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On absolutely representing systems in spaces of infinitely differentiable functions

by

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Abstract. The main part of the paper is devoted to the problem of the existence of absolutely representing systems of exponentials with imaginary exponents in the spaces $C^\infty(G)$ and $C^\infty(K)$ of infinitely differentiable functions where G is an arbitrary domain in $\mathbb{R}^p$, $p \geq 1$, while K is a compact set in $\mathbb{R}^p$ with non-void interior $\dot{K}$ such that $\overline{\dot{K}} = K$. Moreover, absolutely representing systems of exponents in the space $H(G)$ of functions analytic in an arbitrary domain $G \subseteq \mathbb{C}^p$ are also investigated.

1. Introduction. Let H be a linear topological space over the field $\mathbb{C}$. A sequence $X := (x_k)_{k=1}^\infty \subset H$ is called a *representing system* (RS) in H if each element x of H can be represented in the form of a series

$$x = \sum_{k=1}^{\infty} c_k x_k, \quad c_k \in \mathbb{C}, \ k = 1, 2, \ldots, \tag{1.1}$$

converging in H. Let now H be a complete locally convex space (CLCS). A sequence X is said to be an *absolute representing system* (ARS) in H if each $x \in H$ can be represented in the form of a series (1.1) absolutely converging in H. It is evident that every ARS in H is a fortiori an RS. The problem of existence of such systems was investigated in [9].

Suppose that $H = \underrightarrow{\operatorname{ind}}\, H_n$ where for any $n \geq 1$, H_n is a CLCS, $H_n \hookrightarrow H_{n+1}$ and $x_k \in H_1$, $k \geq 1$. If X is an RS (or an ARS) in each H_n then X is an RS (respectively, an ARS) in H. This trivial fact is mentioned in [13, §3, point 1]; a far more difficult question is also posed there: whether X is an RS (or an ARS) in $H = \underleftarrow{\operatorname{proj}}\, H_n$ if X is an RS (respectively, an ARS) in each H_n.

A number of results in this direction for certain function spaces (mainly for the Fréchet space $H = H(G)$ of functions analytic in the domain G with the standard compact-open topology) and for some sequences x_k (mainly of

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exponentials and Mittag-Leffler functions) have been obtained by the author [13] and A. V. Abanin ([1]–[3]). Some of these results are quoted below.

Rather general results on stability of ARS under the passage to projective limits of FN- and DFN-spaces were stated in [4]. Here is the main result of [4] for the case when H_n are FN-spaces, that is, nuclear Fréchet spaces.

(Theorem 2.1 of [4]) (1) *Let H_n be an FN-space with the topology defined by seminorms $\{p_j^n\}_{j=1}^{\infty}$, $n = 1, 2, \ldots$ Let $u_k \in H_n$ for $k, n \geq 1$. Suppose that*

$$\forall j,n \quad \lim_{k\to\infty} \frac{p_j^n(u_k)}{p_{j+1}^n(u_k)} = 0. \tag{1.2}$$

(2) *Let $U = (u_k)_{k=1}^{\infty}$ be an ARS in each H_n, $n = 1, 2, \ldots$*

(3) *Then U is an ARS in $H = \underleftarrow{\operatorname{proj}} H_n$.*

A. V. Abanin [3] noted that the proof of this statement in [4] is erroneous. Namely, the relation $y_p \in R_j^n$ for $p \geq 1$ ([4], p. 202) was obtained incorrectly since $y_p \in H'$ but not necessarily $y_p \in H_n'$. Abanin remarked that the validity of this result remained open. Therefore it will be cited from now on as *conjecture A*.

Let us formulate two similar hypotheses.

Conjecture B. *Let the assumption* (1) *of conjecture A be satisfied and let U be an RS in H_n, $n = 1, 2, \ldots$ Then U is an RS in $H = \underleftarrow{\operatorname{proj}} H_n$.*

Conjecture C. *Let the assumption* (1) *of conjecture A be satisfied and let U be an ARS in H_n, $n = 1, 2, \ldots$ Then U is an RS in $H = \underleftarrow{\operatorname{proj}} H_n$.*

It is clear that any of conjectures A, B implies conjecture C. It is shown below that the latter conjecture is false. Consequently, both conjectures A, B are also untrue.

2. A class of ARS of exponentials in $C^\infty(K)$. Let K be a compact set in $\mathbb{R}^p$, $p \geq 1$, with non-empty interior $\dot{K}$. We assume everywhere below that $K = \overline{\dot{K}}$. Let $C^\infty(K)$ be the space of all complex-valued functions $f \in C^\infty(\dot{K})$ uniformly continuous in $\dot{K}$ together with all partial derivatives. It is clear that $C^\infty(K) = \{f \in C^\infty(\dot{K}) : f$ and all its partial derivatives admit a continuous extension to $K\}$. The Fréchet space topology in $C^\infty(K)$ is defined by the norms

$$\|y\|_n := \sup\{|y^{\alpha}(x)| : x \in \dot{K},\ |\alpha|_p \leq n\}, \quad n = 0, 1, \ldots,$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$, $|\alpha|_p = |\alpha_1| + \ldots + |\alpha_p|$, $\alpha_s \in \mathbb{N}_0 = \{0, 1, \ldots\}$, $s = 0, 1, \ldots, p$.

We say that K is a *Whitney compactum* (W.c.) if $K = \overline{\dot{K}}$ and if $C^\infty(K)$ coincides with its subspace $C^\infty_{\mathrm{W}}(K)$ of *Whitney functions* on K, that is, of traces on K of functions from $C^\infty(\mathbb{R}^p)$. The topology in $C^\infty_{\mathrm{W}}(K)$ is defined by the norms $\|\cdot|_n$ $(n = 0, 1, 2, \ldots)$:

$$\|y|_n = \|y\|_n + \sup\left\{\frac{|(R^n_{x_0}y)(x)|}{|x - x_0|_p^{n-|\alpha|_p}} : x, x_0 \in K,\ x \neq x_0,\ |\alpha|_p \leq n\right\}$$

where $R^n_{x_0}y$ is the nth Taylor remainder of the function y at x_0. It is known that $C^\infty_{\mathrm{W}}(K)$ is an FN-space. Since $C^\infty(K)$ coincides with $C^\infty_{\mathrm{W}}(K)$ for any W.c. K and $\|y|_n \geq \|y\|_n$ for all $y \in C^\infty_W(K)$ it follows that $C^\infty(K)$ is topologically isomorphic to $C^\infty_{\mathrm{W}}(K)$ and therefore $C^\infty(K)$ is an FN-space.

According to [27], K is a W.c. if K is connected and has the property $(\mathcal{P})$ [8, Ch. II, §2.3]. In particular each convex compact set is a W.c. However, an exact geometrical characterization of a W.c. is unknown.

We show in this section that $C^\infty(K)$ has an ARS of exponentials with purely imaginary exponents if and only if K is a W.c.

Let us first state a convergence criterion for series of such exponentials:

$$\sum_{|k|_p=0}^{\infty} c_k \exp\Big(i\sum_{j=1}^{p} \mu_{j,k}x_j\Big), \quad \mu_{j,k} \in \mathbb{R},\ 1 \leq j \leq p,\ k = (k_1, \ldots, k_p). \tag{2.1}$$

A series obtained by termwise differentiation (any number of times) of the series (2.1) with respect to arbitrary variables $x_1, \ldots, x_p$ will be called *associated* with the series (2.1). Let us formulate a number of assertions for a fixed compact set K with $K = \overline{\dot{K}}$:

(1) the series (2.1) converges absolutely in $C^\infty(K)$;

(2) the series (2.1) converges in $C^\infty(K)$;

(3) the series (2.1) and all series associated with it converge absolutely at each point of K;

(4) the series (2.1) and all its associated series converge absolutely at some point of K;

(5) the series (2.1) and all its associated series converge pointwise in K;

(6) the series (2.1) and all its associated series converge at some point of K;

(7) for all $m \geq 0$, $\sum_{|k|_p=0}^{\infty} |c_k| \cdot |\mu_k|_p^m < \infty$, $\mu_k = (\mu_{1,k}, \ldots, \mu_{p,k})$;

(8) for all $m \geq 0$, $\sup_{|k|_p\geq 0} |c_k| \cdot |\mu_k|_p^m < \infty$;

(9) the series (2.1) converges absolutely in $C^\infty(D)$ for each compactum $D \subset \mathbb{R}^p$.

Taking into account the equality $|\exp(i\sum_{j=1}^p \mu_{j,k}x_j)| = 1$ for $k \geq 1$ and $x \in \mathbb{R}^p$, one can easily obtain the following result.

Lemma 2.1. *If $\overline{\dot{K}} = K$ and K is a compact set in $\mathbb{R}^p$, then* (4)⇒(7)⇒(1)⇒(2)⇒(5)⇒(6)⇒(8) *and* (7)⇒(9)⇒(1)⇒(3)⇒(4)⇒(6)⇒(8).

If, additionally,

$$\limsup_{|k|_p\to\infty} \frac{\ln|k|_p}{\ln|\mu_k|_p} < \infty, \tag{2.2}$$

then (8)⇒(7) *and in this case all assertions* (1)–(9) *are equivalent.*

It follows from Lemma 2.1 that assertions (1), (3), (4), (7), (9) are always equivalent.

THEOREM 2.2. *Let K be a Whitney compactum in $\mathbb{R}^p$, $p\ge 1$, and let $T = \{x : a_k < x_k < b_k,\ k=1,\dots,p\}$ be an arbitrary bounded open rectangular parallelepiped containing K. Then*

$$\mathcal{E}_p^T := \left\{\exp\left(2\pi i\sum_{j=1}^p \frac{k_j x_j}{b_j-a_j}\right) : k_s = 0,\pm1,\pm2,\dots;\ s=1,\dots,p\right\}$$

is an ARS in $C^\infty(K)$.

Proof. Let $f\in C^\infty(K)$. Then there exists $g\in C^\infty(\mathbb{R}^p)$ with $g|_K = f$. For any $\eta>0$ we put $K_\eta := \{x\in\mathbb{R}^p : \varrho(x,K)\le\eta\}$ where $\varrho(x,K) := \min\{|x-y|_p : y\in K\}$.

Fix $\varepsilon>0$ so small that $K_{3\varepsilon}\subset T$. It is shown in the proof of Theorem 1.4.1 of [8] (Ch. I, §1.4) that there exists $h\in C^\infty(\mathbb{R}^p)$ such that $h|_{K_\varepsilon}\equiv 1$ and $\operatorname{supp} h\subset K_{3\varepsilon}$. We put $H(x) = h(x)g(x)$. It is evident that $H\in C^\infty(\mathbb{R}^p)$, $H|_K = f$, $H|_{K_\varepsilon}\equiv g$. If for $k=(k_1,\dots,k_p)$ and $x\in\mathbb{R}^p$ we set

$$v_k(x) := \exp\left(2\pi i\sum_{j=1}^p \frac{k_j}{b_j-a_j}\left(x_j - \frac{a_j+b_j}{2}\right)\right),$$

then

$$H(x) \sim \sum_{|k|_p=0}^\infty h_k v_k \tag{2.3}$$

is the Fourier series of $H(x)$. Employing the standard integration by parts to the well known integral representation of the coefficients h_k, and taking into account that $H(x)$ and all its partial derivatives vanish on the boundary of T, we obtain

$$\forall m\ge 0 \quad \sup_{|k|_p\ge 0} |h_k|\cdot|k|_p^m < \infty. \tag{2.4}$$

The series (2.3) can be written in the form

$$\sum_{|k|_p=0}^\infty h_k v_k = \sum_{|k|_p=0}^\infty \widetilde{h}_k \exp\left(2\pi i\sum_{j=1}^p \frac{k_j x_j}{b_j-a_j}\right)$$

where $|h_k| = |\widetilde{h}_k|$ for all k. Since the series (2.3) satisfies the conditions (8) and (2.2) of Lemma 2.1, it converges absolutely in $C^\infty(T)$. It is clear that

$H(x)$ is the sum of the series (2.3) in T, whence for all $x\in K$,

$$f(x) = \sum_{|k|_p=0}^\infty \widetilde{h}_k \exp\left(2\pi i\sum_{j=1}^p \frac{k_j x_j}{b_j-a_j}\right) \tag{2.5}$$

and this series converges absolutely in $C^\infty(K)$.

COROLLARY 2.3. *Let K be a convex compact set in $\mathbb{R}^p$ with non-void interior. Then $C^\infty(K)$ has an ARS of exponentials $\{\exp 2\pi i\langle\alpha k, x\rangle\}_{|k|_p=0}^\infty$, where $\alpha = (\alpha_1,\dots,\alpha_p)\in\mathbb{R}^p$, $\alpha k = \{\alpha_j k_j\}_{j=1}^p$ and $\langle m,x\rangle = \sum_{j=1}^p m_j x_j$.*

By Lemma 2.1 the absolute convergence of an arbitrary series (2.1) in $C^\infty(K)$ for some fixed K with $K=\overline{\overset{\circ}{K}}$ implies its absolute convergence in $C^\infty(D)$ for each compact set D in $\mathbb{R}^p$. Therefore the sum of such a series belongs to $C(\mathbb{R}^p)$. Combining this simple argument with Theorem 2.1 we obtain the following result.

THEOREM 2.4. *Let K be a non-empty compact set in $\mathbb{R}^p$ such that $K = \overline{\overset{\circ}{K}}$. Then the following assertions are equivalent:*

(1) *K is a W.c.;*

(2) *in $C^\infty(K)$ there exists an ARS of exponentials with purely imaginary exponents;*

(3) *for each open bounded rectangular parallelepiped T containing K the corresponding system $\mathcal{E}_p^T$ is an ARS in $C^\infty(K)$.*

REMARK 2.5. It follows from Theorem 2.4 that if K is a W.c., then each $f\in C^\infty(K)$ can be extended to a periodic function $F\in C^\infty(\mathbb{R}^p)$ such that $F(x_1+\alpha_1,\dots,x_p+\alpha_p)\equiv F(x_1,\dots,x_p)$, where $\alpha_j = b_j-a_j$ and $\{x : a_k < x_k < b_k,\ 1\le k\le p\}$ is any fixed open rectangular parallelepiped containing K.

Now we indicate some conditions under which the system $\mathcal{E}_p^T$ is not an ARS in $C^\infty(K)$.

THEOREM 2.6. *Let K be non-empty compact set in $\mathbb{R}^p$ such that $\overline{\overset{\circ}{K}} = K$. Let $T = \{x : a_k < x_k < b_k,\ k=1,\dots,p\}$ be an open bounded rectangular parallelepiped in $\mathbb{R}^p$. Suppose that there exists at least one pair of different points $X^{(1)}$ and $X^{(2)}$ in K such that $X_j^{(1)} = X_j^{(2)} + m_j(b_j-a_j)$, where $m_j\in\mathbb{Z}$, $j=1,\dots,p$. Then the system $\mathcal{E}_p^T$ is not complete in $C^\infty(K)$.*

Proof. Each function $v\in\operatorname{span}\mathcal{E}_p^T$ satisfies $v(X^{(1)}) = v(X^{(2)})$. The same equality holds for all functions from the closure of $\operatorname{span}\mathcal{E}_p^T$. Since $X^{(1)}\ne X^{(2)}$, we have $X_j^1\ne X_j^2$ for some $j\le p$. The function $f_j(x) = x_j$ belongs to $C^\infty(K)$, but $f_j(X^{(1)})\ne f_j(X^{(2)})$.

COROLLARY 2.7. *Let $-\infty < a_j < 0 < b_j < \infty$ and let $\theta_j > 0, 1 \le j \le p$. Then the system*

$$\mathcal{E}^T_{p,\theta} := \left\{\exp\left(2\pi i \sum_{j=1}^{p} \frac{k_j\theta_j x}{b_j - a_j}\right) : k_s \in \mathbb{Z},\ s = 0,1,\ldots,p\right\}$$

is an ARS in $C^\infty(\overline{T})$, where $\overline{T} = \{x : a_k \le x_k \le b_k,\ 1 \le k \le p\}$, if $\theta_j \in (0,1)$ for all $j \le p$. On the other hand $\mathcal{E}^T_{p,\theta}$ is not a complete system in $C^\infty(\overline{T})$ if $\theta_{j_0} \ge 1$ for some $j_0 \le p$.

To end this section we show that no system of exponentials can be a basis in $C^\infty(K)$. Suppose that the system $\mathcal{E}^p_{(\alpha)} := \{\exp(\sum_{j=0}^{p} \alpha_{j,k}x_k)\}_{|k|_p=0}^{\infty}$ with $\alpha_{j,k} \in \mathbb{C}$, $j = 1,\ldots,p$, $k \in \mathbb{Z}^p$, is an ARS in $C^\infty(K)$ for some compactum $K = \overline{\dot{K}}$. Fix an arbitrary $m \le p$. Then $f_m(x) := x_m$ belongs to $C^\infty(K)$ and there exists a series converging to x_m in $C^\infty(K)$:

$$x_m = \sum_{|k|_p=0}^{\infty} d_k \exp\langle\alpha_k, x\rangle, \qquad \alpha_k = \{\alpha_{1,k},\ldots,\alpha_{p,k}\}.$$

We can differentiate this series with respect to x_m:

$$1 = \sum_{|k|_p=0}^{\infty} d_k\alpha_{m,k}\exp\langle\alpha_k, x\rangle, \qquad x \in K.$$

It is evident that $d_{k_0}\alpha_{m,k_0} \ne 0$ at least for one $k_0 \in \mathbb{Z}^p$. After the second differentiation we obtain

$$0 = \sum_{|k|_p=0}^{\infty} d_k(\alpha_{m,k})^2\exp\langle\alpha_k, x\rangle, \qquad x \in K,$$

and this series converges in $C^\infty(K)$. So there exists a non-trivial expansion of zero in $C^\infty(K)$, which means that $\mathcal{E}^p_{(\alpha)}$ is not a basis in $C^\infty(K)$.

3. Negative results. We are now going to obtain a result opposite in a sense to Theorem 2.2. For an arbitrary domain G in $\mathbb{R}^p$ let us introduce the vector space $b(G)$ of all functions defined and bounded in G, and the Fréchet space $C^\infty(G)$ of functions infinitely differentiable in G with the topology defined by the seminorms

$$|y|_{K,n} = \max\{|y^{(\alpha)}(x)| : x \in K,\ |\alpha|_p \le n\}, \quad n = 0,1,\ldots,$$

where K is an arbitrary compact set in G.

THEOREM 3.1. *Let $\mu_{j,k} \in \mathbb{R}$, $j = 1,\ldots,p$; $|k|_p = 0,1,\ldots$; $\mu_k = (\mu_{j,k})_{j=1}^p$; $\mathcal{E}_\mu := \{\exp(i\sum_{j=1}^p \mu_{j,k}x_j)\}$. If G is an arbitrary domain in $\mathbb{R}^p$, then $\mathcal{E}_\mu$ is not an ARS in $C^\infty(G)$. If, additionally, the condition (2.2) holds, then $\mathcal{E}_\mu$ is not an RS in $C^\infty(G)$.*

Proof. 1. Suppose that $\mathcal{E}_\mu$ is an ARS in $C^\infty(G)$ for some real exponents $\mu_{j,k}$. Then each $y \in C^\infty(G)$ can be represented as a series

$$(3.1) \qquad y(x) = \sum_{|k|_p=0}^{\infty} y_k \exp\left(i\sum_{j=1}^{p}\mu_{j,k}x_j\right)$$

converging absolutely in $C^\infty(G)$ and a fortiori in $C^\infty(K_0)$, where K_0 is an arbitrary closed p-dimensional ball contained in G. Due to Lemma 2.1, $\sum_{|k|_p=0}^{\infty}|y_k| < \infty$, whence $\sup_{x\in G}|y(x)| \le \sum_{|k|_p=0}^{\infty}|y_k|$, and $y \in b(G)$. So, if $\mathcal{E}_\mu$ is an ARS in $C^\infty(G)$, then $C^\infty(G) \subseteq b(G)$. But it is easy to show that the latter inclusion is impossible. Indeed, if the domain G is unbounded and $f(x) = x_1^2 + x_2^2 + \ldots + x_p^2$, then $f \in C^\infty(G)$, but $f \notin b(G)$. In case G is a bounded domain, one can fix an arbitrary finite boundary point $\beta = (\beta_k)_{k=1}^p$ of G and consider the function $g(x) = (\sum_{k=1}^p (x_k - \beta_k)^2)^{-1}$. It is clear that $g \in C^\infty(G)$, but $g \notin b(G)$.

2. Suppose now that (2.2) is satisfied, and $\mathcal{E}_\mu$ is an RS in $C^\infty(G)$ for some $\mu \in \mathbb{R}^p$. Then each $y \in C^\infty(G)$ is represented in the form of a series (3.1) converging in $C^\infty(G)$. By Lemma 2.1 this series converges absolutely in $C^\infty(G)$. Therefore $\mathcal{E}_\mu$ is an ARS in $C^\infty(G)$, which contradicts the first part of the proof.

Theorems 2.2 and 3.1 lead to the following result.

THEOREM 3.2. *Conjecture C is false.*

Proof. Fix some non-void bounded convex domain G in $\mathbb{R}^p$. It is well known that there always exists a sequence $(K_n)_{n=1}^\infty$ of convex compact subsets of G such that

$$(3.2) \qquad \forall n \ge 1 \quad K_n \subseteq \dot{K}_{n+1} \subset G = \bigcup_{m=1}^{\infty} K_m.$$

Take an arbitrary parallelepiped $T := \{x : a_k < x < b_k,\ 1 \le k \le p\}$ containing G. By Theorem 2.2, $\mathcal{E}_\mu$ is an ARS in $C^\infty(K_m)$, $m = 1,2,\ldots$ We put $\mu_{j,k} = 2\pi k_j/(b_j - a_j)$, $j = 1,\ldots,p$; $k = 0,\pm1,\pm2,\ldots$ It is clear that there are $\alpha > 0$ and $\beta > 0$ such that $\alpha|k|_p \le |\mu_k|_p \le \beta|k|_p$ for all k with $|k|_p > 0$. Therefore the condition (2.2) holds, and by Theorem 3.1, $\mathcal{E}_\mu$ is not an RS in $C^\infty(G) = \operatorname{proj}_{\longleftarrow} C^\infty(K_n)$.

Let us now show that in the present situation the assumption (1) of Theorem 2.1 from [4] is valid. For any $n, j \ge 1$ we have $p_j^n(y) = \max\{|y^{(\alpha)}(x)| : x \in K_n,\ |\alpha|_p \le j\}$. Let us arrange the systems $k = (k_1,\ldots,k_p)$ into a sequence $\{l\}_1^\infty$ in such a manner that $|k|_p$ does not decrease. Then each k obtains its number $l = l(k)$ where $l \to \infty \Leftrightarrow |k|_p \to \infty$. For $n \ge 1$, $j \ge 0$

and $l = l(k) \geq 1$ we have $u_l = \exp(i\sum_{s=1}^{p} \mu_{s,k} x_s)$ and

$$|p_j^n(u_l)| = \max\{|\mu_{1,k}|^{\alpha_1} \ldots |\mu_{p,k}|^{\alpha_p} : |\alpha|_p \leq j\} = \max\{(k_1|\gamma_1|)^{\alpha_1}, \ldots, (k_p|\gamma_p|)^{\alpha_p} : |\alpha|_p \leq j\},$$

where $\gamma_s = 2\pi/(b_s - a_s)$, $s = 1, \ldots, p$. Thus $|p_j^n(u_l)|$ does not depend on n. Moreover,

$$\forall j \geq 1\ \exists \delta_j \geq 1\ \forall l \geq 1 \qquad |p_{j+1}^n(u_l)| \geq \delta_j |k|_p p_j^n(u_l).$$

So the condition (1.2) is satisfied, and the theorem is proved.

COROLLARY 3.3. *Theorem* 2.1 *of* [4] *is untrue.*

The latter result implies in particular that the validity of the following corollary of Theorem 2.1 from [4] remains open.

(Corollary 2.1 of [4]) *Let G be an arbitrary convex domain in $\mathbb{C}^p$ and let $\{G_n\}_{n=1}^{\infty}$ be a sequence of convex domains in $\mathbb{C}^p$ such that for all $n \geq 1$, $\overline{G}_n \subset G_{n+1} \subset G = \bigcup_{m=1}^{\infty} G_m$. Suppose that $\mathcal{E}_{(\Lambda)} := \{\exp \sum_{j=1}^{p} \lambda_{j,k} z_j\}_{|k|_p=0}^{\infty}$, $\lambda_{j,k} \in \mathbb{C}$, is an ARS in each space $H(G_m)$, $m \geq 1$. Then $\mathcal{E}_{(\Lambda)}$ is an ARS in $H(G)$.*

The author of [4] remarks that the special case of the latter statement with $p = 1$ and $G_n = q_n G$, $q_n \uparrow 1$, was obtained in [13] (Ch. II, §3, Theorem 6). However he did not apparently notice a more general result of the same paper, namely, Theorem 9 [13, p. 109]. This theorem implies, in the case $\varrho = 1$, the validity of Corollary 2.1 of [4] for $p = 1$ and for an arbitrary bounded convex domain G. Yet it is unknown whether such a result is valid for an arbitrary unbounded convex domain G even in the case $p = 1$ (this problem was raised before in [13]). The most general (but not final) results in this direction are due to A. V. Abanin ([1]–[3]) who employed the notion of weakly sufficient sets and the connection between those sets and ARS described in [11, §3], [13, Ch. 1, §2], [14, §8]. The sequence $\{G_n\}_{n=1}^{\infty}$ with the properties described in Corollary 2.1 of [4] will be called an *approximating sequence* for G.

The approximating sequence $\{G_m\}_{m=1}^{\infty}$ of domains G_m with support functions h_m is said to be *suitable* for G if $h_{m+1}(z)/h_m(z) \to 1$ uniformly on the sphere $|z|_p = 1$ as $m \to \infty$.

It is shown in [3] that each approximating sequence for a bounded convex domain G in $\mathbb{C}^p$ is suitable for G. Moreover, there always exists a suitable sequence for an arbitrary convex domain in $\mathbb{C}^p$. Taking into account the well known connection between ARS and weakly sufficient sets [14] one can obtain the following result.

THEOREM (A. V. Abanin [2]). *If $\{G_n\}_{n=1}^{\infty}$ is a suitable sequence for a convex domain $G \subset \mathbb{C}^p$, $p \geq 1$, and if $\mathcal{E}_{(\mu)}$ is an ARS of exponentials in $H(G_m)$ for all $m \geq 1$, then $\mathcal{E}_{(\mu)}$ is an ARS in $H(G)$.*

The author of [4] also investigated the case of projective limits of DFN-spaces, and claimed to prove the following result.

(Theorem 2.2 of [4]) *Let $U = (u_k)_{k=1}^{\infty}$ be an ARS of each DFN-space $H_n = \mathrm{ind}_j B_j^n$ where B_j^n is a B-space with norm $\|\cdot\|_j^n$. Suppose that:*

(a) *the inclusion of B_j^n into B_{j+1}^n is nuclear for all $j, n \geq 1$;*

(b) $\lim_{k\to\infty} \|u_k\|_j^j / \|u_k\|_l^l = 0$ *for all $l > j$.*

Then U is an ARS in $H = \underleftarrow{\mathrm{proj}}\, H_n$.

However the proof has just the same error (see [4, p. 205]) as in the case of the above cited Theorem 2.1 of [4]. Moreover an example disproving Theorem 2.2 of [4] can be constructed with the help of ARS of exponentials with imaginary exponents in some subspaces of $C^{\infty}(K)$ and $C^{\infty}(G)$. Since the description of this example is much longer than that of the example given in the proof of Theorem 3.2 we shall publish it elsewhere. In any case the validity of Corollary 2.2 of Theorem 2.2 of [4] remains open. Some special cases of the latter corollary have been obtained by Yu. F. Korobeĭnik and A. V. Abanin, but the case of an unbounded convex domain G in $\mathbb{C}^p$ is not completely investigated yet.

4. ARS of exponentials in $H(G)$. The author of [4] applied his Theorems 2.1, 2.2 to the proof of existence of ARS of exponentials in the space $H(G)$ of functions analytic in G and in the space $H(K)$ of analytic germs on a compact set K. In this section we obtain some results in this direction for the space $H(G)$ and compare them with those of [4]. Let us begin from the simpler one-dimensional case.

THEOREM 4.1. *The following assertions are equivalent for an arbitrary domain $G \subset \mathbb{C}$:*

(1) *G is convex;*

(2) *$H(G)$ has at least one ARS of exponentials;*

(3) *there exists an ARS $\mathcal{E}_{\Lambda}$ of exponentials in $H(G)$ with $\lim_{k\to\infty} |\lambda_k| = \infty$ which is an ARS in $H(aG)$ for all $0 < a < \infty$.*

Proof. (2)⇒(1). Let $\mathcal{E}_{\Lambda}$ be an ARS in $H(G)$. Then any $y \in H(G)$ is represented by the series

$$y(z) = \sum_{k=1}^{\infty} y_k \exp(\lambda_k z) \tag{4.1}$$

converging absolutely in $H(G)$. All the more the series (4.1) converges absolutely at each point of G. Since the set of all points of absolute convergence of an arbitrary series of the form (4.1) is convex ([6], [21, Ch. III, §1]), the series (4.1) converges absolutely in the convex hull conv G. By [6], [21] it converges uniformly on each compact subset of conv G. So every $f \in H(G)$ admits a single-valued analytic extension onto conv G, which is possible only if $G = \operatorname{conv} G$.

(1)⇒(3). This implication was established long ago. According to Theorem 10 of [11] for an arbitrary convex domain G in $\mathbb{C}$ there exists an ARS $\mathcal{E}_\Lambda$ in $H(G)$ such that $\lim_{k\to\infty}|\lambda_k| = \infty$ and $\mathcal{E}_\Lambda$ is an ARS in $H(aG)$, $0 < a < \infty$. The proof of this theorem has a small gap rectified in [12, pp. 104–108]. Just the same addition to the proof of Theorem 10 of [11] was made in [19, p. 251]. The existence of an ARS of exponentials in $H(G)$ with the required properties arises also from Theorem 2 of [19]. According to that theorem, if $d_\Lambda(z) := \inf\{|z-\lambda_k| : k \geq 1\}$, $\Lambda = \{\lambda_k : k \geq 1\}$ and $\lim_{z\to\infty} d_\Lambda(z) = 0$, then $\mathcal{E}_\Lambda$ is an ARS in each convex domain in $\mathbb{C}$. Finally, the implication (3)⇒(2) is trivial.

Let us now consider a multi-dimensional situation.

THEOREM 4.2. *The following assertions are equivalent for an arbitrary domain G in $\mathbb{C}^p$, $p \geq 1$:*

(1) *the envelope of holomorphy of G is convex;*

(2) *$H(G)$ has at least one ARS of exponentials;*

(3) *there exists an ARS $\mathcal{E}_\Lambda$ of exponentials in $H(G)$ with $\lim_{k\to\infty}|\lambda_k| = \infty$ which is an ARS in $H(aG)$ for all $0 < a < \infty$;*

(4) *the envelope of holomorphy of G coincides with* conv G.

Proof. (2)⇒(4). Let $\mathcal{E}_\Lambda$ be an ARS in $H(G)$. Then every $y \in H(G)$ is represented in the form of a series

$$y(z) = \sum_{k=1}^{\infty} y_k \exp\langle\lambda_k, z\rangle \tag{4.2}$$

absolutely converging in $H(G)$ and a fortiori at each point of G. Just as in the one-dimensional case the set Q of all points of absolute convergence of (4.2) is convex, and the series converges uniformly on each compact subset of int Q. As mentioned above the proof of these assertions for the one-dimensional case can be found in [6], [21]. The proof for $p \geq 2$ is quite analogous. It can be found in [18]; however, it is most probable that this proof was published earlier by other authors. So the sum in (4.2) is analytic in conv G and therefore each function from $H(G)$ admits a single-valued analytic continuation to conv G. Thus conv G is a holomorphic extension of G. Since each convex domain in $\mathbb{C}^p$ is a domain of holomorphy, the domain conv G is the envelope of holomorphy of G.

(1)⇒(3). Applying the method of [11]–[13] to the multi-dimensional situation, V. V. Morzhakov [24, 25] constructed, for an arbitrary convex domain G in $\mathbb{C}^p$, $p \geq 2$, an ARS $\mathcal{E}_\Lambda$ of exponentials such that $\mathcal{E}_\Lambda$ is an ARS in $H(aG)$ for all $0 < a < \infty$, and $\lim_{k\to\infty}|\lambda_k| = \infty$. Moreover, in the same papers and by the same method he obtained a multi-dimensional generalization of the above cited Theorem 2 of [19]: if $\Lambda = \{\lambda_k : k \geq 1\}$, $d_\Lambda(z) := \inf\{|z-\lambda_k|_p : k \geq 1\}$ and $\lim_{|z|_p\to\infty} d_\lambda(z) = 0$, then $\mathcal{E}_\Lambda$ is an ARS in every convex domain in $\mathbb{C}^p$.

Finally, the implications (4)⇒(1) and (3)⇒(2) are trivial.

Let us compare Theorems 4.1 and 4.2 with Theorem 4.1 of [4]. It is easy to show that they are in fact equivalent. However, the proof of the latter is not quite correct, since it employs the false Theorem 2.1 of [4].

It is worth remarking that the equivalence (1)⇔(2) in Theorem 4.2 was obtained earlier in [15] (Theorem 4) in the case when G is an envelope of holomorphy, that is, when G is holomorphically convex.

5. ARS of exponentials in spaces of germs. Let K be a compact set in $\mathbb{C}^p$, $p \geq 1$. Denote by $H(K)$ the space of analytic germs on K, equipped with the inductive topology of Grothendieck–Martineau [22], [10, Ch. XI]. The space $H(K)$ is an LN*-space in the sense of Sebastião e Silva [10, Ch. XI], [26]. Let us write $H(K) = H(\operatorname{conv} K)$ if each germ from $H(K)$ admits a one-to-one analytic continuation to a germ from $H(\operatorname{conv} K)$.

THEOREM 5.1. *If K is an arbitrary compact set in $\mathbb{C}^p$, $p \geq 1$, then the following assertions are equivalent:*

(1) $H(K) = H(\operatorname{conv} K)$;

(2) *$H(K)$ has at least one ARS of exponentials;*

(3) *there exists an ARS of exponentials $\mathcal{E}_\Lambda := \{\exp\langle\lambda_k, z\rangle\}_{k=1}^\infty$ such that $\lim_{k\to\infty}|\lambda_k|_p = \infty$ and $\mathcal{E}_\Lambda$ is an ARS in $H(aK)$ for all $0 < a < \infty$.*

The proof of this theorem employs a nearly evident result described in [13, Ch. I, the beginning of points 3 and 4], as well as in [11, §1, point 2]: if $\{x_k\}_{k=1}^\infty$ is an RS or an ARS in a CLCS H_1 and if L is an epimorphism of H_1 onto a CLCS H_2, then $\{Lx_k\}_{k=1}^\infty$ is an RS (respectively, an ARS) in H_2. In particular, if L is an epimorphism of a CLCS H and $X = (x_k)_{k=1}^\infty$ is some sequence of eigenelements of L which is an RS or an ARS in H, then the sequence $X^L = X \setminus L^{-1}(0)$ is also a RS (or an ARS) in H. These simple arguments enable us to obtain the following useful auxiliary result which was strengthened for the case $p = 1$ in [12, Theorem 2] and is described in [16] for the general case $p \geq 1$.

LEMMA 5.2. *Let G be an arbitrary convex domain in $\mathbb{C}^p$, $p \geq 1$. Suppose that $\mathcal{E}_\Lambda$ is an ARS of exponentials in $H(G)$, and denote by $\mathcal{E}'_\Lambda$ the sequence obtained by removing from $\mathcal{E}_\Lambda$ any finite number of terms. Then $\mathcal{E}'_\Lambda$ is also an ARS in $H(G)$.*

Proof. Let $\lambda_{n_s} = (\lambda_{n_s,1}, \ldots, \lambda_{n_s,p})$, $s = 1, \ldots, N$, be the exponents of the exponentials removed from $\mathcal{E}_\Lambda$. Denote by $\mathcal{P}(D)$ the linear differential operator of finite order with constant coefficients and with characteristic polynomial

$$\mathcal{P}(z) = \prod_{s=1}^{N} \prod_{j=1}^{p} (z_j - \lambda_{n_s,j}).$$

According to Theorem A of [5, Ch. V, 5.17.1], $\mathcal{P}(d)$ is an epimorphism of $H(G)$. Moreover, for all $\lambda \in \mathbb{C}^p$ and $z \in \mathbb{C}^p$,

$$\mathcal{P}(D)(\exp\langle \lambda, z\rangle) = \mathcal{P}(\lambda) \cdot \exp\langle \lambda, z\rangle.$$

So $\exp\langle \lambda_{n_s,j}, z\rangle \in \mathcal{P}(D)^{-1}(0)$ and it remains to employ the general result cited above.

Proof of Theorem 5.1. (2)⇒(1). Let $\mathcal{E}_\Lambda$ be an ARS in $H(K)$ and let F be an arbitrary germ from $H(K)$. If f is any representative of this germ then f can be represented in the form of a series

$$f(z) = \sum_{k=1}^{\infty} f_k \exp\langle \lambda_k, z\rangle \tag{5.1}$$

converging absolutely in $H(G)$ where G is some domain containing K. The series on the right-hand side of (5.1) converges absolutely in G. Hence it converges absolutely in $\operatorname{conv} G$, uniformly on each compactum in $\operatorname{conv} G$, and realizes a single-valued analytic extension of f onto $\operatorname{conv} G$. It is clear that F admits a one-to-one analytic continuation to some germ from $H(\operatorname{conv} K)$.

(1)⇒(3). Without loss of generality one may now assume that K is a convex compactum in $\mathbb{C}^p$. It is always possible to construct a sequence $\{G_n\}_{n=1}^{\infty}$ of bounded convex domains such that for all $n \geq 1$, $\overline{G}_{n+1} \subset G_n$, $K = \bigcap_{n=1}^{\infty} G_n$. Theorem 4.2 provides for each $n \geq 1$ the existence of a sequence $\mathcal{E}_{\Lambda_{(n)}} = \{\exp\langle \lambda_{k,n}, z\rangle\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} |\lambda_{k,n}|_p = \infty$ and $\mathcal{E}_{\Lambda_{(n)}}$ is an ARS in $H(aG_n)$ for all $0 < a < \infty$. Denote by $\mathcal{E}'_{\Lambda_{(n)}}$ the sequence obtained by removing from $\mathcal{E}_{\Lambda_{(n)}}$ all its exponentials with exponents $\lambda_{k,n}$ such that $|\lambda_{k,n}|_p \leq n$. Due to Lemma 5.2, $\mathcal{E}'_{\Lambda_{(n)}}$ is an ARS in $H(aG_n)$ for $0 < a < \infty$. Let us enumerate all functions from $\bigcup_{n=1}^{\infty} \mathcal{E}'_{\Lambda_{(n)}}$ in the form of one sequence $\mathcal{E}_\mu$ in such a manner that $|\lambda_{k,n}|_p$ does not decrease. It is easy to see that $\mathcal{E}_\mu = \{\exp\langle \mu_l, z\rangle\}_{l=1}^{\infty}$ is an ARS in $H(aK)$ for $0 < a < \infty$, and $\lim_{l\to\infty} |\mu_l|_p = \infty$.

Theorem 5.1 is similar to Theorem 4.2 of [4]. However, the proof of the latter is not quite correct by the same reason as the proof of Theorem 4.1 of [4].

6. Concluding remarks. 1. Theorem 2.2 enables us to establish the falseness of Theorem 2.1 of [4]. At the same time it has an independent significance, since it asserts the existence of an ARS $\mathcal{E}_p$ of exponentials with imaginary exponents in the space $C^\infty(K)$ for an arbitrary W.c. K in $\mathbb{R}^p$, $p \geq 1$, and, in particular, for every convex compact set in $\mathbb{R}^p$. Analyzing the proof of Theorem 2.2 it is not difficult to notice that each $f \in C^\infty(K)$ can be represented in the form of a series (2.5) absolutely converging in $C^\infty(K)$ with effectively determined coefficients $\widetilde{h}_n$. In other words, the representation (2.5) can be made explicit. The existence of such an ARS of exponentials in $C^\infty(K)$ enables, in particular, an explicit construction of a particular solution of the equation $\mathcal{P}(D)y = g(x)$ for each $g \in C^\infty(K)$ with the help of the method described in [11, §9], as well as of a partial solution of the Cauchy problem for the same equation using the method of exponential representation of a solution [16, 17, 20]. The corresponding results will be described elsewhere.

2. An analogue of Theorem 2.2 can be obtained for other function spaces, in particular, for the Beurling space $\mathcal{E}_{(\omega)}(K)$ [23]. In this way new examples disproving Theorem 2.1 of [4] can also be constructed.

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Weighted Hardy inequalities and Hardy transforms of weights

by

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Abstract. Many problems in analysis are described as weighted norm inequalities that have given rise to different classes of weights, such as A_p-weights of Muckenhoupt and B_p-weights of Ariño and Muckenhoupt. Our purpose is to show that different classes of weights are related by means of composition with classical transforms. A typical example is the family M_p of weights w for which the Hardy transform is $L_p(w)$-bounded. A B_p-weight is precisely one for which its Hardy transform is in M_p, and also a weight whose indefinite integral is in A_{p+1}.

1. Introduction. If w is a weight on $\mathbb{R}^+ = [0,\infty)$, we define $W(t) = \int_0^t w(x)\,dx$, and $T : X \to Y$ indicates that T is a bounded operator between X and Y, two function spaces on $\mathbb{R}^+$. X^{d} will denote the subset of all nonincreasing and nonnegative functions (briefly, decreasing functions) of X.

We recall that A_p, for $p > 1$, is defined by the condition

$$(A_p) \qquad \sup_I \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the suppremum is taken over all intervals I and, if $p = 1$, by $Mw \le Cw$. Here M is the Hardy–Littlewood maximal function and it is well known (see [Mu1]) that $w \in A_p$ if and only if $M : L_p(w) \to L_p(w)$ $(1 < p < \infty)$.

In [Mu2], the weights w such that $S_1 f(t) = (1/t)\int_0^t f(x)\,dx$ (the Hardy operator) is bounded on $L_p(w)$ $(1 \le p < \infty)$ are described as the weights of class M_p, defined for $1 < p < \infty$ by the estimate

$$(M_p) \qquad \sup_{t>0}\left(\int_t^\infty \frac{w(x)}{x^p}\,dx\right)^{1/p}\left(\int_0^t w(x)^{-p'/p}\,dx\right)^{1/p'} < \infty.$$

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