle y_0 \rangle$. (Then

$\mathrm{Ran}(A) \cap N = (0)$.) Now

$$M_1^{\perp} = \{(y, -Ax, x) : x, y \in H\},$$
$$M_2^{\perp} = H \oplus N \oplus H,$$
$$M_3^{\perp} = \{(x, -Ax, y) : x, y \in H\}.$$

It is not difficult to show that $\mathrm{Alg}\,\mathcal{L}^{\perp} = \mathrm{Alg}\{M_1^{\perp}, M_2^{\perp}, M_3^{\perp}\}$ is the set of operators on $H^{(3)}$ of the form

$$\begin{pmatrix} X+BA & B & 0 \\ 0 & W & 0 \\ 0 & C & X+CA \end{pmatrix}$$

where $B, C, W, X \in \mathcal{B}(H)$ and $WA = AX, W(N) \subseteq N$. Consider the subspace $K = H \oplus \langle y_0 \rangle \oplus H$. Clearly $M_2 \subset K^{\perp} \subset L$ so $K \notin \mathcal{L}^{\perp}$. However, $K \in \mathrm{LatAlg}\,\mathcal{L}^{\perp}$. For if $W \in \mathcal{B}(H)$ satisfies $WA = AX$ for some $X \in \mathcal{B}(H)$, and $W(N) \subseteq N$, then, by a result of C. Foiaş [5], W leaves $\mathrm{Ran}(A^{1/2})$ invariant so leaves $\mathrm{Ran}(A^{1/2}) \cap N = \langle y_0 \rangle$ invariant.

We conclude by discussing some ways by which new M-bases arise from old ones. (Analogous results in this vein for strong M-bases are discussed in [1].) We have already remarked how "slicing" and "selecting" produce new M-bases from old ones. Note that, by Propositions 3.1, 3.2, and 3.4, if $\{M_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of a reflexive Banach space X, then $\{(M_\gamma')^{\perp}\}_{\gamma\in\Gamma}$ is an M-basis of X^*. (If $\{M_\gamma\}_{\gamma\in\Gamma}$ generates the JSL $\mathcal{L}$, then $\{(M_\gamma')^{\perp}\}_{\gamma\in\Gamma}$ need not generate $\mathcal{L}^{\perp}$. The JSL shown in Figure 2 shows this.) The following proposition gives a more general result. But first note that, applying this result once more, and identifying $(M_\beta')^{\perp\perp}$ with M_β' using the canonical mapping of X onto X^{**}, we find that $\{N_\gamma\}_{\gamma\in\Gamma}$ is also an M-basis of X where $N_\gamma = \bigcap_{\beta\neq\gamma} M_\beta'$ for every $\gamma \in \Gamma$ (since $(((M_\gamma')^{\perp})')^{\perp} = (\bigvee_{\beta\neq\gamma}(M_\beta')^{\perp})^{\perp} = \bigcap_{\beta\neq\gamma}(M_\beta')^{\perp\perp}$). Although $M_\gamma \subseteq N_\gamma$ for every $\gamma \in \Gamma$, the M-basis $\{M_1, M_2, M_3\}$ described in Example 3.3 shows that we need not have equality.

PROPOSITION 3.5. *If $\{M_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of a (not necessarily reflexive) Banach space X, then $\{(M_\gamma')^{\perp}\}_{\Gamma}$ is an M-basis of $\bigvee_{\gamma\in\Gamma}(M_\gamma')^{\perp}$ (where $M_\gamma' = \bigvee_{\beta\neq\gamma} M_\beta$ for every $\gamma \in \Gamma$).*

Proof. Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be an M-basis of X. For every subset $G \subseteq \Gamma$, $\bigvee_{\gamma\in G} M_\gamma \subseteq \bigcap_{\beta\notin G} M_\beta'$ so $\bigvee_{\beta\notin G}(M_\beta')^{\perp} \subseteq (\bigvee_{\gamma\in G} M_\gamma)^{\perp}$. Thus if I, J are disjoint subsets of Γ, then

$$\Big(\bigvee_{\gamma\in I}(M_\gamma')^{\perp}\Big) \cap \Big(\bigvee_{\gamma\in J}(M_\gamma')^{\perp}\Big) \subseteq \Big(\bigvee_{\gamma\notin I} M_\gamma\Big)^{\perp} \cap \Big(\bigvee_{\gamma\notin J} M_\gamma\Big)^{\perp} = \Big(\bigvee_{\gamma\in\Gamma} M_\gamma\Big)^{\perp} = (0),$$

so

$$\Big(\bigvee_{\gamma\in I}(M_\gamma')^{\perp}\Big) \cap \Big(\bigvee_{\gamma\in J}(M_\gamma')^{\perp}\Big) = (0).$$

Also, for every $\gamma \in \Gamma$, $\bigvee_{\beta\neq\gamma}(M_\beta')^{\perp} \subseteq M_\gamma^{\perp}$, so

$$\bigcap_{\gamma\in\Gamma}\bigvee_{\beta\neq\gamma}(M_\beta')^{\perp} \subseteq \bigcap_{\gamma\in\Gamma} M_\gamma^{\perp} = \Big(\bigvee_{\gamma\in\Gamma} M_\gamma\Big)^{\perp} = (0),$$

and

$$\bigcap_{\gamma\in\Gamma}\bigvee_{\beta\neq\gamma}(M_\beta')^{\perp} = (0).$$

By definition, it follows that $\{(M_\gamma')^{\perp}\}_{\gamma\in\Gamma}$ is an M-basis of $\bigvee_{\gamma\in\Gamma}(M_\gamma')^{\perp}$. ∎

If $\{f_\gamma\}_{\gamma\in\Gamma}$ is a (vector) M-basis of a Banach space X, then $(\langle f_\gamma\rangle')^{\perp} = \langle f_\gamma^*\rangle$ for every $\gamma \in \Gamma$, so the preceding proposition shows that $\{f_\gamma^*\}_{\gamma\in\Gamma}$ is an M-basis of $\bigvee_{\gamma\in\Gamma} f_\gamma^*$. This has already been observed in [9]. On the other hand, if $\{f_\gamma\}_{\gamma\in\Gamma}$ is a strong M-basis of X, then $\{f_\gamma^*\}_{\gamma\in\Gamma}$ need not be a strong M-basis of $\bigvee_{\gamma\in\Gamma} f_\gamma^*$, even when the latter equals X^* [9]. As for the case of vector M-bases, "coming down" from X^* to X causes no problems.

PROPOSITION 3.6. *Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a family, with at least two elements, of non-zero subspaces of a (not necessarily reflexive) Banach space X. If $\{(M_\gamma')^{\perp}\}_{\gamma\in\Gamma}$ is an M-basis of X^*, then $\{M_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of X.*

Proof. Let $\{(M_\gamma')^{\perp}\}_{\gamma\in\Gamma}$ be an M-basis of X^*. For every $\beta \in \Gamma$, $M_\beta' \subseteq \bigvee_{\gamma\in\Gamma} M_\gamma$ so $(\bigvee_{\gamma\in\Gamma} M_\gamma)^{\perp} \subseteq \bigcap_{\beta\in\Gamma}(M_\beta')^{\perp} = (0)$. Hence $\bigvee_{\gamma\in\Gamma} M_\gamma = X$. Also, $X^* = \bigvee_{\gamma\in\Gamma}(M_\gamma')^{\perp} \subseteq (\bigcap_{\gamma\in\Gamma} M_\gamma')^{\perp}$ implies that $\bigcap_{\gamma\in\Gamma} M_\gamma' = (0)$.

Finally, let I, J be disjoint subsets of Γ. Then

$$\Big(\bigvee_{\gamma\in I} M_\gamma\Big) \cap \Big(\bigvee_{\gamma\in J} M_\gamma\Big) \subseteq \Big(\bigcap_{\gamma\notin I} M_\gamma'\Big) \cap \Big(\bigcap_{\gamma\notin J} M_\gamma'\Big) = (0),$$

so

$$\Big(\bigvee_{\gamma\in I} M_\gamma\Big) \cap \Big(\bigvee_{\gamma\in J} M_\gamma\Big) = (0).$$

This completes the proof. ∎

By [1, Corollary 5.4] "strong M-basis" can be substituted for "M-basis" in the statement of the preceding proposition, provided also that each M_γ

is one-dimensional. The M-basis $\{M_1, M_2, M_3\}$ described earlier in Example 3.3 shows that, without this proviso, this substitution will lead to a statement that is not valid. Indeed, any M-basis of a reflexive space X, with three elements, which is not a strong M-basis, will show this because $\{(M_1')^\perp, (M_2')^\perp, (M_3')^\perp\}$ is a strong M-basis of X^*, for every M-basis $\{M_1, M_2, M_3\}$ of X. For, $\{(M_1')^\perp, (M_2')^\perp, (M_3')^\perp\}$ is an M-basis of X^* by Proposition 3.5 and all that we need additionally to verify, by symmetry, is that

$$((M_1')^\perp \vee (M_2')^\perp) \cap ((M_1')^\perp \vee (M_3')^\perp) = (M_1')^\perp.$$

This follows from the fact that

$$(M_1' \cap M_2') \vee (M_1' \cap M_3') = M_1',$$

(using $M_m \subseteq M_n'$, if $m \neq n$).

Some more obvious ways by which new M-bases arise from old ones are as follows (their verification is left to the reader).

EXAMPLE 3.4. Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a (subspace) M-basis of X.

(1) For every non-empty subset $\Delta \subseteq \Gamma$, $\{M_\gamma\}_{\gamma\in\Delta}$ is an M-basis of $\bigvee_{\gamma\in\Delta} M_\gamma$.

(2) Let $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ be a family of pairwise disjoint non-empty subsets of Γ with $\bigcup_{\lambda\in\Lambda} \Gamma_\lambda = \Gamma$. For each $\lambda \in \Lambda$ put $N_\lambda = \bigvee_{\gamma\in\Gamma_\lambda} M_\gamma$. Then $\{N_\lambda\}_{\lambda\in\Lambda}$ is an M-basis of X.

(3) Let $S : X \to Y$ be a bicontinuous linear bijection of X onto a Banach space Y. Then $\{SM_\gamma\}_{\gamma\in\Gamma}$ is an M-basis of Y.

Combining (1) and (2) immediately above we see that if $\{f_\gamma\}_{\gamma\in\Gamma}$ is a (vector) M-basis of X, then, with $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ as in (2), $\{f_\gamma\}_{\gamma\in\Gamma_\lambda}$ is an M-basis of $\bigvee_{\gamma\in\Gamma_\lambda} f_\gamma$ for every $\lambda \in \Lambda$, and $\{\bigvee_{\gamma\in\Gamma_\lambda} f_\gamma\}_{\lambda\in\Lambda}$ is an M-basis of X. This procedure does not reverse. More precisely, if $\{N_\lambda\}_{\lambda\in\Lambda}$ is an M-basis of X and if, for each $\lambda \in \Lambda$, $\{f_{\gamma,\lambda} : \gamma \in \Gamma_\lambda\}$ is an M-basis of N_λ, it does not follow that $\bigcup_{\lambda\in\Lambda}\{f_{\gamma,\lambda} : \gamma \in \Gamma_\lambda\}$ is an M-basis of X. In fact, as the following example shows, $\bigcup_{\lambda\in\Lambda}\{f_{\gamma,\lambda} : \gamma \in \Gamma_\lambda\}$ need not even be minimal, even when $\{N_\lambda\}_{\lambda\in\Lambda}$ is a strong M-basis of X and, for each $\lambda \in \Lambda$, $\{f_{\gamma,\lambda} : \gamma \in \Gamma_\lambda\}$ is a strong M-basis of N_λ.

EXAMPLE 3.5. Let $A \in \mathcal{B}(H)$ be a positive injective non-invertible operator and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of H satisfying $\mathrm{Ran}(A) \cap \mathrm{span}\{e_n\}_{n=1}^\infty = (0)$. (Such an orthonormal basis exists [4].)

Define subspaces of $H^{(2)}$ by $N_1 = \{(x, Ax) : x \in H\}$ and $N_2 = \{(x, -Ax) : x \in H\}$. Then $\{N_1, N_2\}$ is a strong M-basis of $H^{(2)}$. It is easily verified that $\{(e_n, Ae_n)\}_{n=1}^\infty$ is a strong M-basis of N_1 (with biorthogonal sequence $\{((1+A^2)^{-1}e_n, A(1+A^2)^{-1}e_n)\}_{n=1}^\infty$) and $\{(e_n, -Ae_n)\}_{n=1}^\infty$ is a strong M-basis of N_2 (with biorthogonal sequence $\{((1+A^2)^{-1}e_n, -A(1+A^2)^{-1}e_n)\}_{n=1}^\infty$).

However, the family of vectors $\{(e_n, Ae_n)\}_{n=1}^\infty \cup \{(e_n, -Ae_n)\}_{n=1}^\infty$ is not even minimal in $H^{(2)}$. In fact, if F, G are disjoint sets of positive integers, then

$$\Big(\bigvee_{n\notin F} (e_n, Ae_n)\Big) \vee \Big(\bigvee_{m\notin G} (e_m, -Ae_m)\Big) = H^{(2)}.$$

For, let $(x, y) \in H^{(2)}$ with $((x, y) \mid (e_n, Ae_n)) = 0$ for every $n \notin F$ and $((x, y) \mid (e_m, -Ae_m)) = 0$, for every $m \notin G$. Then $x + Ay \in \bigvee_{n\in F} e_n$ and $x - Ay \in \bigvee_{m\in G} e_m$. Thus, subtracting gives $Ay \in \mathrm{Ran}(A) \cap \mathrm{span}\{e_n\}_{n=1}^\infty$ so $y = 0$. Then $x \in (\bigvee_{n\in F} e_n) \cap (\bigvee_{m\in G} e_m) = (0)$, so $x = 0$.

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Fractional Sobolev norms and structure of Carnot–Carathéodory balls for Hörmander vector fields

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Abstract. We study the notion of fractional L^p-differentiability of order $s \in (0,1)$ along vector fields satisfying the Hörmander condition on $\mathbb{R}^n$. We prove a modified version of the celebrated structure theorem for the Carnot–Carathéodory balls originally due to Nagel, Stein and Wainger. This result enables us to demonstrate that different $W^{s,p}$-norms are equivalent. We also prove a local embedding $W^{1,p} \subset W^{s,q}$, where q is a suitable exponent greater than p.

1. Introduction. It is well known that the classical theory of Sobolev spaces plays an important role in many problems concerning partial differential equations. It has also been realized in the last years that an essential tool in the study of second order differential operators arising from degenerate vector fields on $\mathbb{R}^n$ is the construction of generalized Sobolev spaces suitably related to the fields.

To motivate our discussion we recall some simple features of first order Sobolev spaces. Given a family $X_1, \ldots, X_m$ of (at least Lipschitz continuous) vector fields on $\mathbb{R}^n$, $X_j = \sum_{k=1}^n a_{j,k}(x)\partial/\partial x_k$, a natural generalization of the usual $W^{1,p}$ space can be defined by means of the norm

$$\|u\|_{W_X^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $Xu = (X_1u, \ldots, X_mu)$ denotes the "degenerate gradient", $X_ju = \sum a_{j,k}\partial_k u$. If we assume that the fields are smooth and satisfy the Hörmander condition (see (5)), then a Sobolev-type embedding holds for the space $W_X^{1,p}$. Namely, representing a function u as a "convolution" by means of the fundamental solution Γ of $\sum X_j^2$, using the estimates of Γ and $X\Gamma$ (see Nagel, Stein and Wainger [47] and Sánchez-Calle [52]), together with the continuity of some "fractional integration op-

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