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Invariant operators and pluriharmonic functions on symmetric irreducible Siegel domains

by

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Abstract. Let $\mathcal{D}$ be a symmetric irreducible Siegel domain. Pluriharmonic functions satisfying a certain rather weak growth condition are characterized by $r+2$ operators ($r+1$ in the tube case), r being the rank of the underlying symmetric cone.

0. Introduction. Let $\mathcal{D}$ be a symmetric Siegel domain. There exists a solvable Lie group S which acts simply transitively as a group of biholomorphisms on $\mathcal{D}$. In [DHP] and [DHMP] we studied the class of S-invariant real elliptic degenerate second order operators on $\mathcal{D}$ which annihilate holomorphic functions and, consequently, their real and imaginary parts: the pluriharmonic functions. Such operators will be called *admissible*. A well known example of an operator in this class is the Laplace–Beltrami operator corresponding to the Bergman metric.

Our particular interest in second order, degenerate elliptic operators is caused by the fact that for such an operator there is a very well understood potential theory. Indeed, the theory of harmonic functions with respect to an S-invariant operator satisfying the Hörmander condition was studied in [DH], [DHP]. The origin of this research goes back to H. Furstenberg, Y. Guivarc'h and A. Raugi who developed a probabilistic approach to harmonic functions on groups. We adapted their methods to left-invariant operators on certain solvable groups (a generalization of the NA subgroups in the Iwasawa decomposition) ([D], [DH]). Together with R. Penney we applied these methods to groups acting on Siegel domains [DHP]. The basic result of the theory is the description of bounded L-harmonic functions as Poisson integrals on a certain nilpotent subgroup $N(L)$ of S.

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For an admissible L on a Siegel domain the boundary $N(L)$ always contains a group $N(\Phi)$ which acts simply transitively on the Shilov boundary. Also, there is a simple algebraic description of operators L for which $N(L) = N(\Phi)$. The most important consequence of all these facts is the existence of a number of real (Poisson) kernels on the Shilov boundary (similar to the Poisson–Szegő kernel) which reproduce bounded pluriharmonic functions from their boundary values. However, this class cannot be characterized as the space of zeros of a *single admissible operator*, except for the easiest example of the upper half-plane. The operator exhibited by Forelli [F] many years ago does not give rise to any interesting potential theory.

Having all that in mind we may very well ask the following questions:

1. Can pluriharmonic functions be described by systems of admissible operators and, more generally, is there any reasonable description of the zeros of such a system?

2. What happens if we impose growth conditions on functions, other than boundedness?

In this paper we study the zeros of a particular system of admissible operators on symmetric irreducible Siegel domains with a particular growth condition.

The group $S = N(\Phi)S_0$ is a semidirect product of a step two nilpotent Lie group $N(\Phi)$ and a group S_0. The action of $N(\Phi)$ on $\mathcal{D}$ extends to a simply transitive action on the Bergman–Shilov boundary $\partial\mathcal{D}$ of $\mathcal{D}$. The group S_0 is a linear triangular group which acts simply transitively on the corresponding symmetric cone Ω. Identifying functions on $\mathcal{D}$ with those on $S = N(\Phi)S_0$ we consider three types of growth conditions:

$$(H^2) \qquad \sup_{s\in S_0} \int_{N(\Phi)} |F((\zeta,x)s)|^2\, d\zeta\, dx < \infty,$$

$$(H_0^2) \qquad \int_K \int_{N(\Phi)} |F((\zeta,x)s)|^2\, d\zeta\, dx\, dm_{\mathrm{R}}(s) < \infty$$

for every compact $K \subset S_0$ (where m_{R} is the right Haar measure), and

$$(H_\psi^2) \qquad \int_\Omega \int_{N(\Phi)} |F((\zeta,x)\cdot iu)|^2 \psi(u)\, d\zeta\, dx\, du < \infty$$

for a positive continuous function ψ on $\Omega = S_0 \cdot e$.

The admissible operators are made of basic building blocks Δ_{kl} (see [DHMP]) that are the unique S-invariant operators such that in complex coordinates $z_1, \ldots, z_m$ around a fixed point p, we have

$$(0.1) \qquad \Delta_{jk} f(p) = \partial_{z_j} \partial_{\overline{z}_k} f(p).$$

Every admissible L is of the form

$$L = \sum_{j,k=1}^{m} a_{jk} \Delta_{jk}$$

with $a_{jk} = \overline{a}_{jk}$.

When shifted to the group S, the operators Δ_{jk} are expressed in terms of the Lie algebra $\mathcal{S}$ of S (see Theorem (1.18)) and the role played by various Δ_{jk} is by no means the same. Making use of some algebra, we were able to prove that given an elliptic admissible L on an irreducible domain there are two more operators $\mathcal{L}, \mathbf{H}$ (or one more for the tube case) such that if F satisfies (H^2) and $LF = \mathcal{L}F = \mathbf{H}F = 0$ then F is the real part of a holomorphic (H^2) function [DHMP]. In the proof we heavily exploited the fact that functions F annihilated by an elliptic admissible operator L which satisfy (H^2) are integrals against the Poisson kernel P^L of an L^2 function on the boundary ([DH], [DHP]).

In this paper we study the L-harmonic functions satisfying (H_0^2). This seems to be a natural generalization of (H^2), because, on the one hand, (H_0^2) fits in well with the use of the partial group Fourier transform along $N(\Phi)$, and on the other hand, the situation becomes sufficiently complicated to be interesting. In particular, the Poisson kernel P^L is of no use here. The main motivation to look at (H_0^2), (H_ψ^2) comes from [RV], where such spaces are used to study holomorphic discrete series of the corresponding semisimple Lie group. It seems to be an interesting problem which (preferably very small) systems of admissible operators are needed to yield pluriharmonicity provided (H_0^2), (H_ψ^2) hold, and what the role of various Δ_{jk} is in that.

It turns out that for tube domains we can find $r+1$ admissible operators $\Delta_1, \ldots, \Delta_r, L_0$ such that if $\Delta_1 F = \ldots = \Delta_r F = L_0 F = 0$ and F satisfies (H_0^2), then F is pluriharmonic, but its conjugate function does not have to satisfy (H_0^2) (Theorem (4.14)) unlike for the (H^2) condition. The operators $\Delta_1, \ldots, \Delta_r$ are single basic building blocks (0.1) closely related to the r-dimensional Abelian group acting on the nilradical of S (see (1.19) and (3.2)). It seems that they play the same role as $\mathbf{H} = \Delta_1 + \ldots + \Delta_r$ and the Poisson kernel did together for the (H^2) case. This role is not completely clear to the authors and should be further investigated.

In general, the condition (H_0^2) does not suffice to prove that the partial Fourier transform of F must be supported by $\Omega \cup -\Omega$, but (H_ψ^2) with appropriate ψ does. Moreover, the function conjugate to F then satisfies (H_ψ^2) as well (Theorem (4.24)).

For a Siegel domain of type two we consider only condition (H_ψ^2) and we prove that there are $r+2$ admissible operators $\Delta_1, \ldots, \Delta_r, L_0, \mathcal{L}$ such that if $\Delta_1 F = \ldots = \Delta_r F = L_0 F = \mathcal{L}F = 0$ and F satisfies (H_ψ^2) then it is the real part of a holomorphic function satisfying (H_ψ^2) (Theorem (5.3)).

The conditions we consider here are of different nature from those studied earlier in order to characterize pluriharmonic functions. Namely, we do not assume anything about the behaviour of the functions at the Shilov boundary. All the assumptions made previously ([L1], [L2], [BBG], [DHMP]) included at least existence of boundary values and in [L1], [L2], [BBG] some more conditions were imposed.

It is worth mentioning that the simplifying role of the Poisson kernel was not well understood at the very beginning of the work [DHMP]. In the present paper we use some ideas on which these first calculations were based. Some other inspirations came during the visits of the first author at Kiel, Nancy and Metz. She is grateful to Jean Philippe Anker, Didier Arnal, Jean Louis Clerc, Jean Ludwig and Detlef Müller for their warm hospitality.

1. Preliminaries. Assume that V is an algebra and a Euclidean space with scalar product $\langle\cdot,\cdot\rangle$. If for all elements x, y and z in V we have

$$xy = yx, \quad x(x^2y) = x^2(xy), \quad \langle xy, z\rangle = \langle y, zx\rangle,$$

then V is a Euclidean Jordan algebra. Every such V is in a unique way a direct sum of simple Euclidean algebras. In this paper V is always simple. Let e be the unit element of V. Let

$$\Omega = \operatorname{int}\{x^2 : x \in V\}$$

be the associated symmetric cone ([FK], Theorem III.2.1). Every symmetric cone in a Euclidean vector space is of this form ([FK], Theorem III.3.1).

The most representative exemple is the space of symmetric $r\times r$ matrices, the cone being the set of positive definite matrices.

We start with some facts about Jordan algebras which we need later. For further material we refer to [FK]. We fix a *Jordan frame* $\{c_1, \ldots, c_r\}$, i.e. a system of orthogonal idempotents

$$c_i^2 = c_i, \quad c_ic_j = 0 \quad \text{if } i \neq j,$$

which is *complete*, i.e. $c_1 + \ldots + c_r = e$ and none of $c_1, \ldots, c_r$ can be written as the sum of two non-zero idempotents. All the Jordan frames in V have the same length r called the *rank* of V.

Let

$$V = \bigoplus_{1\le i\le j\le r} V_{ij} \tag{1.1}$$

be the *Peirce decomposition* of V ([FK], Theorem IV.2.1). This means that V is the orthogonal direct sum (1.1), and furthermore the following properties hold:

$$\begin{aligned} &V_{ij}\cdot V_{ij} \subset V_{ii} + V_{jj},\\ &V_{ij}\cdot V_{jk} \subset V_{ik} \quad \text{if } i \neq k,\\ &V_{jk}\cdot V_{jl} \subset V_{kl} \quad \text{if } k < l,\\ &V_{ij}\cdot V_{kl} = \{0\} \quad \text{if } \{i,j\}\cap\{k,l\} = \emptyset. \end{aligned} \tag{1.2}$$

Moreover, all the spaces V_{ij}, $i < j$, have the same dimension and for each $j = 1, \ldots, r$ we have $V_{jj} = \mathbb{R}c_j$.

Let G be the connected component of the group $G(\Omega)$ of all transformations in $\mathrm{GL}(V)$ which leave Ω invariant. Its Lie algebra will be denoted by $\mathcal{G}$. An element $X \in \mathcal{G}$ acts on V in the usual way: $Xx = \frac{d}{dt}\exp tX\cdot x|_{t=0}$. It has a convenient description in terms of the Jordan algebra V. The linear transformation of V given by multiplication by x is denoted by $L(x)$, i.e.

$$L(x)y = xy.$$

For every $x \in V$, we have $L(x) \in \mathcal{G}$. The elements $c_1, \ldots, c_r$ provide a simultaneous diagonalization of the Abelian subalgebra $\mathcal{A}$ of $\mathcal{G}$ consisting of elements

$$H = L(a), \quad \text{where} \quad a = \sum_{j=1}^r a_jc_j \in \bigoplus_i V_{ii}.$$

Namely

$$Hx = L(a)x = \lambda_{ij}(H)x, \quad x \in V_{ij}, \tag{1.3}$$

with $\lambda_{ij}(H) = \frac{1}{2}(a_i + a_j)$. For $H = L(a) \in \mathcal{A}$ and $i < j$ we define

$$\alpha_{ij}(H) = \tfrac{1}{2}(a_j - a_i), \quad \mathcal{N}_{ij} = \{X \in \mathcal{G} : [H, X] = \alpha_{ij}(H)X,\ \forall_{H\in\mathcal{A}}\}.$$

Then, for $i < j$,

$$\mathcal{N}_{ij} = \{z \,\square\, c_i : z \in V_{ij}\}, \quad \mathcal{N}_{ji} = \{z \,\square\, c_j : z \in V_{ij}\},$$

where $z \,\square\, c_i := \frac{1}{2}L(z) + [L(z), L(c_i)]$ ([FK], Proposition VI.3.3). Moreover,

$$(z \,\square\, c_i)^* = z \,\square\, c_j \quad \text{for } z \in V_{ij}. \tag{1.4}$$

Let

$$\mathcal{S}_0 = \bigoplus_{i<j} \mathcal{N}_{ij} \oplus \mathcal{A} =: \mathcal{N} \oplus \mathcal{A}. \tag{1.5}$$

Then $\mathcal{S}_0$ is a subalgebra of $\mathcal{G}$. There is an orthonormal basis of V corresponding to the Peirce decomposition such that in this basis the elements of $\mathcal{A}$ are diagonal and the elements of $\mathcal{N}$ are upper triangular with zeros on the diagonal. We also consider

$$\mathcal{S}_0^* = \bigoplus_{i<j} \mathcal{N}_{ji} \oplus \mathcal{A},$$

which consists of the adjoints of elements in $\mathcal{S}_0$ with respect to $\langle\cdot,\cdot\rangle$. The group $S_0 = \exp \mathcal{S}_0$ acts simply transitively on Ω ([FK], Chap. VI).

Let $V^{\mathbb{C}} = V + iV$ be the complexification of V. We extend the action of $G(\Omega)$ to $V^{\mathbb{C}}$.

In addition to $V^{\mathbb{C}}$ suppose that we are given a complex vector space $\mathcal{Z}$. Let $\Phi : \mathcal{Z}\times\mathcal{Z} \to V^{\mathbb{C}}$ be a Hermitian symmetric bilinear mapping. We assume that

$$\Phi(\zeta,\zeta) \in \overline{\Omega},\quad \zeta\in\mathcal{Z},\qquad \Phi(\zeta,\zeta)=0 \quad\text{implies}\quad \zeta = 0.$$

The *Siegel domain* associated with these data is defined as

$$\mathcal{D} = \{(\zeta,z)\in\mathcal{Z}\times V^{\mathbb{C}} : \Im z - \Phi(\zeta,\zeta)\in\Omega\}.$$

There is an algebraic representation $\sigma : G(\Omega)\ni g\mapsto \sigma(g)\in \mathrm{GL}(\mathcal{Z})$ such that

$$g\Phi(\zeta,\omega) = \Phi(\sigma(g)\zeta,\sigma(g)\omega), \tag{1.6}$$

and the transformation $(\zeta,z)\mapsto(\sigma(g)\zeta, gz)$ is a biholomorphic automorphism of $\mathcal{D}$ (see [KW]). The elements $\zeta\in\mathcal{Z}$, $x\in V$ and $g\in G(\Omega)$ act on $\mathcal{D}$ in the following way:

$$\begin{aligned}\zeta\cdot(\omega,z) &= (\zeta+\omega, z+2i\Phi(\omega,\zeta)+i\Phi(\zeta,\zeta)),\\ x\cdot(\omega,z) &= (\omega, z+x),\\ g\cdot(\omega,z) &= (\sigma(g)\omega, gz).\end{aligned} \tag{1.7}$$

The first two actions generate a two-step nilpotent (or Abelian if $\mathcal{Z}=0$) group $N(\Phi)$ of biholomorphic automorphisms of $\mathcal{D}$:

$$(\zeta,x)(\omega,y) = (\zeta+\omega, x+y+2\Im\Phi(\zeta,\omega)). \tag{1.8}$$

All three actions (g restricted to S_0) generate a solvable Lie group $S = N(\Phi)S_0$, the group $N(\Phi)$ being a normal subgroup of S. Since the representation σ is algebraic, the action of $\sigma(\mathcal{A})$ is diagonalizable over $\mathbb{R}$,

$$\mathcal{Z} = \bigoplus_{j=1}^{r}\mathcal{Z}_j \quad\text{with}\quad \sigma(H)\zeta = \frac{\lambda_j(H)}{2}\zeta,\ \zeta\in\mathcal{Z}_j, \tag{1.9}$$

where $\lambda_1,\dots,\lambda_r$ is the dual basis to $c_1,\dots,c_r$. Moreover, all the spaces $\mathcal{Z}_j$ have the same dimension. (The standard proof of (1.9) is e.g. in [DHMP].)

Therefore, the Lie algebra $\mathcal{S}$ of S has the decomposition

$$\mathcal{S} = \mathcal{N}(\Phi)\oplus\mathcal{S}_0 = \Big(\bigoplus_{j=1}^{r}\mathcal{Z}_j\Big)\oplus\Big(\bigoplus_{i\le j}V_{ij}\Big)\oplus\Big(\bigoplus_{i<j}\mathcal{N}_{ij}\Big)\oplus\mathcal{A}.$$

The adjoint action of $\mathcal{A}$ preserves all the subspaces $\mathcal{Z}_j$, V_{ij}, $\mathcal{N}_{ij}$. More

precisely, if $H\in\mathcal{A}$, then

$$\begin{aligned}[H,X] &= \frac{\lambda_j(H)}{2}X && \text{for } X\in\mathcal{Z}_j,\\ [H,X] &= \frac{\lambda_i(H)+\lambda_j(H)}{2}X && \text{for } X\in V_{ij},\\ [H,X] &= \frac{\lambda_j(H)-\lambda_i(H)}{2}X && \text{for } X\in\mathcal{N}_{ij}.\end{aligned} \tag{1.10}$$

Given $\lambda\in V$ let

$$H_\lambda(\zeta,\omega) := 4\langle\lambda,\Phi(\zeta,\omega)\rangle.$$

For $\lambda\in\Omega$ the Hermitian form H_λ is not degenerate. If $\lambda=\sum_{j=1}^r\lambda_jc_j$, the form H_λ decomposes nicely as

$$H_\lambda(\zeta,\omega) = \sum_{j=1}^{r} H_{\lambda_j}(\zeta_j,\omega_j), \tag{1.11}$$

where $\zeta=\zeta_1+\ldots+\zeta_r$, $\omega=\omega_1+\ldots+\omega_r$, $\omega_j,\zeta_j\in\mathcal{Z}_j$ (again see e.g. [DHMP]).

Now we are going to describe a suitable orthonormal basis of $\mathcal{G}$. Let $\{e_{ij}^\alpha\}$ be an orthonormal basis of V_{ij} corresponding to the Peirce decomposition with the identification $e_{ii}^\alpha = c_i$. Let

$$\begin{aligned}X_{ij}^\alpha &\in V_{ij}, && i\le j,\ 1\le\alpha\le\dim V_{ij},\\ Y_{ij}^\alpha &\in\mathcal{N}_{ij}, && i<j,\ 1\le\alpha\le\dim\mathcal{N}_{ij}=\dim V_{ij},\end{aligned}$$

be the left-invariant vector fields on S corresponding to e_{ij}^α and $2e_{ij}^\alpha\,\square\,c_i$, respectively. This means that we identify $X_{ij}^\alpha(e)$ with the vector $e_{ij}^\alpha\in V_{ij}$, $i\le j$ ($X_{jj}^\alpha = X_j$), and Y_{ij}^α, $i<j$, with the transformation $2e_{ij}^\alpha\,\square\,c_i\in\mathcal{N}_{ij}$ defined in (1.4). Analogously, let $\widetilde{X}_{ij}^\alpha$ be the left-invariant vector field on $N(\Phi)$ corresponding to e_{ij}^α. Let H_j be the left-invariant vector field corresponding to $L(c_j)$.

In $\mathcal{Z}$ we choose coordinates compatible with the decomposition (1.9). Moreover,

(1.12) let $e_{j\alpha}$, $\alpha=1,\dots,\dim\mathcal{Z}_j$, be a basis of $\mathcal{Z}_j$ such that $H_{c_j}(e_{j\alpha},e_{j\beta}) = \delta_{\alpha\beta}$,

i.e. for

$$\zeta = \sum_{\alpha=1}^{\dim\mathcal{Z}_j}\zeta_{j\alpha}e_{j\alpha},\qquad \omega=\sum_{\alpha=1}^{\dim\mathcal{Z}_j}\omega_{j\alpha}e_{j\alpha}$$

we have

$$H_{c_j}(\zeta,\omega) = \sum_\alpha \zeta_{j\alpha}\overline{\omega}_{j\alpha}. \tag{1.13}$$

Let $\zeta_{j\alpha} = x_{j\alpha} + iy_{j\alpha}$ and let $\mathcal{X}_j^\alpha, \mathcal{Y}_j^\alpha$ be the left-invariant vector fields on S corresponding to $\partial_{x_{j\alpha}}$ and $\partial_{y_{j\alpha}}$, respectively. As before, $\widetilde{\mathcal{X}}_j^\alpha, \widetilde{\mathcal{Y}}_j^\alpha$ are the left-invariant vector fields on $N(\Phi)$ corresponding to $\partial_{x_{j\alpha}}$ and $\partial_{y_{j\alpha}}$.

The basis

$$\mathcal{X}_j^\alpha,\ \mathcal{Y}_j^\alpha,\ X_{ij}^\alpha,\ Y_{ij}^\alpha,\ X_j,\ H_j \tag{1.14}$$

is orthonormal with respect to the Riemannian form g on S which is the image of the Bergman metric under the identification $S \ni s \mapsto s \cdot ie \in \mathcal{D}$. For the proof of (1.14) see e.g. [DHMP]. Also the complex structure $\mathcal{J}$ on S, transported from $\mathcal{D}$, is computed there in terms of the basis (1.14). It is given by

$$\begin{aligned} \mathcal{J}(X_j) &= H_j, & \mathcal{J}(X_{ij}^\alpha) &= Y_{ij}^\alpha, \\ \mathcal{J}(H_j) &= -X_j, & \mathcal{J}(Y_{ij}^\alpha) &= -X_{ij}^\alpha \end{aligned} \tag{1.15}$$

and

$$\mathcal{J}(\mathcal{X}_j^\alpha) = \mathcal{Y}_j^\alpha, \qquad \mathcal{J}(\mathcal{Y}_j^\alpha) = -\mathcal{X}_j^\alpha. \tag{1.16}$$

Let X be one of the vector fields $\mathcal{X}_j^\alpha$, X_{ij}^α, X_j, and let ∂_x be the partial derrivative corresponding to X at the unit element e of S, i.e. $\partial_x f(e) = Xf(e)$ (e denotes both the unit element of the group S and of the Jordan algebra V, but this does not lead to confusion). Extending $\partial_x + \mathcal{J}\partial_x$ to a left-invariant vector field we obtain

$$Z = X + i\mathcal{J}X. \tag{1.17}$$

Let ∇_X be the Riemannian connection corresponding to the biholomorphically invariant Riemannian metric on $\mathcal{D}$.

In order to describe pluriharmonic functions we are going to use the operators

$$\Delta_Z = Z\overline{Z} - \nabla_Z \overline{Z} = X^2 + (\mathcal{J}X)^2 - \nabla_X X - \nabla_{\mathcal{J}X}\mathcal{J}X.$$

Δ_Z is the unique left-invariant operator agreeing with $(\partial_x + \mathcal{J}\partial_x)(\partial_x - \mathcal{J}\partial_x)$ at e. In [DHMP] the following theorem is proved.

(1.18) THEOREM. *Let*

$$\Delta_j = \Delta_{X_j + iH_j}, \qquad \mathcal{L}_j^\alpha = \Delta_{\mathcal{X}_j^\alpha + i\mathcal{Y}_j^\alpha}, \qquad \Delta_{ij}^\alpha = \Delta_{X_{ij}^\alpha + iY_{ij}^\alpha}.$$

Then

$$\begin{aligned} \Delta_j &= X_j^2 + H_j^2 - H_j, \\ \mathcal{L}_j^\alpha &= (\mathcal{X}_j^\alpha)^2 + (\mathcal{Y}_j^\alpha)^2 - H_j, \\ \Delta_{ij}^\alpha &= (X_{ij}^\alpha)^2 + (Y_{ij}^\alpha)^2 - H_j. \ \blacksquare \end{aligned} \tag{1.19}$$

In what follows we will use the terms *cone operators* for Δ_j, Δ_{ij}^α and *boundary operators* for $\mathcal{L}_j^\alpha$.

Now we present some technical lemmas concerning Jordan algebras, which will be used later. For that we need a notion of determinant $\det x$ of $x \in V$. There are a number of equivalent descriptions of it (see Chapters II and III of [FK]). One possible definition is that $\det x$ is the determinant of $L(x)$ restricted to the subalgebra $\mathbb{R}[x]$ generated by the powers of x. (For $n \times n$ real symmetric matrices $\det x$ is the determinant of the matrix x.) $\det x \neq 0$ iff $\dim \mathbb{R}[x] = r$ iff x is invertible in the Jordan algebra V. The set $\{x : \det x \neq 0\}$ is open and dense in V.

We consider the subalgebras

$$P_k = \bigoplus_{i,j \le k} V_{ij}, \qquad P^k = \bigoplus_{i,j \ge k} V_{ij}.$$

Let $\pi_k : V \to P_k$ and $\pi^k : V \to P^k$ be the corresponding orthogonal projections and let

$$\Delta_k(x) = \det \pi_k x, \qquad \Delta^k(x) = \det \pi^k x.$$

Clearly, for the Jordan algebra of real symmetric $n \times n$ matrices, $\Delta_1(x), \ldots$ $\ldots, \Delta_r(x)$ are the principal minors of x.

The sets $J = \{x : \forall_{1 \le k \le r} \Delta_k(x) \neq 0\}$ and $J' = \{x : \forall_{1 \le k \le r} \Delta^k(x) \neq 0\}$ are open dense in V.

Given $z \in V_{1j} \oplus \ldots \oplus V_{j-1,j} \oplus V_{j,j+1} \oplus \ldots \oplus V_{jr}$, let

$$\tau(z) = \exp(2z \,\square\, c_j) \tag{1.20}$$

where, as before, $z \,\square\, c_j = \frac{1}{2}L(z) + [L(z), L(c_j)]$. Notice that by (1.4), $2z \,\square\, c_j \in \mathcal{N}$ for $z \in \bigoplus_{k=j+1}^r V_{jk}$, and $2z \,\square\, c_j \in \mathcal{N}^* = \bigoplus_{i<j} \mathcal{N}_{ji}$ for $z \in \bigoplus_{k=1}^{j-1} V_{kj}$.

(1.21) LEMMA [FK]. *For $x \in J$ there exist*

$$z^j \in \bigoplus_{k=j+1}^{r} V_{jk}, \qquad 1 \le j \le r-1,$$

and real numbers $a_1, \ldots, a_r \neq 0$ such that

$$x = \tau(z^1) \ldots \tau(z^{r-1}) \Big(\sum_{k=1}^{r} a_k c_k \Big),$$

the elements z^j and the numbers a_j being unique. Similarly, for $x \in J'$ there exist

$$z_j \in \bigoplus_{k=1}^{j-1} V_{kj}, \qquad 2 \le j \le r,$$

and real numbers $a'_1, \ldots, a'_r \neq 0$ such that

$$x = \tau(z_r) \ldots \tau(z_2) \Big(\sum_{k=1}^{r} a'_k c_k \Big),$$

the elements z_j and the numbers a_j being unique. Moreover, for every $y \in N = \exp \mathcal{N}$, the restriction $\mathrm{Ad}_y|_V$ is of the form

$$\mathrm{Ad}_y|_V = \tau(z^1)\ldots\tau(z^{r-1}), \quad \textit{where} \quad z^j \in \bigoplus_{k=j+1}^{r} V_{jk},\ 1 \le j \le r-1.$$

Proof. The first statement (for $x \in \Omega$) is Theorem VI.3.5 of [FK]. The generalization for $x \in J$ or $x \in J'$ is straightforward. The third statement is Theorem VI.3.6 in the same book. ∎

The third statement of Lemma (1.21) provides a convenient system of coordinates in the group $N = \exp \mathcal{N}$. Moreover, since $\mathcal{N}^*$ consists of adjoints of elements of $\mathcal{N}$ with respect to $\langle\cdot,\cdot\rangle$, the above lemma shows that every $x \in J'$ is of the form

$$x = \mathrm{Ad}_y^*\Big(\sum_{k=1}^{r} a_k c_k\Big)$$

for some $y \in N$ and non-zero numbers $a_1, \ldots, a_r$. We will also write x as

$$x = \mathrm{Ad}_y^* \,\mathrm{Ad}_a \Big(\sum_{k=1}^{r} \varepsilon_k c_k\Big) \tag{1.22}$$

with $\mathrm{Ad}_a = L(\sum_{k=1}^{r} |a_k| c_k)$, $\varepsilon_k = \operatorname{sgn} a_k$.

(1.23) Lemma [DHMP]. *Let y_{jl} be the V_{jl}-component of y in the Peirce decomposition* (1.1). *Then*

$$\Big\langle \sum_k a_k c_k, \mathrm{Ad}_y\, c_j \Big\rangle = a_j + \frac{1}{2}\sum_{l>j} a_l |y_{jl}|^2. \ ∎$$

2. Regularity of the partial Fourier transform along $N(\Phi)$. Let F be a function on S such that for every compact set $K \subset S_0$,

$$\int_K \int_{N(\Phi)} |F((\zeta, x)s)|^2 \, d\zeta\, dx\, ds < \infty. \tag{2.1}$$

In this chapter we concentrate on the partial Fourier transform of F along $N(\Phi)$. We recall that $N(\Phi)$ is step two nilpotent. If it is Abelian all goes through with obvious simplifications.

We start with some basic facts about Fourier analysis on $N(\Phi)$. All what we need has been elaborated in [OV] for the $N(\Phi)$ which arise in the theory of general Siegel domains of type II.

Let (,) be the Hermitian scalar product in which the basis (1.12) is orthonormal. We define a Hermitian transformation $M_\lambda : \mathcal{Z} \to \mathcal{Z}$ by

$$H_\lambda(\zeta, \omega) = (M_\lambda \zeta, \omega), \quad \zeta, \omega \in \mathcal{Z}, \tag{2.2}$$

and consider the set

$$\widetilde{\Lambda} = \{\lambda \in V : H_\lambda \text{ is not degenerate}\} = \{\lambda \in V : \det M_\lambda \neq 0\}.$$

Since for $\lambda \in \Omega$, $H_\lambda(\zeta,\zeta) > 0$, $\det M_\lambda$ is a non-zero polynomial of λ and so $\widetilde{\Lambda}$ is an open set of full measure. The set $\widetilde{\Lambda}$ carries the Plancherel measure (see [OV])

$$\varrho(\lambda) d\lambda = |\det M_\lambda| d\lambda. \tag{2.3}$$

For every $\lambda \in \Lambda$ we define a complex structure $\mathcal{J}_\lambda$ which corresponds to λ and determines the representation space $\mathcal{H}_\lambda$. Let $|M_\lambda|$ be the positive Hermitian transformation such that $|M_\lambda|^2 = M_\lambda^2$. Then

$$\mathcal{J}_\lambda = i|M_\lambda|^{-1} M_\lambda.$$

If $\lambda \in \Omega$, then $\mathcal{J}_\lambda = iI =: \mathcal{J}$, i.e. it coincides with the ordinary complex structure in $\mathcal{Z}$. Now, $\mathcal{J}_\lambda$ has a description in an appropriate basis. Namely, there is a λ-measurable choice of a basis $e_1^\lambda, \ldots, e_m^\lambda$, orthonormal with respect to $(\cdot,\cdot)$ such that

$$H_\lambda(e_j^\lambda, e_k^\lambda) = \sigma_j \delta_{jk}$$

with $\sigma_j = \pm 1$. In the basis $e_1^\lambda, \ldots, e_m^\lambda, \mathcal{J}e_1^\lambda, \ldots, \mathcal{J}e_m^\lambda$ of $\mathcal{Z}$ (over $\mathbb{R}$) we have

$$\mathcal{J}_\lambda(e_j^\lambda) = \sigma_j \mathcal{J} e_j^\lambda \quad \text{and} \quad \mathcal{J}_\lambda(\mathcal{J}e_j^\lambda) = -\sigma_j e_j^\lambda.$$

Let $B_\lambda = \Im H_\lambda$. A direct calculation shows that

$$B_\lambda(\mathcal{J}_\lambda e_j^\lambda, e_k^\lambda) = \delta_{jk}$$

and so

$$B_\lambda(\mathcal{J}_\lambda \zeta, \zeta) > 0 \quad \text{if } \zeta \neq 0. \tag{2.4}$$

Now we are ready to define a version of a unitary irreducible representation U^λ (the *Fock representation*) associated with $\lambda \in \Lambda$. Let $\mathcal{H}_\lambda$ be the set of all C^∞ functions F on $\mathcal{Z}$ which are holomorphic with respect to the complex structure $\mathcal{J}_\lambda$ such that

$$F(\cdot)\varrho(\lambda)^{1/2} e^{-(\pi/2)B_\lambda(\mathcal{J}_\lambda\cdot,\cdot)} \in L^2(\mathcal{Z}, dz),$$

where $dz = dx_1 dy_1 \ldots dx_m dy_m$ and the coordinates $z_j = x_j + iy_j$ are defined with respect to the basis (1.12). In the basis $e_1^\lambda, \ldots, e_m^\lambda$ we write ζ as

$$\zeta = \sum \zeta_j e_j^\lambda.$$

Then, by construction of the basis e_j^λ, $d\zeta = \varrho(\lambda) dz$, the appropriate scalar product in $\mathcal{H}_\lambda$ and the representation U_λ are defined by

$$(F_1, F_2)_\lambda = \int_{\mathcal{Z}} F_1(\zeta) \overline{F}_2(\zeta) e^{-\pi B_\lambda(\mathcal{J}_\lambda \zeta, \zeta)}\, d\zeta$$

and

$$U^\lambda(\zeta, x) F(\omega) = e^{-2\pi i\langle\lambda, x\rangle - (\pi/2)|\zeta|^2 + \pi\omega\overline{\zeta}} F(\omega - \zeta) \tag{2.5}$$

with $\omega\overline{\zeta} = B_\lambda(\mathcal{J}_\lambda\omega,\zeta) + iB_\lambda(\omega,\zeta)$, $|\zeta|^2 = \zeta\overline{\zeta}$. The orthonormal basis of $\mathcal{H}_\lambda$ which changes measurably with λ is as follows.

Given a multi-index $\alpha = (\alpha_1,\ldots,\alpha_m)$ let

$$\xi_\alpha^\lambda = \frac{\pi^{|\alpha|/2}}{\sqrt{\alpha!}}\prod_j \widetilde{\zeta}_j^{\alpha},$$

where $\widetilde{\zeta}_j = \zeta_j$ if $\sigma_j = 1$ and $\widetilde{\zeta}_j = \overline{\zeta}_j$ if $\sigma_j = -1$, $\alpha! = \alpha_1!\ldots\alpha_m!$, $|\alpha| = \sum_{j=1}^m \alpha_j$. Then every ξ_α^λ is holomorphic with respect to the complex structure $\mathcal{J}_\lambda$ and the family $\{\xi_\alpha^\lambda\}$ forms a $(\ ,\ )_\lambda$-orthonormal basis. Indeed,

$$\begin{aligned}(\xi_\alpha^\lambda,\xi_\beta^\lambda)_\lambda &= \int_Z \xi_\alpha^\lambda\overline{\xi}_\beta^\lambda e^{-\pi B_\lambda(\mathcal{J}_\lambda\zeta,\zeta)}\,d\zeta\\ &= \frac{\pi^{(|\alpha|+|\beta|)/2}}{\sqrt{\alpha!\beta!}}\int_Z\prod_j \widetilde{\zeta}_j^{\alpha_j}\overline{\widetilde{\zeta}}_j^{\beta_j}e^{-\pi\sum_j|\zeta_j|^2}\,d\zeta.\end{aligned}$$

Given $s\in S$ let

$$F_s(\zeta,x) = F((\zeta,x)s).$$

Since $F_s\in L^2(N(\Phi))$ (see (2.13)), $U_{F_s}^\lambda$ is defined for almost every λ and it is a Hilbert–Schmidt operator. Let

$$\widehat{F}(\lambda,\alpha,\beta,s) = (U_{F_s}^\lambda\xi_{\alpha\lambda},\xi_{\beta\lambda}). \tag{2.6}$$

By the Plancherel formula we have

$$\int_V|\widehat{F}(\lambda,\alpha,\beta,s)|^2\varrho(\lambda)\,d\lambda \le \int_V\|U_{F_s}^\lambda\|_{\mathrm{HS}}^2\varrho(\lambda)\,d\lambda = \|F_s\|_{L^2}^2. \tag{2.7}$$

We see that $\widehat{F}(\lambda,\alpha,\beta,s)$ is a measurable function of both variables λ, s and by (2.7) we have an analog of (2.1) for the Fourier transform, i.e.

$$\int_K\int_V|\widehat{F}(\lambda,\alpha,\beta,s)|^2\varrho(\lambda)\,d\lambda\,ds<\infty \quad \text{for every compact } K\subset S_0. \tag{2.8}$$

We are going to prove that for almost every λ, $\widehat{F}(\lambda,\alpha,\beta,s)$ is smooth as a function of s. Then we look at the differential operators (1.19) on the Fourier transform side.

Let $Y_1,\ldots,Y_n$ be a basis of $\mathcal{S}_0$, and let $I=(I_1,\ldots,I_n)$ be a multi-index and $Y^I = Y_1^{I_1}\ldots Y_n^{I_n}$. Notice that for almost every λ, and every α,β, $Y^I\widehat{F}(\lambda,\alpha,\beta,s)$ has a well defined meaning as a distribution on S_0. For smoothness of $Y^I\widehat{F}(\lambda,\alpha,\beta,s)$ we start by proving that the derivatives of F with respect to s exist in $L^2(N(\Phi))$. Let

$$\|F(\cdot,s)\|_{L^2} = \int_{N(\Phi)}|F((\zeta,x)s)|^2\,d\zeta\,dx.$$

(2.9) Lemma. *Let F be harmonic with respect to a real, elliptic, second order, left-invariant operator on S and suppose F satisfies condition* (2.1). *Then*

$$\lim_{t\to0}\left\|\frac{Y^IF(\cdot,s\exp tY)-Y^IF(\cdot,s)}{t}-YY^IF(\cdot,s)\right\|_{L^2}=0, \tag{2.10}$$

and for every compact $K\subset S$,

$$\lim_{t\to0}\int_K\left\|\frac{Y^IF(\cdot,s\exp tY)-Y^IF(\cdot,s)}{t}-YY^IF(\cdot,s)\right\|_{L^2}ds=0, \tag{2.11}$$

where Y is one of $Y_1,\ldots,Y_n$.

Proof. Since

$$\frac{Y^IF((\zeta,x)s\exp tY)-Y^IF((\zeta,x)s)}{t}=\frac1t\int_0^t YY^IF((\zeta,x)s\exp rY)\,dr,$$

it suffices to estimate the L^2-norm of

$$\begin{aligned}&\frac1t\int_0^t|YY^IF((\zeta,x)s\exp rY)-YY^IF((\zeta,x)s)|\,dr\\ &\qquad=\frac1t\int_0^t\left|\int_0^r Y^2Y^IF((\zeta,x)s\exp uY)\,du\right|dr.\end{aligned} \tag{2.12}$$

Let B be a ball around e in S. By the "left-invariant" Harnack inequality, for t sufficiently small, we have

$$|Y^2Y^IF((\zeta,x)s\exp uY)|\le c\int_B|F((\zeta,x)s(\zeta_1,x_1)w)|\,d\zeta_1\,dx_1\,dw. \tag{2.13}$$

Therefore,

$$\begin{aligned}&\|Y^2Y^IF(\cdot,s\exp uY)\|_{L^2}^2\\ &\quad\le cr^2\int_{N(\Phi)}\Big(\int_B|F((\zeta,x)s(\zeta_1,x_1)w)|\,d\zeta_1\,dx_1\,dw\Big)^2\,d\zeta\,dx\\ &\quad\le cr^2|B|\int_{N(\Phi)}\int_B|F((\zeta,x)s(\zeta_1,x_1)w)|^2\,d\zeta_1\,dx_1\,dw\,d\zeta\,dx\\ &\quad\le cr^2|B|\int_B\int_{N(\Phi)}|F((\zeta,x)sw)|^2\,d\zeta\,dx\,dw\,d\zeta_1\,dx_1.\end{aligned}$$

If s belongs to a compact set K, the integral on the right side is bounded by a constant depending on K. Hence

$$\|Y^2Y^IF(\cdot,s\exp uY)\|_{L^2}\le c(K)r$$

and so (2.10) and (2.11) are proved.

(2.14) LEMMA. *Let F be as in Lemma (2.9). For every multi-index I, every α,β and every $s\in S_0$,*

$$Y^I\widehat{F}(\lambda,\alpha,\beta,s)=(U^\lambda_{(Y^IF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda \tag{2.15}$$

for almost every λ. Moreover, if Y is one of $Y_1,\dots,Y_n$ then

$$\lim_{t\to 0}\int_V\left|\frac{Y^I\widehat{F}(\lambda,\alpha,\beta,s\exp tY)-Y^I\widehat{F}(\lambda,\alpha,\beta,s)}{t}-(U^\lambda_{(YY^IF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda\right|^2\varrho(\lambda)\,d\lambda=0 \tag{2.16}$$

and for every compact $K\subset S_0$,

$$\lim_{t\to 0}\int_K\int_V\left|\frac{Y^I\widehat{F}(\lambda,\alpha,\beta,s\exp tY)-Y^I\widehat{F}(\lambda,\alpha,\beta,s)}{t}-(U^\lambda_{(YY^IF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda\right|^2\varrho(\lambda)\,d\lambda\,ds=0. \tag{2.17}$$

Proof. It suffices to prove that (2.15) implies (2.16) and (2.17), and to use induction. First we notice that, by (2.15),

$$\frac{Y^I\widehat{F}(\lambda,\alpha,\beta,s\exp tY)-Y^I\widehat{F}(\lambda,\alpha,\beta,s)}{t}-(U^\lambda_{(YY^IF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda = (U^\lambda_{((Y^IF)_{s\exp tY}-(Y^IF)_s)/t-(YY^IF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda.$$

But by the Plancherel formula the $L^2(\varrho(\lambda))$-norm of the right side is dominated by

$$\left\|\frac{Y^IF(\cdot,s\exp tY)-Y^IF(\cdot,s)}{t}-YY^IF(\cdot,s)\right\|_{L^2}.$$

Hence (2.16) and (2.17) follow from Lemma (2.9). Now, from (2.17) we deduce (2.15) for YY^I, which justifies induction. ■

(2.18) LEMMA. *Let F be as in (2.6) and let $\widetilde{X}$ be a central element of $\mathcal{N}(\Phi)$. Then for all α,β,s,*

$$(U^\lambda_{(XF)_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda=2\pi i\langle\lambda,\mathrm{Ad}_s\widetilde{X}\rangle(U^\lambda_{F_s}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda \tag{2.19}$$

for a.e. λ.

Proof. For a C^∞_c function ϕ we have

$$(U^\lambda_{\widetilde{X}\phi}\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda=2\pi i\langle\lambda,\widetilde{X}\rangle(U^\lambda_\phi\xi_{\lambda\alpha},\xi_{\lambda\beta})_\lambda$$

for every $\lambda\in\Lambda$. Let $\phi\in C^\infty_c(N(\Phi))$, $0\le\phi\le 1$, $\phi=1$ on a neighbourhood B of 0 and $\phi_n(\zeta,x)=\phi\circ\delta_{n^{-1}}(\zeta,x)=\phi(n^{-1}\zeta,n^{-2}x)$. Clearly $0\le\phi_n\le 1$, $\phi_n=1$ on $B_n=\delta_nB$ and $\phi_n\to 1$. Moreover, $\widetilde{X}\phi_n(\zeta,x)=n^{-2}(\widetilde{X}\phi)_n(\zeta,x)$ because δ_n is an automorphism of $N(\Phi)$. We approximate $F((\zeta,x)s)$ in $L^2(N(\Phi))$ by $\phi_n(\zeta,x)F((\zeta,x)s)=F_n((\zeta,x)s)=F^n_s(\zeta,x)$, which is in $C^\infty_c(N(\Phi))$. Since $(XF^n)_s=(\mathrm{Ad}_s\widetilde{X})F^n_s$, (2.19) holds for F^n_s. To conclude, it suffices to show that $(XF^n)_s$ converges (in $L^2(N(\Phi))$) to $(XF)_s$. This follows from

$$\int_{N(\Phi)}|X(\phi_nF)((\zeta,x)s)-XF((\zeta,x)s)|^2\,dx\,d\zeta \le\int_{B_n^c}(|X(\phi_nF)((\zeta,x)s)|^2+2|XF((\zeta,x)s)|^2)\,dx\,d\zeta$$

and the fact that by the Harnack inequality both F_s and $(XF)_s$ are in $L^2(N(\Phi))$. ■

(2.20) COROLLARY. *Assume that F satisfies (2.1) and is annihilated by an elliptic, second order, left-invariant operator on S. Then for all α,β,s and almost every λ,*

$$(\Delta_jF)^\wedge(\lambda,\alpha,\beta,s)=(-4\pi^2\langle\lambda,\mathrm{Ad}_s\widetilde{X}_j\rangle^2+H_j^2-H_j)\widehat{F}(\lambda,\alpha,\beta,s),$$
$$(\Delta^\alpha_{ij}F)^\wedge(\lambda,\alpha,\beta,s)=(-4\pi^2\langle\lambda,\mathrm{Ad}_s\widetilde{X}^\alpha_{ij}\rangle^2+(Y^\alpha_{ij})^2-H_j)\widehat{F}(\lambda,\alpha,\beta,s).\ ■$$

3. Fourier transform of a function annihilated by cone operators. In this section we apply the operators (1.19) to functions which satisfy condition (2.1). We assume that

$$L_0F=0 \tag{3.1}$$

for a second order, left-invariant, admissible, real elliptic differential operator L_0 on S;

$$\Delta_jF=0\quad\text{for } j=1,\dots,r; \tag{3.2}$$

and there are strictly positive γ^α_{ij} such that

$$L_0F=\sum_{i<j}\gamma^\alpha_{ij}\Delta^\alpha_{ij}F=0. \tag{3.3}$$

Notice that for non-tube domains, (3.2) and (3.3) do not imply (3.1).

Formula (2.1) allows us to take the partial Fourier transform of F along $N(\Phi)$ and consider the Δ_j, L_0 on the Fourier transform side. By a subsequent application of $\Delta_1,\dots,\Delta_r$ and L_0 we obtain the following theorem.

(3.4) THEOREM. *Assume that F satisfies (2.1) and (3.1)–(3.3). Then for all α,β,s and almost every λ,*

$$\widehat{F}(\lambda,\alpha,\beta,s)=d_-(\lambda,\alpha,\beta)e^{-2\pi\langle\lambda,s\cdot e\rangle}+d_+(\lambda,\alpha,\beta)e^{2\pi\langle\lambda,s\cdot e\rangle}. \tag{3.5}$$

REMARK. In the (H^2) case, i.e. for Poisson integrals of L^2-functions, $\widehat{F}(\lambda,\alpha,\beta,s)$ is bounded as a function of s (for λ,α,β fixed). Therefore, the equation

$$\Big(\sum_{j=1}^{r}\Delta_j\Big)F=0$$

read on the Fourier transform side implies $(\Delta_jF)^\wedge=0$ for $j=1,\ldots,r$, and consequently $\widehat{F}(\lambda,\alpha,\beta,s)=0$ if $\lambda\notin\Omega\cup-\Omega$ (see [DHMP]). Then (3.5) follows from the formula for the Fourier transform of the Poisson kernel inside $\Omega\cup-\Omega$. If only (2.1) is satisfied we have to apply Δ_j's one by one and L_0 at the end to obtain (3.5). Moreover, all these do not imply that $\widehat{F}(\lambda,\alpha,\beta,s)=0$ for $\lambda\notin\Omega\cup-\Omega$.

If $\mathcal{Z}=0$, i.e. in the case of a tube domain, Theorem (3.4) should be formulated in the following way:

If F satisfies (2.1), (3.2), (3.3), *then for all s and almost every λ,*

$$\widehat{F}(\lambda,s)=d_-(\lambda)e^{-2\pi\langle\lambda,s\cdot e\rangle}+d_+(\lambda)e^{2\pi\langle\lambda,s\cdot e\rangle}.$$

As we shall see in Theorem (4.1), for a tube domain this implies pluriharmonicity. In other words, if on a tube domain F satisfies (2.1) and is annihilated by $r+1$ operators $L_0,\Delta_1,\ldots,\Delta_r$, then F is pluriharmonic.

Since the proof of Theorem (3.4) is long and technical we have added an appendix with the proof of the same assertion for the case when Ω is the cone of 2×2 symmetric positive definite matrices. It illustrates the main idea of the proof.

We fix α,β. Let W be the set of λ's such that $\widehat{F}(\lambda,\alpha,\beta,\cdot)\in C^\infty(S_0)$ and moreover,

$$(3.6)\qquad (-4\pi^2\langle\lambda,\mathrm{Ad}_s\widetilde{X}_j\rangle^2+H_j^2-H_j)\widehat{F}(\lambda,\alpha,\beta,s)=0,\qquad j=1,\ldots,r,$$

$$(3.7)\qquad \sum_{\substack{i<j\\ \alpha}}\gamma_{ij}^\alpha(-4\pi^2\langle\lambda,\mathrm{Ad}_s\widetilde{X}_{ij}^\alpha\rangle^2+(Y_{ij}^\alpha)^2-H_j)\widehat{F}(\lambda,\alpha,\beta,s)=0.$$

By Corollary (2.20), W is of full measure. Assume now that $\lambda=\mathrm{Ad}^*_{y_0}\lambda_0$ with $\lambda_0\in\sum_{j=1}^r V_{jj}$ and let $f(\lambda,s)=\widehat{F}(\lambda,\alpha,\beta,y_0^{-1}s)$. Then, for $\lambda\in W$, we have

$$(3.8)\qquad (-4\pi^2\langle\lambda_0,\mathrm{Ad}_s\widetilde{X}_j\rangle^2+H_j^2-H_j)f(\lambda,s)=0,\qquad j=1,\ldots,r,$$

$$(3.9)\qquad \sum_{i<j}\gamma_{ij}^\alpha(-4\pi^2\langle\lambda_0,\mathrm{Ad}_s\widetilde{X}_{ij}^\alpha\rangle^2+(Y_{ij}^\alpha)^2-H_j)f(\lambda,s)=0.$$

Let

$$W_j(\lambda_0,y)=2\pi\langle\lambda_0,\mathrm{Ad}_y\widetilde{X}_j\rangle,\qquad W_{ij}^\alpha(\lambda_0,y)=2\pi\langle\lambda_0,\mathrm{Ad}_y\widetilde{X}_{ij}^\alpha\rangle.$$

It follows from Lemma (1.23) that there is a neighbourhood U of e in N such that $W_j(\lambda_0,y)\neq0$ for $j=1,\ldots,r$ and $y\in U$.

(3.10) THEOREM. *Assume that $f(\lambda,s)$ satisfies* (3.8). *Then*

$$(3.11)\qquad f(\lambda,ya)=\sum_\delta d_\delta(\lambda,y)e^{\sum_{j=1}^r\delta_jW_j(\lambda_0,y)a_j},\qquad y\in U,\ a\in A,$$

where $\delta=(\delta_1,\ldots,\delta_r)$, $\delta_j=\pm1$. Moreover, for every δ, $d_\delta(\lambda_0,y)$ is a smooth function of y.

Proof. We proceed by induction proving that if the first k equations (3.8) are satisfied, then

$$(3.12)\qquad f(\lambda,ya)=\sum_\delta d_\delta(\lambda,ya_{k+1}\ldots a_r)e^{\sum_{j=1}^k\delta_jW_j(\lambda_0,y)a_j},\qquad y\in U,$$

where $\delta=(\delta_1,\ldots,\delta_r)$, $\delta_j=\pm1$ and for every δ, $d_\delta(\lambda,ya_{k+1}\ldots a_r)$ is a smooth function of ya. Assume (3.12), the first step being $k=0$. Applying the $k+1$ equations to f we obtain

$$\sum_\delta((-W_{k+1}(\lambda_0,y)^2+\partial^2_{a_{k+1}})d_\delta(\lambda,ya_{k+1}\ldots a_r))e^{\sum_{j=1}^r\delta_jW_j(\lambda_0,y)a_j}=0$$

for $y\in U$, $a\in A$. (Notice that $H_j^2-H_j=a_j^2\partial^2_{a_j}$.) Now if for three numbers c_1,c_2 and $\eta\neq0$ we have

$$(3.13)\qquad c_1e^{-\eta a}+c_2e^{\eta a}=0$$

for every positive a, then $c_1=c_2=0$. Applying this principle k times we obtain

$$(-W_{k+1}(\lambda_0,y)^2+\partial^2_{a_{k+1}})d_\delta(\lambda,ya_{k+1}\ldots a_r)=0$$

for every $\delta=(\delta_1,\ldots,\delta_k)$, $a\in A$ and $y\in U$. Therefore, since $W_{k+1}(\lambda_0,y)$ has constant sign on U we have

$$(3.14)\qquad d_\delta(\lambda,ya_{k+1}\ldots a_r)=d_{(\delta,1)}(\lambda,ya_{k+2}\ldots a_r)e^{W_{k+1}(\lambda_0,y)a_{k+1}}+d_{(\delta,-1)}(\lambda,ya_{k+2}\ldots a_r)e^{-W_{k+1}(\lambda_0,y)a_{k+1}}.$$

Moreover, for $a'_{k+1}\neq a_{k+1}$, (3.14) is

$$d_\delta(\lambda,ya'_{k+1}\ldots a_r)=d_{(\delta,1)}(\lambda,ya_{k+2}\ldots a_r)e^{W_{k+1}(\lambda_0,y)a'_{k+1}}+d_{(\delta,-1)}(\lambda,ya_{k+2}\ldots a_r)e^{-W_{k+1}(\lambda_0,y)a'_{k+1}}.$$

The above system of two equations has the solution

$$d_{(\delta,1)}(\lambda,ya_{k+2}\ldots a_r)=\frac{1}{\det}(d_\delta(\lambda,ya_{k+1}\ldots a_r)e^{-W_{k+1}(\lambda_0,y)a'_{k+1}}-d_\delta(\lambda,ya'_{k+1}a_{k+2}\ldots a_r)e^{-W_{k+1}(\lambda_0,y)a_{k+1}}),$$

$$d_{(\delta,-1)}(\lambda,ya_{k+2}\ldots a_r)=\frac{1}{\det}(d_\delta(\lambda,ya'_{k+1}a_{k+1}\ldots a_r)e^{W_{k+1}(\lambda_0,y)a_{k+1}}-d_\delta(\lambda,ya_{k+1}\ldots a_r)e^{W_{k+1}(\lambda_0,y)a'_{k+1}}),$$

where

$$\det = e^{W_{k+1}(\lambda_0,y)(a_{k+1}-a'_{k+1})} - e^{W_{k+1}(\lambda_0,y)(a'_{k+1}-a_{k+1})} \neq 0,$$

which proves that $d_{(\delta,1)}$, $d_{(\delta,-1)}$ are smooth on U. ■

The next step will be to eliminate y from $d_\delta(\lambda,y)$, i.e. to prove that $d_\delta(\lambda,y)=d_\delta(\lambda)$, $y\in U$, and moreover, that $d_\delta(\lambda)\neq 0$ only if $\delta=(1,\ldots,1)$ or $\delta=(-1,\ldots,-1)$. To do it we use (3.9). First we introduce some notation. Let

$$P_k=\bigoplus_{i,j\le k} V_{ij},\qquad \mathcal{N}_k=\bigoplus_{i<j\le k}\mathcal{N}_{ij},\qquad N_k=\exp\mathcal{N}_k,$$

$$\widetilde{\mathcal{N}}=\bigoplus_{i<r}\mathcal{N}_{ir},\qquad \widetilde{N}=\exp\widetilde{\mathcal{N}}.$$

Also we need some identities for derivatives of W_j, W^α_{ij}.

(3.15) Lemma. *We have*

$$\widetilde{Y}^\alpha_{ij}W_k=\begin{cases}0 & \text{if } k\neq i,\\ W^\alpha_{ij} & \text{if } k=i,\end{cases}\tag{3.16}$$

and

$$\widetilde{Y}^\alpha_{ij}W^\alpha_{ij}=W_j.\tag{3.17}$$

Proof. We have

$$\widetilde{Y}^\alpha_{ij}W_k(\lambda,y)=2\pi\frac{d}{dt}\langle\lambda,\mathrm{Ad}_{y\exp t\widetilde{Y}^\alpha_{ij}}\widetilde{X}_k\rangle\Big|_{t=0} =2\pi\frac{d}{dt}\langle\lambda,\mathrm{Ad}_y\,\tau(te^\alpha_{ij})\widetilde{X}_k\rangle\Big|_{t=0},$$

where τ is as in (1.20). But, in view of Lemma VI.3.1 of [FK],

$$\tau(te^\alpha_{ij})e_k=\begin{cases}e_k & \text{if } k\neq i,\\ te^\alpha_{ij}+2t^2L(e-c_i)L(e^\alpha_{ij})^2e_i & \text{if } k=i.\end{cases}$$

and (3.16) follows. For (3.17) we write again

$$\widetilde{Y}^\alpha_{ij}W^\alpha_{ij}(\lambda,y)=2\pi\frac{d}{dt}\langle\lambda,\mathrm{Ad}_{y\exp t\widetilde{Y}^\alpha_{ij}}\widetilde{X}^\alpha_{ij}\rangle\Big|_{t=0}=2\pi\frac{d}{dt}\langle\lambda,\mathrm{Ad}_y\,\tau(te^\alpha_{ij})\widetilde{X}^\alpha_{ij}\rangle\Big|_{t=0}.$$

By Lemma VI.3.1 of [FK], we have

$$\tau(te^\alpha_{ij})e^\alpha_{ij}=e^\alpha_{ij}+2(e-c_i)(te^\alpha_{ij}\cdot e^\alpha_{ij}) =e^\alpha_{ij}+2(e-c_i)\frac{1}{2}t(c_i+c_j)=e^\alpha_{ij}+tc_j,$$

which implies (3.17). ■

(3.18) Lemma. *Assume that $f(\lambda,ya)$ is given by (3.11) and additionally satisfies (3.9). Then for the family of functions d_δ the following identity holds*:

$$\sum_{\substack{i<j\le r\\ \alpha,\delta}}\gamma^\alpha_{ij}\big(a_j(\delta_i-\delta_j)W_j(\lambda_0,y)d_\delta(\lambda,y)+a_ja_i^{-1}(\widetilde{Y}^\alpha_{ij})^2d_\delta(\lambda,y) + 2a_j\delta_i(\widetilde{Y}^\alpha_{ij}d_\delta(\lambda,y))W^\alpha_{ij}(\lambda_0,y)\big)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}=0.\tag{3.19}$$

Proof. Since $H_j=a_j\partial_j$, we have

$$H_j\big(d_\delta(\lambda,y)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}\big) =a_j\delta_jW_j(\lambda_0,y)d_\delta(\lambda,y)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}\tag{3.20}$$

and

$$Y^\alpha_{ij}\big(d_\delta(\lambda,y)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}\big) =a_j^{1/2}a_i^{-1/2}(\widetilde{Y}^\alpha_{ij}d_\delta(\lambda,y)+\delta_iW^\alpha_{ij}(\lambda_0,y)a_id_\delta(\lambda,y))e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}$$

and

$$\begin{aligned}(Y^\alpha_{ij})^2\big(d_\delta(\lambda,y)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}\big) &=a_ja_i^{-1}((\widetilde{Y}^\alpha_{ij})^2d_\delta(\lambda,y)+\delta_iW_j(\lambda_0,y)a_id_\delta(\lambda,y)\\ &\quad+2\delta_ia_iW^\alpha_{ij}(\lambda_0,y)\widetilde{Y}^\alpha_{ij}d_\delta(\lambda,y)\\ &\quad+a_i^2W^\alpha_{ij}(\lambda_0,y)^2d_\delta(\lambda,y))e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}.\end{aligned}\tag{3.21}$$

Putting (3.20) and (3.21) together we obtain

$$\begin{aligned}(-a_ja_iW^\alpha_{ij}(\lambda_0,y)^2+(Y^\alpha_{ij})^2-H_j)\big(d_\delta(\lambda,y)e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}\big) &=(a_j(\delta_i-\delta_j)W_j(\lambda_0,y)d_\delta(\lambda,y)+a_ja_i^{-1}(\widetilde{Y}^\alpha_{ij})^2d_\delta(\lambda,y)\\ &\quad+2a_j\delta_i(\widetilde{Y}^\alpha_{ij}d_\delta(\lambda,y))W^\alpha_{ij}(\lambda_0,y))e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k},\end{aligned}$$

which proves (3.19). ■

(3.22) Theorem. *Let U be a neighbourhood of e in N such that $W_j(\lambda_0,y)\neq 0$ for $j=1,\ldots,r$, $y\in U$. Let d_δ, $\delta=(\delta_1,\ldots,\delta_r)$, $\delta_j=\pm1$, be a system of smooth functions defined on U which satisfy the equation*

$$\sum_{\substack{i<j\le r\\ \alpha,\delta}}\gamma^\alpha_{ij}(a_j(\delta_i-\delta_j)W_j(\lambda_0,y)d_\delta(y)+a_ja_i^{-1}(\widetilde{Y}^\alpha_{ij})^2d_\delta(y) +2a_j\delta_i(\widetilde{Y}^\alpha_{ij}d_\delta(y))W^\alpha_{ij}(\lambda_0,y))e^{\sum_{k=1}^r\delta_kW_k(\lambda_0,y)a_k}=0\tag{3.23}$$

with $\gamma^\alpha_{ij}>0$. Then there is a neighbourhood $U'\subset U$ such that every d_δ is a constant function on U' and moreover $d_\delta\neq0$ only if $\delta=(1,\ldots,1)$ or $\delta=(-1,\ldots,-1)$.

Proof. The proof is by induction with respect to r. The induction step includes also the initial case $r=2$. Let

$$I_1=\sum_{\substack{i<j<r\\ \alpha,\delta,\delta_r=1}}\gamma_{ij}^\alpha(a_j(\delta_i-\delta_j)W_j(\lambda_0,y)d_\delta(y)+a_ja_i^{-1}(\widetilde{Y}_{ij}^\alpha)^2d_\delta(y)$$
$$+2a_j\delta_i(\widetilde{Y}_{ij}^\alpha d_\delta(y))W_{ij}^\alpha(\lambda_0,y))e^{\sum_{k=1}^{r-1}\delta_kW_k(\lambda_0,y)a_k}.$$

I_2 is defined precisely in the same way as I_1 but with $\delta_r=-1$. Let

$$I_3=\sum_{\substack{i<r\\ \alpha,\delta,\delta_r=1}}\gamma_{ir}^\alpha((\delta_i-\delta_r)W_r(\lambda_0,y)d_\delta(y)+a_i^{-1}(\widetilde{Y}_{ir}^\alpha)^2d_\delta(y)$$
$$+2\delta_i(\widetilde{Y}_{ir}^\alpha d_\delta(y))W_{ir}^\alpha(\lambda_0,y))e^{\sum_{k=1}^{r-1}\delta_kW_k(\lambda_0,y)a_k}.$$

I_4 is defined in the same way as I_3 with $\delta_r=-1$. Therefore, (3.23) becomes

$$I_1e^{W_r(\lambda_0,y)a_r}+I_2e^{-W_r(\lambda_0,y)a_r}+a_r(I_3e^{W_r(\lambda_0,y)a_r}+I_4e^{-W_r(\lambda_0,y)a_r})=0.$$

Dividing both sides by

$$\begin{cases}a_re^{W_r(\lambda_0,y)a_r} & \text{if } W_r(\lambda_0,y)>0,\\ a_re^{-W_r(\lambda_0,y)a_r} & \text{if } W_r(\lambda_0,y)<0,\end{cases}$$

and letting $a_r\to\infty$, we obtain $I_3=0$ or $I_4=0$, respectively. Now, if $a_r\to\infty$ we conclude that $I_1=0$ or $I_2=0$, respectively. Hence we obtain $I_2+a_rI_4=0$ in the first case, and $I_1+a_rI_3=0$ in the second, which yields $I_2=I_4=0$ or $I_1=I_3=0$, respectively.

Let $\delta'=(\delta_1,\ldots,\delta_{r-1})$, $\delta_j=\pm1$ and $\delta=(\delta',1)$. We write N as

$$N=\widetilde{N}N_{r-1}. \tag{3.24}$$

Then

$$d_\delta(y)=d_\delta(uy'),\quad u\in\widetilde{N},\ y'\in N_{r-1}. \tag{3.25}$$

Then we fix u and consider $I_1=0$ as an equation on the group $\widetilde{N}$.

We obtain a new system of functions $d_{\delta'}(y')=d_\delta(uy')$. We want to conclude that $d_{\delta'}$ are constant and moreover $d_{\delta'}\neq0$ only if $\delta'=(1,\ldots,1)$ or $\delta'=(-1,\ldots,-1)$.

We claim that the $d_{\delta'}$'s satisfy equation (3.23) on a neighbourhood U' of e in N_{r-1} for a $\lambda_0'\in\sum_{j=1}^{r-1}V_{jj}$ instead of λ_0. The λ_0' is defined by

$$\lambda_0'=\sum_{j=1}^{r-1}\lambda_jc_j,\quad\text{where}\quad\lambda_0=\sum_{j=1}^{r}\lambda_jc_j. \tag{3.26}$$

First we prove that for $j<r$,

$$W_{ij}^\alpha(\lambda_0,uy')=W_{ij}^\alpha(\operatorname{Ad}_u\lambda_0,y')=W_{ij}^\alpha(\lambda_0',y') \tag{3.27}$$

(with the convention $W_{jj}^\alpha=W_j$). We write $u=\exp Y$ and notice that

$$\langle\lambda_0-\operatorname{Ad}_u^*\lambda_0,X\rangle=\langle\lambda_0,X-e^{\operatorname{ad}Y}X\rangle=0. \tag{3.28}$$

Indeed, by (1.10), $[\widetilde{N},V]\subset\bigoplus_{j<r}V_{jr}=\widetilde{V}$, which gives (3.28). On the other hand, if $j<r$ then $\operatorname{Ad}_{y'}\widetilde{X}_{ij}^\alpha\in P_{r-1}=\bigoplus_{i,j\le r-1}V_{ij}$ and so

$$W_{ij}^\alpha(\operatorname{Ad}_u^*\lambda_0,y')=2\pi\langle\lambda_0,\operatorname{Ad}_{y'}\widetilde{X}_{ij}^\alpha\rangle=2\pi\langle\lambda_0',\operatorname{Ad}_{y'}\widetilde{X}_{ij}^\alpha\rangle=W_{ij}^\alpha(\lambda_0',y'),$$

which is (3.27). Moreover, since $\widetilde{Y}_{ij}^\alpha$ is left-invariant we have

$$\widetilde{Y}_{ij}^\alpha d_{\delta'}(y')=(\widetilde{Y}_{ij}^\alpha d_{\delta'})(uy').$$

Finally, we choose neighbourhoods $\widetilde{U}$, U' of e in $\widetilde{N}$ and N_{r-1}, respectively such that $\widetilde{U}U'\subset U$ and for $y'\in U'$ we have

$$\sum_{\substack{i<j\le r\\ \alpha,\delta'}}\gamma_{ij}^\alpha(a_j(\delta_i-\delta_j)W_j(\lambda_0',y')d_{\delta'}(y')+a_ja_i^{-1}(\widetilde{Y}_{ij}^\alpha)^2d_{\delta'}(y') \tag{3.29}$$
$$+2a_j\delta_i(\widetilde{Y}_{ij}^\alpha d_{\delta'}(y))W_{ij}^\alpha(\lambda_0',y'))e^{\sum_{k=1}^{r-1}\delta_kW_k(\lambda_0',y')a_k}=0.$$

Hence, by our inductive hypothesis,

$$d_{\delta'}(y')=d_{(\delta',1)}(uy')=d_{(\delta',1)}(u)$$

and $d_{\delta'}(u)=0$ if $\delta'\neq(1,\ldots,1)$ or $(-1,\ldots,-1)$. Considering $I_2=0$ instead of $I_1=0$ we obtain the same conclusion for the system $d_{(\delta',-1)}$.

Now we make use of the fact that $I_3=0$. Since $\delta'=(1,\ldots,1)=\mathbf{1}$ or $\delta'=(-1,\ldots,-1)=-\mathbf{1}$, for $\delta=(\delta',1)$, we have

$$\sum_{\substack{i<r\\ \alpha}}(\gamma_{ir}^\alpha((\delta_i-\delta_r)W_r(\lambda_0,y)d_{(\mathbf{1},1)}(u)+a_i^{-1}(\widetilde{Y}_{ir}^\alpha)^2d_{(\mathbf{1},1)}(u)$$
$$+2\delta_i(\widetilde{Y}_{ir}^\alpha d_{(\mathbf{1},1)}(u))W_{ir}^\alpha(\lambda_0,y)))$$
$$=\sum_{\substack{i<r\\ \alpha}}\gamma_{ir}^\alpha(a_i^{-1}(\widetilde{Y}_{ir}^\alpha)^2d_{(\mathbf{1},1)}(u)+2\delta_i(\widetilde{Y}_{ir}^\alpha d_{(\mathbf{1},1)}(u))W_{ir}^\alpha(\lambda_0,y))=0$$

and

$$\sum_{\substack{i<r\\ \alpha}}(\gamma_{ir}^\alpha((\delta_i-\delta_r)W_r(\lambda_0,y)d_{(-\mathbf{1},1)}(u)+a_i^{-1}(\widetilde{Y}_{ir}^\alpha)^2d_{(-\mathbf{1},1)}(u)$$
$$+2\delta_i(\widetilde{Y}_{ir}^\alpha d_{(-\mathbf{1},1)}(u))W_{ir}^\alpha(\lambda_0,y)))$$
$$=\sum_{\substack{i<r\\ \alpha}}(\gamma_{ir}^\alpha(-2W_r(\lambda_0,y)d_{(-\mathbf{1},1)}(u)+a_i^{-1}(\widetilde{Y}_{ir}^\alpha)^2d_{(-\mathbf{1},1)}(u)$$
$$+2\delta_i(\widetilde{Y}_{ir}^\alpha d_{(-\mathbf{1},1)}(u))W_{ir}^\alpha(\lambda_0,y)))=0.$$

The above identities hold for $y = uy'$, $u \in \widetilde{U}$, $u' \in U'$. Proceeding as before (i.e. letting $a_j \to \infty$ separately), we find that for every i,

$$\sum_\alpha (\widetilde{Y}_{ir}^\alpha)^2 d_{(1,1)}(u) = 0, \qquad \sum_\alpha (\widetilde{Y}_{ir}^\alpha)^2 d_{(-1,1)}(u) = 0.$$

Therefore,

$$\begin{aligned} &\sum_{\substack{i<r\\ \alpha}} \gamma_{ir}^\alpha \widetilde{Y}_{ir}^\alpha d_{(1,1)}(u) W_{ir}^\alpha(\lambda_0, y) = 0,\\ &\sum_{\substack{i<r\\ \alpha}} \gamma_{ir}^\alpha (W_r(\lambda_0, y) d_{(-1,1)}(u) + \widetilde{Y}_{ir}^\alpha d_{(-1,1)}(u) W_{ir}^\alpha(\lambda_0, y)) = 0. \end{aligned} \tag{3.30}$$

Since the above equalities hold for every $y' \in U'$, we may put $y' = e$. By Lemma (1.23) we have

$$W_r(\lambda_0, u) = \lambda_r$$

according to the decomposition (3.24) and (3.25). We also need a more precise formula for $W_{ir}^\alpha(\lambda_0, u)$. By (3.16) and the fact that $\widetilde{N}$ is Abelian,

$$W_{ir}^\alpha(\lambda_0, u) = \widetilde{Y}_{ir}^\alpha W_i^\alpha(\lambda_0, u) = \partial_{u_{ir}}^\alpha W_i^\alpha(\lambda_0, u).$$

Hence Lemma (1.23) implies that

$$W_{ir}^\alpha(\lambda_0, u) = \lambda_r u_{ir}^\alpha.$$

If we again take into account that $\widetilde{N}$ is Abelian, (3.30) becomes

$$\sum_{\substack{i<r\\ \alpha}} \gamma_{ir}^\alpha \partial_{u_{ir}}^\alpha d_{(1,1)}(u) u_{ir}^\alpha = 0, \quad u \in \widetilde{U},$$

$$\sum_{\substack{i<r\\ \alpha}} \gamma_{ir}^\alpha \partial_{u_{ir}}^\alpha d_{(1,1)}(u) u_{ir}^\alpha = -\Big(\sum_{\substack{i<r\\ \alpha}} \gamma_{ir}^\alpha\Big) d_{(-1,1)}(u), \quad u \in \widetilde{U}.$$

Continuity of d_δ implies that $d_{(1,1)}$ is constant and $d_{(-1,1)} = 0$. A standard proof of that is included in Lemma (3.31) below. ■

(3.31) LEMMA. *Let U be a neighbourhood of 0 in $\mathbb{R}^k$ and let d be a continuous function which satisfies*

$$\sum_{j=1}^k b_j \partial_{u_j} d(u) u_j = -cd, \quad u \in U, \tag{3.32}$$

with $b_j > 0$, $c \ge 0$. Then d is constant and $d = 0$ if $c > 0$.

Proof. We solve the equation (3.32) by the characteristic method. Let $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$ with $\gamma_j(t) = u_j e^{b_j t}$. Then

$$\dot{\gamma}_j(t) = b_j \gamma_j(t) \tag{3.33}$$

with the initial condition $\gamma(0) = (u_1, \ldots, u_r)$. By (3.32), (3.33),

$$\frac{d}{dt} d(\gamma(t)) = \sum_j \partial_{y_j} d(\gamma_j(t)) \dot{\gamma}_j(t) = -cd(\gamma(t)).$$

Therefore, $d(\gamma(t)) = e^{-ct} d(\gamma(0))$. All the characteristics meet at 0, i.e.

$$\lim_{t \to -\infty} \gamma(t) = 0.$$

If $c = 0$, then d is constant along the characteristics so by continuity of d, $d(\gamma(0)) = d(u) = d(0)$. For $c > 0$ assume that there is $u = \gamma(0)$ such that $d(u) \neq 0$. Then $\lim_{t \to -\infty} d(\gamma(t)) = \pm\infty$, which contradicts the continuity of d. ■

Now putting together Theorem (3.10), Lemma (3.18) and Theorem (3.22), we are able to prove the main theorem of this section.

Proof of Theorem (3.4). Applying Theorem (3.22) to $d_\delta(y) = d_\delta(\lambda, y)$ we see that

$$f(\lambda, ya) - d_1(\lambda) e^{\sum_{j=1}^r W_j(\lambda_0, y) a_j} - d_{-1}(\lambda) e^{-\sum_{j=1}^r W_j(\lambda_0, y) a_j} = 0 \tag{3.34}$$

for y in a neighbourhood U of $e \in N$ and all $a \in A$. On the other hand, the left hand side of (3.34) is defined on the whole of S_0 and annihilated there by the operators (3.8) and (3.9), i.e. by an elliptic operator which is the sum of all of them. Therefore (3.34) holds for $y \in N$, $a \in A$. Since $W_j(\lambda_0, y_0 y) = W_j(\lambda, y)$ and $\widehat{F}(\lambda, \alpha, \beta, s) = f(\lambda, y_0 s)$ we obtain

$$\widehat{F}(\lambda, \alpha, \beta, ya) = d_-(\lambda, \alpha, \beta) e^{-\sum_{j=1}^r W_j(\lambda, y) a_j} + d_+(\lambda, \alpha, \beta) e^{\sum_{j=1}^r W_j(\lambda, y) a_j}.$$

But

$$\begin{aligned} \sum_{j=1}^r W_j(\lambda, y) a_j &= 2\pi \sum_{j=1}^r \langle \mathrm{Ad}_y^* \lambda, \mathrm{Ad}_a X_j \rangle = 2\pi \Big\langle \mathrm{Ad}_{ya}^* \lambda, \sum_{j=1}^r X_j \Big\rangle \\ &= 2\pi \langle \mathrm{Ad}_{ya}^* \lambda, e \rangle = 2\pi \langle \lambda, ya \cdot e \rangle \end{aligned}$$

and Theorem (3.4) is proved. ■

4. Pluriharmonic functions on symmetric tube domains. In this section we are going to characterize pluriharmonic functions which satisfy (2.1) on a tube domain. More precisely, our goal is to prove that provided (2.1), a real-valued function F is pluriharmonic iff (3.2) and (3.3) hold. Clearly, only sufficiency of (3.2) and (3.3) is of interest. In view of Theorem (3.4), (3.2) and (3.3) imply that the partial Fourier transform of F is of the form

$$F(\widehat{\lambda}, s) = d_-(\lambda) e^{-2\pi \langle \lambda, s \cdot e \rangle} + d_+(\lambda) e^{2\pi \langle \lambda, s \cdot e \rangle}. \tag{4.1}$$

Therefore, we have to show that (4.1) implies pluriharmonicity. This is our Theorem (4.10). However, we will show that (2.1), (3.2), (3.3) do not imply the existence of a function F' conjugate to F that satisfies (2.1). (F' is called a *conjugate function* to F if $F + iF'$ is holomorphic. Clearly two functions conjugate to F differ by a constant.) More can be said about functions which satisfy a stronger condition than (2.1), i.e. functions which belong to a certain weighted L^2-space on $\mathcal{D} = V + i\Omega$ (see Theorem (4.16)); some of those spaces are of interest for the representation theory of semisimple Lie groups ([FK], [RV]).

For a tube domain $\mathcal{D} = V + i\Omega$, (2.1) becomes

$$\int_K \int_V |F(xs)|^2 \, dx \, ds < \infty \quad \text{for every compact set } K \subset S_0, \tag{4.2}$$

or, equivalently,

$$\int_K \int_V |F(x+iu)|^2 \, dx \, du < \infty \quad \text{for every compact set } K \subset \Omega. \tag{4.3}$$

We shall identify $\mathcal{D}$ with S by $z = xs \cdot ie$, $x \in V$, $s \in S_0$ and functions on S and $\mathcal{D}$, respectively.

It is not difficult to see that if a holomorphic function F satisfies (4.3), then the partial Fourier transform of F along V is of the form

$$F(\widehat{\lambda}, u) = d(\lambda) e^{-2\pi\langle\lambda,u\rangle}$$

(see e.g. [RV]) and for every compact set K included in Ω we have

$$\int_K \int_V |d(\lambda)|^2 e^{-4\pi\langle\lambda,u\rangle} \, d\lambda \, du < \infty. \tag{4.4}$$

Conversely, if $d(\lambda)$ satisfies (4.4) then

$$F(x+iu) = \int_V d(\lambda) e^{-2\pi\langle\lambda,u\rangle} e^{2\pi i\langle\lambda,x\rangle} \, d\lambda \tag{4.5}$$

is holomorphic. (The Fourier transform in (4.5) and in all what follows is meant in the L^2-sense.) Holomorphy of F follows from Theorem 2.10 of [RV]. This theorem characterizes holomorphic functions which belong to $\mathcal{H}_\psi$, i.e. satisfy

$$\int_\Omega \int_V |F(x+iu)|^2 \psi(u) \, dx \, du < \infty, \tag{4.6}$$

for a positive continuous function ψ on Ω. Let

$$I_\psi(\lambda) = \int_\Omega e^{-4\pi\langle\lambda,u\rangle} \psi(u) \, du, \qquad V_\psi = \{\lambda : I_\psi(\lambda) < \infty\}.$$

Then V_ψ is a convex set and I_ψ is continuous on V_ψ. Theorem 2.10 of [RV] says that (4.5) gives one-one correspondence between $\mathcal{H}_\psi$ and $L^2_\psi(V_\psi)$, the latter being the space of measurable functions on V_ψ such that

$$\int_{V_\psi} |d(\lambda)|^2 I_\psi(\lambda) \, d\lambda < \infty.$$

If (4.4) holds, then for a suitable ψ,

$$\int_\Omega \int_V |d(\lambda)|^2 e^{-4\pi\langle\lambda,u\rangle} \psi(u) \, d\lambda \, du < \infty,$$

whence we may conclude that F is holomorphic.

Let now F be a pluriharmonic function which satisfies (4.2). By Theorem (3.4), its partial Fourier transform is of the form

$$F(\widehat{\lambda}, u) = d_-(\lambda) e^{-2\pi\langle\lambda,u\rangle} + d_+(\lambda) e^{2\pi\langle\lambda,u\rangle} \tag{4.7}$$

with

$$\int_K \int_V |F(\widehat{\lambda}, u)|^2 \, d\lambda \, du < \infty \tag{4.8}$$

for every compact $K \subset \Omega$ and

$$d_-(-\lambda) = \overline{d_+(\lambda)} \tag{4.9}$$

((4.9) means nothing more than that F is real-valued).

It is natural to expect that (4.7)–(4.9) give a characterization of Fourier transforms of such pluriharmonic functions. Indeed, we have the following theorem:

(4.10) THEOREM. *Let $d_-(\lambda)$, $d_+(\lambda)$ be measurable functions,*

$$d_-(-\lambda) = \overline{d_+(\lambda)},$$

and let

$$g(\lambda, u) = d_-(\lambda) e^{-2\pi\langle\lambda,u\rangle} + d_+(\lambda) e^{2\pi\langle\lambda,u\rangle}.$$

Assume that for every compact set $K \subset \Omega$,

$$\int_K \int_V |g(\lambda, u)|^2 \, d\lambda \, du < \infty.$$

Then

$$F(x+iu) = \int_V g(\lambda, u) e^{2\pi i\langle\lambda,x\rangle} \, d\lambda$$

is pluriharmonic.

REMARK. Notice that we do not assume that each of

$$d_-(\lambda) e^{-2\pi\langle\lambda,u\rangle}, \quad d_+(\lambda) e^{2\pi\langle\lambda,u\rangle}$$

satisfies (4.8) separately and, in fact, this is not true for the partial Fourier transform of a pluriharmonic function. As an example of such a situation we take g defined on $\mathbb{R}+i\mathbb{R}^+$ by

$$g(\lambda,u)=\frac{1}{\lambda}\phi(\lambda)e^{-2\pi\langle\lambda,u\rangle}-\frac{1}{\lambda}\phi(\lambda)e^{2\pi\langle\lambda,u\rangle}, \tag{4.11}$$

where $\phi\in C_c(\mathbb{R})$, $\phi(-\lambda)=\overline{\phi(\lambda)}$. Therefore, we cannot treat $d_-(\lambda)e^{-2\pi\langle\lambda,u\rangle}$ and $d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}$ separately and draw the conclusion immediately from the Rossi–Vergne Theorem, but we must proceed in a more delicate way.

Let

$$G(x+iu)=\int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\phi(\lambda)e^{-2\pi\langle\lambda,u\rangle}-\frac{1}{\lambda}\phi(\lambda)e^{2\pi\langle\lambda,u\rangle}\right)e^{2\pi i\lambda x}\,d\lambda.$$

All the functions G' conjugate to G are of the form

$$G'(x+iu)=\int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\phi(\lambda)e^{-2\pi\langle\lambda,u\rangle}+\frac{1}{\lambda}\phi(\lambda)e^{2\pi\langle\lambda,u\rangle}\right)e^{2\pi i\lambda x}\,d\lambda+c,$$

where the integration stands for the Fourier transform of a tempered distribution. None of the G''s satisfies condition (4.3).

Proof (of Theorem (4.10)). First we prove that for every ball $B_\varepsilon(0)\subset V$ both

$$g_-^\varepsilon(\lambda,u)=\mathbf{1}_{B_\varepsilon^c(0)}d_-(\lambda)e^{-2\pi\langle\lambda,u\rangle},\qquad g_+^\varepsilon(\lambda,u)=\mathbf{1}_{B_\varepsilon^c(0)}d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}$$

satisfy (4.8). If so, then

$$F_-^\varepsilon(x+iu)=\int_V g_-^\varepsilon(\lambda,u)e^{2\pi i\langle\lambda,x\rangle}\,dx\quad\text{is holomorphic,}$$

$$F_+^\varepsilon(x+iu)=\int_V g_+^\varepsilon(\lambda,u)e^{2\pi i\langle\lambda,x\rangle}\,dx\quad\text{is antiholomorphic.}$$

Moreover,

$$\overline{F_+^\varepsilon}(x+iu)=F_-^\varepsilon(x+iu).$$

Hence $F_-^\varepsilon+F_+^\varepsilon$ is pluriharmonic. Moreover,

$$\int_K\int_V|F(x+iu)-(F_-^\varepsilon(x+iu)+F_+^\varepsilon(x+iu))|^2\,dx\,du$$
$$=\int_K\int_V|g(\lambda,u)-(g_-^\varepsilon(\lambda,u)-g_+^\varepsilon(\lambda,u))|^2\,d\lambda\,du$$
$$=\int_K\int_{B_\varepsilon(0)}|g(\lambda,u)|^2\,d\lambda\,du.$$

The last integral tends to zero as $\varepsilon\to0$, which proves that F is pluriharmonic.

Now we turn to the proof that g_-^ε, g_+^ε satisfy (4.8). Given $u,u'\in\Omega$, $u\neq u'$, we have

$$g(\lambda,u)e^{-2\pi\langle\lambda,u'\rangle}=d_-(-\lambda)e^{-2\pi\langle\lambda,u+u'\rangle}+d_+(\lambda)e^{2\pi\langle\lambda,u-u'\rangle},$$
$$g(\lambda,u')e^{-2\pi\langle\lambda,u\rangle}=d_-(-\lambda)e^{-2\pi\langle\lambda,u+u'\rangle}+d_+(\lambda)e^{2\pi\langle\lambda,u'-u\rangle}.$$

Hence

$$d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}=\frac{g(\lambda,u)-g(\lambda,u')e^{2\pi\langle\lambda,u'-u\rangle}}{1-e^{4\pi\langle\lambda,u'-u\rangle}}. \tag{4.12}$$

We are going to prove that for every $\varepsilon>0$ the right hand side of (4.12) restricted to $B_\varepsilon^c(0)$ satisfies (4.8). Let $e_1,\ldots,e_n$ be an orthonormal basis of V and let

$$U_j^+=\{\lambda\in V:\langle\lambda,e_j\rangle>\varepsilon/n\},\qquad U_j^-=\{\lambda\in V:\langle\lambda,e_j\rangle<-\varepsilon/n\}.$$

Then

$$B_\varepsilon^c(0)\subset\bigcup_{j=1}^n(U_j^+\cup U_j^-). \tag{4.13}$$

So it is enough to show that for every j,

$$d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}\mathbf{1}_{U_j^+},\qquad d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}\mathbf{1}_{U_j^-}$$

satisfy (4.8). Let η be such that for every $u\in K$, $u+B_\eta(0)\subset\Omega$. Let $u'=u-(\eta/2)e_j$ and $\lambda\in U_j^+$. Then

$$e^{2\pi\langle\lambda,u'-u\rangle}=e^{-\eta\pi\langle\lambda,e_j\rangle}\le e^{-\eta\varepsilon\pi/n},\qquad e^{4\pi\langle\lambda,u'-u\rangle}\le e^{-2\eta\varepsilon\pi/n}.$$

Hence

$$\left|\frac{g(\lambda,u)-g(\lambda,u')e^{2\pi\langle\lambda,u'-u\rangle}}{1-e^{4\pi\langle\lambda,u'-u\rangle}}\right|\le\frac{|g(\lambda,u)|+|g(\lambda,u')|}{1-e^{-2\pi\eta\varepsilon/n}},$$

which shows that

$$\int_K\int_{U_j^+}|d_+(\lambda)|^2e^{4\pi\langle\lambda,u\rangle}\,d\lambda\,du<\infty.$$

For $\xi\in U_j^-$ we choose u' such that $u'-u=(\eta/2)e_j$ and proceed as before.

Finally, by (4.13),

$$\int_K\int_{V\setminus B_\varepsilon(0)}|d_+(\lambda)|^2e^{4\pi\langle\lambda,u\rangle}\,d\lambda\,du<\infty.\ \blacksquare$$

As a corollary we obtain the main theorem of this section.

(4.14) THEOREM. *Let $\mathcal{D}$ be a tube domain and let F be a real-valued function satisfying* (4.2). *Then F is pluriharmonic if, and only if,* (3.2) *and* (3.3) *hold, but its conjugate function does not have to satisfy* (4.2). ■

Let P_ψ be the space of pluriharmonic functions satisfying (4.6). The example (4.11) shows that without some further assumptions on ψ we cannot expect that a function F belonging to P_ψ is the real value of a holomorphic function satisfying the same growth condition (4.6). Therefore, it is natural to ask for which ψ, $P_\psi = \Re\mathcal{H}_\psi$, i.e. when P_ψ consists of the real parts of functions belonging to $\mathcal{H}_\psi$, or equivalently, for which ψ there is an adjoint F' which belongs to P_ψ. A precise characterization of such ψ is not known, but homogeneous ψ (i.e. $\psi(\lambda u) = \lambda^\alpha \psi(u)$, $\alpha > -n$) are known to have this property. Those particular weights are of interest from the point of view of representation theory of semisimple Lie groups (see [RV], [FK]). We are able to prove more. If

$$V_\psi = \{\lambda : I_\psi(\lambda) < \infty\} \subset \Omega \tag{4.15}$$

then $P_\psi = \Re\mathcal{H}_\psi$ (see Theorem (4.23)). More precisely, if $V_\psi \subset \Omega$, then $\operatorname{supp} d_- \subset \Omega$, $\operatorname{supp} d_+ \subset -\Omega$, and so $d_-(\lambda)e^{-2\pi\langle\lambda,u\rangle}$, $d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}$ satisfy (4.6) separately.

(4.16) THEOREM. *Let ψ be a positive continuous function such that $V_\psi \subset \Omega$. Assume that*

$$g(\lambda, u) = d_-(\lambda)e^{-2\pi\langle\lambda,u\rangle} + d_+(\lambda)e^{2\pi\langle\lambda,u\rangle}$$

satisfies

$$\int_\Omega\int_V |g(\lambda,u)|^2\psi(u)\,d\lambda\,du < \infty. \tag{4.17}$$

Then

$$\begin{aligned} d_-(\lambda) &= 0 \quad \textit{for a.e. } \lambda \notin \Omega,\\ d_+(\lambda) &= 0 \quad \textit{for a.e. } \lambda \notin -\Omega. \end{aligned} \tag{4.18}$$

Proof. If the supports of d_- and d_+ are disjoint then (4.17) implies

$$\int_\Omega\int_V (|d_-(\lambda)|^2 e^{-4\pi\langle\lambda,u\rangle} + |d_+(\lambda)|^2 e^{4\pi\langle\lambda,u\rangle})\psi(u)\,d\lambda\,du < \infty,$$

i.e. for almost every λ,

$$|d_-(\lambda)|^2 I_\psi(\lambda) + |d_+(\lambda)|^2 I_\psi(-\lambda) < \infty \tag{4.19}$$

and so $I_\psi(\lambda) < \infty$ for a.e. $\lambda \in \operatorname{supp} d_- \cup -\operatorname{supp} d_+$. This implies (4.18).

Assume now that there is an open set

$$U \subset \operatorname{supp} d_- \cap \operatorname{supp} d_+. \tag{4.20}$$

We are going to prove that then

$$\int_\Omega \psi(u)\,du < \infty, \tag{4.21}$$

which suffices to conclude (4.18). Indeed, since by (4.17),

$$\int_\Omega\int_V (|d_-(\lambda)|^2 e^{-4\pi\langle\lambda,u\rangle} + |d_+(\lambda)|^2 e^{4\pi\langle\lambda,u\rangle} + 2\Re d_-(\lambda)\overline{d^+(\lambda)})\psi(u)\,d\lambda\,du < \infty,$$

taking into account (4.21) we have (4.19) for almost every $\lambda \in \Omega$ and (4.18) follows.

Therefore it remains to prove that (4.20) implies (4.21). Let $K = \overline{\Omega} \cap \{u : |u| = 1\}$ and let $e_1, \ldots, e_n \in U$ be a basis of V such that $d_-(e_j) \neq 0$, $d_+(e_j) \neq 0$ and

$$\int_V (|d_-(e_j)|^2 e^{-4\pi\langle e_j,u\rangle} + |d_+(e_j)|^2 e^{4\pi\langle e_j,u\rangle} + 2\Re d_-(e_j)\overline{d^+(e_j)}) \times \psi(u)\,du < \infty \tag{4.22}$$

for all $j = 1, \ldots, n$. Given $\varepsilon > 0$ let

$$K_{j,\varepsilon}^+ = \{u \in K : \langle e_j, u\rangle > \varepsilon\}, \qquad K_{j,\varepsilon}^- = \{u \in K : \langle e_j, u\rangle < -\varepsilon\}.$$

We claim that there is an $\varepsilon > 0$ such that

$$K \subset \bigcup_j (K_{j,\varepsilon}^+ \cup K_{j,\varepsilon}^-). \tag{4.23}$$

Assume not. Then for every n there is $u_n \in K$ such that $|\langle e_j, u_n\rangle| \le 1/n$ for every j. Taking into account a convergent subsequence u_{n_k}, we conclude that $0 \in K$, which is not possible.

For ε satisfying (4.23) let

$$\Omega_j^+ = \{\lambda u : u \in K_{j,\varepsilon}^+,\ \lambda > 0\}, \qquad \Omega_j^- = \{\lambda u : u \in K_{j,\varepsilon}^-,\ \lambda > 0\}.$$

Clearly, $\Omega \subset \bigcup_j (\Omega_j^+ \cup \Omega_j^-)$ and so we have to prove that (4.20) implies that for every j both $\int_{\Omega_j^+} \psi(u)\,du$, $\int_{\Omega_j^-} \psi(u)\,du$ are finite.

If $u \in \Omega_j^+$, then $\langle e_j, u\rangle > \varepsilon|u|$. Therefore, if B is a ball centred at 0 and with a sufficiently large radius then

$$\tfrac{1}{2}|d_+(e_j)|^2 e^{4\pi\langle e_j,u\rangle} \ge |2\Re d_-(e_j)\overline{d_+(e_j)}|, \qquad u \in \Omega_j^+ \cap B^c.$$

Hence, by (4.22),

$$\int_{\Omega_j^+\cap B^c} \tfrac{1}{2}|d_+(e_j)|^2 e^{4\pi\langle e_j,u\rangle}\psi(u)\,du < \infty$$

and so

$$\int_{\Omega_j^+\cap B^c} \psi(u)\,du < \infty.$$

Since our assumptions on ψ imply that $\int_{\Omega\cap B} \psi(u)\,du < \infty$ for every ball B, we conclude that $\int_{\Omega_j^+} \psi(u)\,du < \infty$. For Ω_j^- we proceed in the same way. ∎

(4.24) THEOREM. *Let F be a real-valued function which satisfies* (4.6), (3.2) *and* (3.3). *Assume additionally that* (4.15) *holds. Then F is not only pluriharmonic, but also the conjugate function F' can be chosen to satisfy* (4.6). ∎

5. Pluriharmonic functions on symmetric Siegel domains of type two. Let $\mathcal{D}$ be a symmetric Siegel domain of type two. As before, we identify S with $\mathcal{D}$ by $(\zeta,x)s\cdot ie$ and the functions on S and $\mathcal{D}$, respectively. This means that if F is a function on $\mathcal{D}$, then

$$F((\zeta,x)s)=F((\zeta,x)s\cdot ie) \tag{5.1}$$

is the corresponding function on S. We use the same notation for both functions always having in mind the identification (5.1). Given $u\in\Omega$ or $s\in S_0$ let

$$F_u(\zeta,x)=F((\zeta,x)\cdot iu),$$
$$F_s(\zeta,x)=F((\zeta,x)s\cdot ie)=F((\zeta,x)\cdot(is\cdot e))=F_{s\cdot e}(\zeta,x).$$

Therefore, we may write

$$\widehat{F}(\lambda,\alpha,\beta,u)=\widehat{F}(\lambda,\alpha,\beta,s)=(U^\lambda_{F_s}\xi^\lambda_\alpha,\xi^\lambda_\beta)=(U^\lambda_{F_u}\xi^\lambda_\alpha,\xi^\lambda_\beta)\ \text{with}\ u=s\cdot e.$$

Assume that ψ is a continuous positive function on Ω that satisfies (4.15). In particular, ψ is homogeneous, i.e. $\psi(\lambda u)=\lambda^\alpha\psi(u)$, $\alpha>-n$ (see [RV]). In this section we consider functions on $\mathcal{D}$ such that

$$\int_\Omega\int_{N(\Phi)}|F((\zeta,x)\cdot iu)|^2\psi(u)\,d\zeta\,dx\,du<\infty. \tag{5.2}$$

Our main goal is to prove the following theorem.

(5.3) THEOREM. *Let F be a real-valued function on $\mathcal{D}$ which satisfies* (3.2), (3.3), (5.2) *and such that*

$$\mathcal{L}F=\sum_{\substack{j=1\\ \alpha}}^{r}\mathcal{L}^\alpha_j F=0. \tag{5.4}$$

Then F is the real part of a holomorphic function h such that

$$\int_\Omega\int_{N(\Phi)}|h((\zeta,x)\cdot iu)|^2\psi(u)\,d\zeta\,dx\,du<\infty.$$

First notice that the support of the partial Fourier transform of every F which satisfies the assumptions of Theorem (5.3) reduces to $\Omega\cup-\Omega$.

(5.5) LEMMA. *Assume that F satisfies* (3.1)–(3.3) *and* (5.2). *Then*

$$\widehat{F}(\lambda,\alpha,\beta,u)=0\quad \text{if}\ \lambda\notin\Omega\cup-\Omega.$$

Proof. The conclusion follows immediately from (5.2), Theorem (3.4) and Theorem (4.16). ∎

Let K be the subgroup of G leaving $e\in V$ invariant. In the proof of Theorem (5.3), the action of the group K on $N(\Phi)$ given by

$$k\cdot(\zeta,x)=(\sigma(k)\zeta,kx)$$

will be crucial. By (1.6), $(\zeta,x)\mapsto k\cdot(\zeta,x)$ is an automorphism of $N(\Phi)$ and moreover dk_e restricted to $\mathcal{Z}$ is orthogonal with respect to g_e. This follows from the fact that the action of K on $\mathcal{D}$ is holomorphic and from the identification

$$(\zeta,0)\mapsto(\zeta,0)\cdot ie=(\zeta,ie).$$

Let

$$\widetilde{\mathcal{L}}=\sum_{\substack{j=1\\ \alpha}}^{r}((\widetilde{\mathcal{X}}^\alpha_j)^2+(\widetilde{\mathcal{Y}}^\alpha_j)^2). \tag{5.6}$$

Since $\widetilde{\mathcal{X}}^\alpha_j,\widetilde{\mathcal{Y}}^\alpha_j$ form an orthonormal basis of $\mathcal{Z}$, the action of K commutes with $\widetilde{\mathcal{L}}$, i.e. for a function f on $N(\Phi)$, we have

$$\widetilde{\mathcal{L}}(f\cdot k)=\widetilde{\mathcal{L}}\cdot k. \tag{5.7}$$

Indeed, since k is an automorphism, $f\mapsto\widetilde{\mathcal{L}}(f\cdot k)$ defines a left-invariant operator and, since dk_e is orthogonal,

$$\widetilde{\mathcal{L}}(f\cdot k)(e)=dk_e(\widetilde{\mathcal{L}})f(e)=\widetilde{\mathcal{L}}f(e).$$

We are going to use (5.7) on the Fourier transform side. For that we need the polar coordinates in V, i.e. coordinates which fit in very well with the action of K.

Any λ in V can be written as

$$\lambda=k\lambda_0,\quad k\in K,\ \lambda_0=\sum_{j=1}^{r}\lambda_jc_j,\ \lambda_j\in\mathbb{R}. \tag{5.8}$$

The numbers $\lambda_1,\ldots,\lambda_r$ are unique provided $\lambda_1\le\ldots\le\lambda_r$, but k is not unique. Let

$$R=\Big\{\lambda=\sum_{j=1}^{r}\lambda_jc_j:\lambda_j\in\mathbb{R}\Big\},$$
$$R^+=\Big\{\lambda_0=\sum_{j=1}^{r}\lambda_jc_j:\lambda_1<\ldots<\lambda_r\Big\},$$
$$M=\{k\in K:\forall_{\lambda\in R}\ k\lambda=\lambda\}.$$

The k in (5.8) is determined up to its left M-coset (see [FK]).

We are going to use coordinates (5.8) in the sense of the following integration formula:

(5.9) THEOREM ([FK], Theorem VI.2.3). *Let f be a function integrable on V. Then*
$$\int_V f(\lambda)\,d\lambda = c_0 \int_{K\times R^+} f(k\lambda_0)\chi(\lambda_0)\,dk\,d\lambda_0,$$
where $\chi(\lambda_0)=\prod_{j<k}(\lambda_k-\lambda_j)^d$, $d=\dim V_{ij}$.

Notice that the image of $K\times R^+$ via $(k,\lambda_0)\mapsto k\lambda_0$ is of full measure in V. Theorem (5.9) provides the following Plancherel formula:

$$(5.10)\qquad \int_{K\times R^+} \|U_F^{k\lambda_0}\|^2\varrho(k\lambda_0)\chi(\lambda_0)\,dk\,d\lambda_0 = \int_V \|U_F^\lambda\|^2\varrho(\lambda)\,d\lambda = \|F\|^2_{L^2(N(\Phi))}$$

for $F\in L^2(N(\Phi))$ or, if we take into account that $\varrho(k\lambda_0)=\varrho(\lambda_0)$,

$$(5.11)\qquad \int_{K\times R^+} \|U_F^{k\lambda_0}\|^2\varrho(\lambda_0)\chi(\lambda_0)\,dk\,d\lambda_0 = \|F\|^2_{L^2(N(\Phi))},\qquad F\in L^2(N(\Phi)).$$

This formula suggests looking at the family $\mathcal{H}_{k\lambda_0}$ rather than $\mathcal{H}_\lambda$. Moreover, this approach has the advantage that we may define a basis $\xi_\alpha^{k\lambda_0}$ which depends smoothly on $k\lambda_0$ and has the property that

$$(5.12)\qquad \xi_\alpha^{k_1k\lambda_0}(\zeta)=\xi_\alpha^{k\lambda_0}(\sigma(k_1^{-1})\zeta).$$

Let ξ_α^e be the orthonormal basis of $\mathcal{H}_e$, $e=c_1+\ldots+c_r$, defined in Section 2. For $k\in K$, $\lambda_0\in R^+$ we put

$$(5.13)\qquad \xi_\alpha^{k\lambda_0}(\zeta)=\xi_\alpha^e(M^{1/2}_{|\lambda_0|}\sigma(k^{-1})\zeta)=\xi_\alpha^{\lambda_0}(\sigma(k^{-1})\zeta).$$

Then (5.12) is clearly satisfied. Moreover, since $J_{k\lambda_0}=iI$ when $k\lambda_0\in\Omega$, and $J_{k\lambda_0}=-iI$ when $k\lambda_0\in-\Omega$, $\xi_\alpha^{k\lambda_0}$ is holomorphic with respect to the appropriate holomorphic structure and by (1.6), (2.2) we have

$$\begin{aligned}(\xi_\alpha^{k\lambda_0},\xi_\beta^{k\lambda_0})_{k\lambda_0} &= \int_Z \xi_\alpha^{\lambda_0}(\sigma(k^{-1})w)\overline{\xi_\beta^{\lambda_0}(\sigma(k^{-1})w)}e^{-\pi H_{k\lambda_0}(w,w)}\varrho(\lambda_0)\,dw\\ &= \int_Z \xi_\alpha^e(M^{1/2}_{|\lambda_0|}w)\overline{\xi_\beta^e(M^{1/2}_{|\lambda_0|}w)}e^{-\pi H_{\lambda_0}(w,w)}\varrho(\lambda_0)\,dw\\ &= (\xi_\alpha^e,\xi_\beta^e)_e.\end{aligned}$$

Hence we have obtained an orthonormal basis of $\mathcal{H}_{k\lambda_0}$.

Let

$$(5.14)\qquad \Phi_{\alpha,\beta}^{k\lambda_0}(\zeta,x)=(U_{(\zeta,x)}^{k\lambda_0}\xi_\alpha^{k\lambda_0},\xi_\beta^{k\lambda_0})_{k\lambda_0}.$$

Then by (2.5) and (5.13),

$$(5.15)\qquad \Phi_{\alpha,\beta}^{k\lambda_0}=\Phi_{\alpha,\beta}^{\lambda_0}\cdot\sigma(k^{-1}).$$

Hence by (5.7) and by (2.30) of [DHMP],

$$(5.16)\qquad \widetilde{\mathcal{L}}\Phi_{\alpha,\beta}^{k\lambda_0}=(\widetilde{\mathcal{L}}\Phi_{\alpha,\beta}^{\lambda_0})\cdot\sigma(k^{-1})=-2\pi\sum_{j=1}^r|\lambda_j|(2|\alpha_j|+d)\Phi_{\alpha,\beta}^{k\lambda_0}.$$

Now,

$$(5.17)\qquad \widehat{F}(k,\lambda_0,\alpha,\beta,s)=(U_{F_s}^{k\lambda_0}\xi_\alpha^{k\lambda_0},\xi_\beta^{k\lambda_0})_{k\lambda_0}$$

is well defined. Proceeding as in Section 2 but using the "new" Plancherel formula (5.11) we are able to prove that for every α,β and almost every $(k,\lambda_0)\in K\times R^+$,
$$\widehat{F}(k,\lambda_0,\alpha,\beta,\cdot)\in C^\infty$$
and that

$$(5.18)\qquad \begin{aligned}&(-4\pi^2\langle k\lambda_0,\mathrm{Ad}_s\widetilde{X}_j\rangle^2+H_j^2-H_j)\widehat{F}(k,\lambda_0,\alpha,\beta,s)=0,\qquad j=1,\ldots,r,\\ &\sum_{\substack{i<j\\ \alpha}}\gamma_{ij}^\alpha(-4\pi^2\langle k\lambda_0,\mathrm{Ad}_s\widetilde{X}_{ij}^\alpha\rangle^2+(Y_{ij}^\alpha)^2-H_j)\widehat{F}(k,\lambda_0,\alpha,\beta,s)=0.\end{aligned}$$

At this point we fix k,λ_0,α,β such that (5.18) is satisfied. Proceeding as in Section 3 we conclude that for every α,β and almost every k,λ_0,

$$(5.19)\qquad \begin{aligned}\widehat{F}&(k,\lambda_0,\alpha,\beta,s)\\ &=d_-(k,\lambda_0,\alpha,\beta)e^{-2\pi\langle k\lambda_0,s\cdot e\rangle}+d_+(k,\lambda_0,\alpha,\beta)e^{2\pi\langle k\lambda_0,s\cdot e\rangle}.\end{aligned}$$

Let
$$\xi_0^\perp=\{\xi\in\mathcal{H}_\lambda:(\xi_0,\xi)_\lambda=0\}.$$
To conclude pluriharmonicity of F it remains to prove that $U_{F_s}^\lambda|_{\xi_0^\perp}=0$, i.e.
$$d_\pm(k,\lambda_0,\alpha,\beta)=0\quad\text{ if } \alpha\ne0.$$

We start with the following theorem.

(5.20) THEOREM. *If F satisfies (3.2), (3.3), (5.2) and (5.4), then*

$$(5.21)\qquad U_{F_e}^\lambda|_{\xi_0^\perp}=0.$$

Proof. For $f\in L^2(N(\Phi))\cap L^1(N(\Phi))$, we have

$$(5.22)\qquad (U_f^{k\lambda_0}\xi_\alpha^{k\lambda_0},\xi_\beta^{k\lambda_0})=\int_{N(\Phi)} f(\zeta,x)\Phi_{\alpha,\beta}^{k\lambda_0}(\zeta,x)\,d\zeta\,dx=:\widehat{f}(k,\lambda_0,\alpha,\beta)$$

Let $\phi_n\in C_c^\infty(N(\Phi))$ be the sequence of functions defined in the proof of Lemma (2.18). As before, we approximate $F_s(\zeta,x)$ by
$$F_s^n(\zeta,x)=F^n((\zeta,x)s)=\phi_n(\zeta,x)F_s(\zeta,x).$$

Notice that $(\mathcal{L}F)_e = \widetilde{\mathcal{L}}F_e$. By (5.22) and (5.16) we have

$$(5.23)\quad (\widetilde{\mathcal{L}}F_e^n)^\wedge(k,\lambda_0,\alpha,\beta) = \int_{N(\Phi)} F_e^n(\zeta,x)\widetilde{\mathcal{L}}\Phi_{\alpha,\beta}^{k\lambda_0}(\zeta,x)\,d\zeta\,dx$$
$$= -2\pi\sum_{j=1}^r |\lambda_j|(2|\alpha_j|+d)\widehat{F}_e^n(k,\lambda_0,\alpha,\beta).$$

On the other hand

$$(5.24)\quad (\widetilde{\mathcal{L}}F_e^n)^\wedge(k,\lambda_0,\alpha,\beta)$$
$$= \int_{N(\Phi)} \phi_n(\zeta,x)\widetilde{\mathcal{L}}F_e(\zeta,x)\Phi_{\alpha,\beta}^{k\lambda_0}(\zeta,x)\,d\zeta\,dx + I^n(k,\lambda_0,\alpha,\beta),$$

where

$$I^n(k,\lambda_0,\alpha,\beta) = ((\widetilde{\mathcal{L}}\phi_n)F_e)^\wedge(k,\lambda_0,\alpha,\beta)$$
$$+ 2\sum_{\substack{j=1\\ \alpha}}^r ((\widetilde{\mathcal{X}}_j^\alpha\phi_n)(\widetilde{\mathcal{X}}_j^\alpha F_e))^\wedge(k,\lambda_0,\alpha,\beta)$$
$$+ 2\sum_{\substack{j=1\\ \alpha}}^r ((\widetilde{\mathcal{Y}}_j^\alpha\phi_n)(\widetilde{\mathcal{Y}}_j^\alpha F_e))^\wedge(k,\lambda_0,\alpha,\beta).$$

But, by (1.9) and (5.4),

$$(5.25)\quad \widetilde{\mathcal{L}}F_e(\zeta,x) = d\sum_{j=1}^r (H_jF)_e(\zeta,x)$$
$$= d\sum_{j=1}^r (\partial_{a_j}F)((\zeta,x)s)|_{s=e} = d\sum_{j=1}^r (\partial_{a_j}F)_e(\zeta,x).$$

Hence putting (5.23)–(5.25) together, we obtain

$$-2\pi\sum_{j=1}^r |\lambda_j|(2|\alpha_j|+d)\widehat{F}_e^n(k,\lambda_0,\alpha,\beta)$$
$$= d\sum_{j=1}^r \int_{N(\Phi)} \phi_n(\zeta,x)(\partial_{a_j}F)_e(\zeta,x)\Phi_{\alpha,\beta}^{k\lambda_0}(\zeta,x)\,d\zeta\,dx + I^n(k,\lambda_0,\alpha,\beta).$$

Now letting $n\to\infty$ we have

$$(5.26)\quad -2\pi\sum_{j=1}^r |\lambda_j|(2|\alpha_j|+d)\widehat{F}(k,\lambda_0,\alpha,\beta,e)$$
$$= d\sum_{j=1}^r ((\partial_{a_j}F)_e)^\wedge(k,\lambda_0,\alpha,\beta) = d\sum_{j=1}^r \partial_{a_j}\widehat{F}(k,\lambda_0,\alpha,\beta,s)|_{s=e}.$$

To obtain (5.26) we make use of the Harnack inequality. Let $D=\mathcal{X}_j^\alpha,\mathcal{Y}_j^\alpha,H_j$, $\widetilde{D}=\widetilde{\mathcal{X}}_j^\alpha,\widetilde{\mathcal{Y}}_j^\alpha$ and let B be a neighbourhood of e in S. First, by the Harnack inequality (c denotes various constants),

$$|DF(\zeta,x)| \le c\int_B |F((\zeta,x)(\zeta_1,x_1)s)|\,d\zeta_1\,dx_1\,ds$$
$$\le c\Big(\int_B |F((\zeta,x)(\zeta_1,x_1)s)|^2\,d\zeta_1\,dx_1\,ds\Big)^{1/2}.$$

Secondly,

$$\int_{N(\Phi)} |DF(\zeta,x)|^2\,d\zeta\,dx \le c\int_B\int_{N(\Phi)} |F((\zeta,x)(\zeta_1,x_1)s)|^2\,d\zeta\,dx\,d\zeta_1\,dx_1\,ds < \infty$$

and

$$|\phi_n(\zeta,x)DF(\zeta,x)| \le |DF(\zeta,x)|.$$

Finally, ϕ_n, $\widetilde{D}\phi_n$, $\widetilde{\mathcal{L}}\phi_n$ are bounded independently of n and

$$\widetilde{D}\phi_n,\widetilde{\mathcal{L}}\phi_n\to 0 \quad \text{as } n\to\infty.$$

Therefore, the Plancherel formula (5.11) implies that for a subsequence n_p we have

$$\lim_{n_p\to\infty} I^{n_p}(k,\lambda_0,\alpha,\beta) = 0,$$
$$\lim_{n_p\to\infty} \widehat{F}_e^{n_p}(k,\lambda_0,\alpha,\beta) = \widehat{F}(k,\lambda_0,\alpha,\beta,e),$$
$$\lim_{n_p\to\infty} (\phi_{n_p}(\partial_{a_j}F)_e)^\wedge(k,\lambda_0,\alpha,\beta) = ((\partial_{a_j}F)_e)^\wedge(k,\lambda_0,\alpha,\beta).$$

In view of (2.15) (proved in the context of the Plancherel formula (5.11)), we have

$$((\partial_{a_j}F)_e)^\wedge(k,\lambda_0,\alpha,\beta) = \partial_{a_j}\widehat{F}(k,\lambda_0,\alpha,\beta,s)|_{s=e},$$

which proves (5.26).

Now we make use of (5.19) substituting the formula for $\widehat{F}(k,\lambda_0,\alpha,\beta,s)$ into (5.26). Since

$$2\pi\langle k\lambda_0, s\cdot e\rangle = \sum_{j=1}^r W_j(k\lambda_0,y)a_j,$$

we have

$$\sum_{j=1}^r \partial_{a_j}\widehat{F}(k,\lambda_0,\alpha,\beta,s)|_{s=e} = -\Big(\sum_{j=1}^r W_j(k\lambda_0,e)\Big)d_-(k,\lambda_0,\alpha,\beta)e^{-2\pi\langle k\lambda_0,e\rangle}$$
$$+\Big(\sum_{j=1}^r W_j(k\lambda_0,e)\Big)d_+(k,\lambda_0,\alpha,\beta)e^{2\pi\langle k\lambda_0,e\rangle}.$$

But

$$\sum_{j=1}^{r} W_j(k\lambda_0, e) = 2\pi \sum_{j=1}^{r} \langle k\lambda_0, \widetilde{X}_j\rangle = 2\pi\langle k\lambda_0, e\rangle = 2\pi\langle \lambda_0, e\rangle = 2\pi \sum_{j=1}^{r} \lambda_j.$$

Taking into account the supports of d_- and d_+, for $\lambda_0 \in \Omega \cup -\Omega$ we obtain

$$2\pi\Big(\sum_{j=1}^{r} |\lambda_j|(2|\alpha_j| + d)\Big) d_\pm(k, \lambda_0, \alpha, \beta) = 2\pi d\Big(\sum_{j=1}^{r} |\lambda_j|\Big) d_\pm(k, \lambda_0, \alpha, \beta),$$

which shows that $\alpha_j = 0$. This proves (5.21). ■

Proof of Theorem (5.3). The first step is to show that for $\lambda \in \Omega \cup -\Omega$ and every $s \in S_0$,

$$(5.27) \qquad U^\lambda_{F_s}|_{\xi_0^\perp} = 0.$$

Given $s \in S_0$, let

$$(5.28) \qquad F'((\zeta, x)w) = F(s(\zeta, x)w).$$

Since all the operators are left-invariant, F' satisfies (3.2), (3.3) and (5.4). Moreover,

$$\int_\Omega \int_{N(\Phi)} |F'((\zeta, x)\cdot iu)|^2 \psi(s\cdot u)\, d\zeta\, dx\, du$$
$$= \int_\Omega \int_{N(\Phi)} |F((\sigma(s)\zeta, sx)\cdot isu)|^2 \psi(s\cdot u)\, d\zeta\, dx\, du$$
$$= \det \sigma(s)^{-1} (\det s^{-1})^2 \int_\Omega \int_{N(\Phi)} |F((\zeta, x)\cdot iu)|^2 \psi(u)\, d\zeta\, dx\, du.$$

It follows that F' satisfies (5.2) with $\psi_s(u) = \psi(su)$. Notice that $I_{\psi_s}(\lambda) = I_\psi((s^{-1})^*\lambda) \det s^{-1}$. Indeed,

$$\int_\Omega \psi(su) e^{-2\pi\langle\lambda, u\rangle}\, du = \det s^{-1} \int_\Omega \psi(u) e^{-2\pi\langle\lambda, s^{-1}u\rangle}\, du.$$

So $V_{\psi_s} \subset \Omega$. All this shows that we may apply Theorem (5.20) to F' to conclude that

$$U^\lambda_{F'_e}|_{\xi_0^\perp} = 0.$$

But, since $F_s = F'_e \cdot \mathrm{Ad}_{s^{-1}}$ by (5.28), $U^\lambda_{F_s}$ can be expressed through $U^{s^*\lambda}_{F'_e}$. More precisely,

$$(5.29) \qquad (U^\lambda_{F_s}\xi, \eta)_\lambda = (U^{s^*\lambda}_{F'_e} s\cdot\xi, s\cdot\eta)_{s^*\lambda}, \qquad \xi, \eta \in \mathcal{H}_\lambda,$$

where

$$(5.30) \qquad s\cdot\xi(\zeta) = \xi(\sigma(s)\zeta).$$

Let us prove (5.29). First, by (2.2), $M_{s^*\lambda} = \sigma(s)^* M_\lambda \sigma(s)$ and so

$$(5.31) \qquad \varrho(s^*\lambda) = |\det_{\mathbb{C}} M_{s^*\lambda}| = |\det_{\mathbb{C}} \sigma(s)|^2 |\det_{\mathbb{C}} M_\lambda| = \det_{\mathbb{R}} \sigma(s) \varrho(\lambda).$$

Moreover, inside $\Omega \cup -\Omega$ the action of s does not change the complex structure, i.e. $J_\lambda = J_{s^*\lambda}$. Therefore, in view of (5.31), we have

$$(s\cdot\xi, s\cdot\eta)_{s^*\lambda} = \int_Z \xi(\sigma(s)w)\overline{\eta(\sigma(s)w)} e^{-\pi H_{s^*\lambda}(w,w)} \varrho(s^*\lambda)\, dw$$
$$= \int_Z \xi(w)\overline{\eta(w)} e^{-\pi H_\lambda(w,w)} \det_{\mathbb{R}} \sigma(s^{-1}) \varrho(s^*\lambda)\, dw = (\xi, \eta)_\lambda,$$

i.e. the action (5.30) is an isometry. In particular, it maps $\xi_0^\perp$ in $\mathcal{H}_\lambda$ onto $\xi_0^\perp$ in $\mathcal{H}_{s^*\lambda}$. Moreover, by (2.5),

$$U^\lambda_{(\sigma(s)\zeta, sx)}\xi(w)$$
$$= e^{-2\pi i\langle\lambda, sx\rangle - (\pi/2)H_\lambda(\sigma(s)\zeta, \sigma(s)\zeta) + \pi H_\lambda(w, \sigma(s)\zeta)} \xi(w - \sigma(s)\zeta)$$
$$= e^{-2\pi i\langle s^*\lambda, x\rangle - (\pi/2)H_{s^*\lambda}(\zeta,\zeta) + \pi H_{s^*\lambda}(\sigma(s^{-1})w, \zeta)} (s\cdot\xi)(\sigma(s^{-1})w - \zeta).$$

Hence

$$(U^\lambda_{(\sigma(s)\zeta, sx)}\xi, \eta)_\lambda = (U^{s^*\lambda}_{(\zeta,x)} s\cdot\xi, s\cdot\eta)_{s^*\lambda},$$

which implies (5.29) and (5.27).

To prove pluriharmonicity of F we use the Fourier inversion formula which by (5.27) becomes

$$F_u(\zeta, x) = \int_{\overline{\Omega}\cup-\overline{\Omega}} \mathrm{Tr}(U^\lambda_{(-\zeta,-x)} U^\lambda_{F_u}) \varrho(\lambda)\, d\lambda$$
$$= \int_{\overline{\Omega}\cup-\overline{\Omega}} (U^\lambda_{F_u}\xi_0, U^\lambda_{(\zeta,x)}\xi_0) \varrho(\lambda)\, d\lambda$$
$$= \int_{\overline{\Omega}} \sum_\beta d_-(0, \beta, \lambda) e^{-2\pi\langle\lambda, u\rangle} (\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0) \varrho(\lambda)\, d\lambda$$
$$+ \int_{-\overline{\Omega}} \sum_\beta d_+(0, \beta, \lambda) e^{2\pi\langle\lambda, u\rangle} (\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0) \varrho(\lambda)\, d\lambda.$$

The first integral gives a holomorphic function and the second its conjugate. Indeed, in view of (2.5), there is a holomorphic function h_β on $\mathcal{D}$ such that

$$(\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0) e^{-2\pi\langle\lambda, u\rangle} = h_\beta((\zeta, x)\cdot iu).$$

Moreover,

$$d_+(0, \beta, \lambda) e^{2\pi\langle\lambda, u\rangle} (\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0) = \overline{d_-(0, \beta, \lambda) e^{-2\pi\langle\lambda, u\rangle} (\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0)}.$$

So it is sufficient to prove that

$$(5.32) \qquad \int_{\overline{\Omega}} \sum_\beta |d_-(0, \beta, \lambda)| \cdot |(\xi_\beta, U^\lambda_{(\zeta,x)}\xi_0)| e^{-2\pi\langle\lambda, u\rangle} \varrho(\lambda)\, d\lambda < \infty.$$

We estimate (5.32) by

$$\int_{\overline{\Omega}}\Big(\sum_{\beta}|d_-(0,\beta,\lambda)|^2e^{-2\pi\langle\lambda,u\rangle}\Big)^{1/2}\Big(\sum_{\beta}|(\xi_\beta,U^\lambda_{(\zeta,x)}\xi_0)|^2\Big)^{1/2}e^{-\pi\langle\lambda,u\rangle}\varrho(\lambda)\,d\lambda$$
$$=\int_{\overline{\Omega}}\|U^\lambda_{F_{\frac12 u}}\xi_0\|\cdot\|U^\lambda_{(\zeta,x)}\xi_0\|e^{-\pi\langle\lambda,u\rangle}\varrho(\lambda)\,d\lambda.$$

The last integral is dominated by

$$\Big(\int_{\overline{\Omega}}\|U^\lambda_{F_{\frac12 u}}\xi_0\|^2\varrho(\lambda)\,d\lambda\Big)^{1/2}\Big(\int_{\overline{\Omega}}e^{-2\pi\langle\lambda,u\rangle}\varrho(\lambda)\,d\lambda\Big)^{1/2}$$
$$\le\|F_{\frac12 u}\|_{L^2}\Big(\int_{\overline{\Omega}}e^{-2\pi\langle\lambda,u\rangle}\varrho(\lambda)\,d\lambda\Big)^{1/2},$$

which gives (5.32). ∎

6. Appendix. Proof of Theorem (3.4) for the cone of 2×2 positive definite symmetric matrices. Let $\widehat{F}(\lambda,s)$ be the partial Fourier transform of F along V, let $\lambda=\mathrm{Ad}^*_{y_0}\lambda_0$, $\lambda_0\in V_{11}\oplus V_{22}$ and let $f(\lambda,s)=\widehat{F}(\lambda,y_0^{-1}s)$. Then $f(\lambda,s)$ satisfies three equations:

$$\text{(6.1)}\qquad \begin{aligned}&(-4\pi^2\langle\lambda_0,\mathrm{Ad}_s\widetilde{X}_j\rangle^2+H_j^2-H_j)f(\lambda,s)=0,\qquad j=1,2,\\&(-4\pi^2\langle\lambda_0,\mathrm{Ad}_s\widetilde{X}_{12}\rangle^2+(Y_{12})^2-H_2)f(\lambda,s)=0,\end{aligned}$$

with $\lambda_0=\lambda_1c_1+\lambda_2c_2$, $\lambda_j\ne 0$. Let

$$W_j(\lambda_0,y)=2\pi\langle\lambda_0,\mathrm{Ad}_y\widetilde{X}_j\rangle,\qquad j=1,2,$$
$$W_{12}(\lambda_0,y)=2\pi\langle\lambda_0,\mathrm{Ad}_y\widetilde{X}_{12}\rangle.$$

Then a direct calculation shows that

$$\text{(6.2)}\quad W_1(\lambda_0,y)=\lambda_1+\tfrac12\lambda_2y^2,\qquad W_2(\lambda_0,y)=\lambda_2,\qquad W_{12}(\lambda_0,y)=\lambda_2y.$$

Hence $\partial_yW_1=W_{12}$ and $\partial_yW_{12}=W_2$. Moreover,

$$X_j=a_j\partial_{x_j},\qquad X_{12}=a_1^{1/2}a_2^{1/2}\partial_{x_{12}},$$
$$H_j=a_j\partial_{a_j},\qquad Y_{12}=a_1^{-1/2}a_2^{1/2}\partial_y,$$

and so (6.1) becomes

$$\text{(6.3)}\qquad (-W_j(\lambda_0,y)^2+\partial^2_{a_j})f(\lambda,s)=0,\qquad j=1,2,$$

$$\text{(6.4)}\qquad (-a_1W_{12}(\lambda_0,y)^2+a_1^{-1}\partial_y^2-\partial_{a_2})f(\lambda,s)=0.$$

Given $\lambda_0=\lambda_1c_1+\lambda_2c_2$ there is a neighbourhood U of 0 in $N=\mathbb{R}$ such that $W_1(\lambda_0,y)$ is either strictly positive or strictly negative for $y\in U$. Solving equation (6.3) on U we get

$$f(\lambda,ya_1a_2)=\sum_{\substack{\delta=\delta_1,\delta_2\\ \delta_j=\pm1}}d_\delta(\lambda,y)e^{\sum_{j=1}^2\delta_jW_j(\lambda_0,y)a_j}$$

for $y\in U$ and arbitrary a_1,a_2. Moreover, $d_{\delta_1,\delta_2}(\lambda,y)$ is smooth with respect to $y\in U$. Now we apply (6.4) to get

$$\text{(6.5)}\qquad \begin{aligned}\sum_\delta(a_1^{-1}\partial_y^2d_\delta(\lambda,y)&+2\delta_1\partial_yd_\delta(\lambda,y)W_{12}(\lambda_0,y)\\&+\delta_1d_\delta(\lambda,y)W_2(\lambda_0,y)\\&-\delta_2d_\delta(\lambda,y)W_2(\lambda_0,y))e^{\sum_{j=1}^2\delta_jW_j(\lambda_0,y)a_j}=0,\qquad y\in U.\end{aligned}$$

But since a_1,a_2 are arbitrary and $W_j(\lambda_0,y)$ has constant sign on U, letting first $a_2\to\infty$, then $a_1\to\infty$, we conclude that

$$\text{(6.6)}\qquad \partial_y^2d_\delta(\lambda,y)=0$$

and

$$\text{(6.7)}\qquad 2\delta_1\partial_yd_\delta(\lambda,y)W_{12}(\lambda_0,y)=(\delta_2-\delta_1)d_\delta(\lambda,y)W_2(\lambda_0,y).$$

Now, if $\delta_1=\delta_2$ we have $\partial_yd_\delta(\lambda,y)y=0$, i.e. $d_\delta(\lambda,y)=d_\delta(\lambda,0)$. If $\delta_1=-\delta_2$, then (6.6) and (6.7) become

$$\partial_y^2d_\delta(\lambda,y)=0,\qquad \partial_yd_\delta(\lambda,y)y=-d_\delta(\lambda_0,y),$$

hence d_δ must be 0. Therefore,

$$\text{(6.8)}\qquad f(\lambda,ya)=d_{(1,1)}(\lambda)e^{\sum_{j=1}^2a_jW_j(\lambda_0,y)}+d_{(-1,-1)}(\lambda)e^{-\sum_{j=1}^2a_jW_j(\lambda_0,y)}$$

for $y\in U$ and arbitrary a. But the right hand side of (6.8) is well defined on NA and satisfies (6.3) and (6.4). Hence (6.8) holds for every $y\in N$.

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Stochastic representation of reflecting diffusions corresponding to divergence form operators

by

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Abstract. We obtain a stochastic representation of a diffusion corresponding to a uniformly elliptic divergence form operator with co-normal reflection at the boundary of a bounded C^2-domain. We also show that the diffusion is a Dirichlet process for each starting point inside the domain.

0. Introduction and notation. Let D be the following non-empty bounded domain in $\mathbb{R}^d$:

$$D = \{x \in \mathbb{R}^d : \Phi(x) > 0\} \quad \text{with} \quad \partial D = \{x \in \mathbb{R}^d : \Phi(x) = 0\}, \tag{0.1}$$

where $\Phi \in C_b^2(\mathbb{R}^d)$ satisfies $|\nabla\Phi(x)| \geq 1$ for all $x \in \partial D$, and let $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ belong to the class $\mathcal{A}(\lambda, \Lambda)$ of all measurable, symmetric matrix-valued functions which satisfy the ellipticity condition

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x, \xi \in \mathbb{R}^d \tag{0.2}$$

for some $0 < \lambda \leq \Lambda$ (we employ the summation convention over repeated indices). Consider the operator

$$A = D_j\big(\tfrac{1}{2}a^{ij}(\cdot)D_i\big)$$

and let p be a weak Neumann function for A on D (see Section 2). Using the estimates on p proved in Gushchin [13] we first construct a family $\{P^x : x \in D\}$ of probability measures on $C([0,T];\overline{D})$ such that the finite-dimensional distributions of P^x are determined by p and then we investigate the structure of the canonical process X under the measures P^x.

More precisely, let γ_a denote the co-normal vector field on ∂D, i.e. $\gamma_a^i(x) = (1/2)a^{ij}(x)\boldsymbol{n}_j(x)$ for $i = 1, \ldots, d$, where $\boldsymbol{n}(x) = \nabla\Phi(x)/|\nabla\Phi(x)|$ is the unit inward normal to ∂D. We prove that X is a Dirichlet process in the sense of Föllmer [5] under P^x for every $x \in D$ and its components admit

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