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On operator bands

by

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Abstract. A multiplicative semigroup of idempotent operators is called an operator band. We prove that for each $K > 1$ there exists an irreducible operator band on the Hilbert space l^2 which is norm-bounded by K. This implies that there exists an irreducible operator band on a Banach space such that each member has operator norm equal to 1.

Given a positive integer r, we introduce a notion of weak r-transitivity of a set of bounded operators on a Banach space. We construct an operator band on l^2 that is weakly r-transitive and is not weakly $(r+1)$-transitive.

We also study operator bands $\mathcal{S}$ satisfying a polynomial identity $p(A,B) = 0$ for all non-zero $A, B \in \mathcal{S}$, where p is a given polynomial in two non-commuting variables. It turns out that the polynomial $p(A,B) = (AB - BA)^2$ has a special role in these considerations.

1. Introduction. Let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on a (real or complex) Banach space X. A subset $\mathcal{S}$ of $\mathcal{B}(X)$ is said to be *irreducible* if the only closed subspaces of X invariant under all members of $\mathcal{S}$ are $\{0\}$ and X. Otherwise, $\mathcal{S}$ is called *reducible*. A set $\mathcal{S}$ of $\mathcal{B}(X)$ is said to be *triangularizable* if there is a chain of closed subspaces that are invariant under every member of $\mathcal{S}$ and this chain is maximal in the lattice of all closed subspaces of X.

An operator A on a vector space V is called *idempotent* if $A^2 = A$. A semigroup $\mathcal{S}$ of idempotents on V is called an *operator band.* If V is a Banach space, we also assume that all operators in $\mathcal{S}$ are bounded. Reducibility of operator bands on Hilbert spaces has recently been studied in [2], [5], and [6]. In [2] an irreducible operator band on the Hilbert space l^2 has been constructed. After having such an example it is natural to ask about the existence of irreducible operator bands with some additional properties. Sections 2 and 3 are devoted to this question. In Section 2 we construct an irreducible operator band on l^2 which is norm-bounded. This implies that

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there exist an irreducible operator band $\mathcal{S}$ on l^2 and an equivalent norm on l^2 with respect to which each member of $\mathcal{S}$ has operator norm equal to 1. In Section 3 we introduce a notion of weak r-transitivity of a set of bounded operators on a Banach space, where r is a given positive integer. We construct an operator band on l^2 that is weakly r-transitive and is not weakly $(r+1)$-transitive.

In [6] it is shown that every operator band $\mathcal{S}$ on a Hilbert space satisfying $(AB-BA)^2=0$ for all $A,B\in\mathcal{S}$ is triangularizable. This result motivates the study of operator bands $\mathcal{S}$ satisfying a polynomial identity $p(A,B)=0$ for all non-zero $A,B\in\mathcal{S}$, where p is a given polynomial in two non-commuting variables. The results of Section 4 show that the polynomial $p(A,B)=(AB-BA)^2$ has a special role in these considerations.

A reference for what follows is [7]. It should be noted that the definitions and remarks below are not needed to understand Theorems 2.2 and 2.3 and their proofs.

Define a relation $\preceq$ on an operator band $\mathcal{S}$ by

$$A\preceq B \Leftrightarrow ABA=A.$$

Then $\preceq$ is a pre-order on $\mathcal{S}$ (it is reflexive and transitive). This (in fact, each) pre-order determines an equivalence relation $\sim$ on $\mathcal{S}$ by

$$A\sim B \Leftrightarrow A\preceq B \text{ and } B\preceq A.$$

Let $\mathcal{C}_A$ denote the equivalence class of $A\in\mathcal{S}$. Then $\mathcal{C}_A$ is a subband of $\mathcal{S}$. We refer to the equivalence classes as *components* of $\mathcal{S}$. Define the multiplication on the set $\mathcal{S}/\sim$ of all components of $\mathcal{S}$ by

$$\mathcal{C}_A\mathcal{C}_B=\mathcal{C}_{AB}.$$

This operation is well defined and $\mathcal{S}/\sim$ is an abelian band under it. The band pre-order on $\mathcal{S}/\sim$ is a partial order. We denote it by $\leq$. It is easy to see that

$$\mathcal{C}_A\leq\mathcal{C}_B \Leftrightarrow A\preceq B.$$

An *ideal* of a semigroup $\mathcal{S}$ is a subset of $\mathcal{S}$ which is closed under right and left multiplications by elements of $\mathcal{S}$. An ideal generated by one element of $\mathcal{S}$ is said to be a *principal ideal.* A *principal-ideal band* is a band with identity in which every ideal is principal. Principal-ideal matrix bands have been studied in [3].

2. Norm-bounded irreducible operator bands. In [2] an irreducible operator band on the Hilbert space l^2 has been constructed. Essentially, this construction is based upon the following operators on l^2.

Given $k\times k$ matrices A and B, let $P_{A,B}$ be the $3k\times 3k$ matrix

$$P_{A,B}=\begin{bmatrix}A\\A\\I\end{bmatrix}\begin{bmatrix}B & -B & I\end{bmatrix}=\begin{bmatrix}AB & -AB & A\\AB & -AB & A\\B & -B & I\end{bmatrix},$$

where I is the identity matrix of order k. Let $T_{A,B}$ be the infinite block-diagonal matrix

$$T_{A,B}=\operatorname{diag}\{D_0,D_1,D_2,\ldots\},$$

where the block D_i is equal to $P_{A,B}$ if the number i is representable in the ternary system by 0's and 1's only, and D_i equals the identity matrix of order $3k$ otherwise. We regard $T_{A,B}$ as an operator on l^2. One readily shows that

$$\|T_{A,B}\|^2\leq 4\|AB\|^2+2\|A\|^2+2\|B\|^2+1\leq(2\|A\|^2+1)(2\|B\|^2+1). \tag{1}$$

It is easy to see that $T_{A,B}T_{C,D}=T_{A,D}$ for all $k\times k$ matrices A,B,C and D. In particular, $T_{A,B}^2=T_{A,B}$.

After the publication of [2], M. D. Choi posed the question of whether there exists an irreducible operator band on a Hilbert space that is also norm bounded. This problem can be reformulated in the following way:

PROBLEM 2.1. *Let $\mathcal{S}$ be an operator band on a Hilbert space such that for some $K\geq 1$ we have $\|S\|\leq K$ for all $S\in\mathcal{S}$. Is $\mathcal{S}$ necessarily reducible?*

If we also assume that $K=1$, then every member of $\mathcal{S}$ is Hermitian (see e.g. [1, Proposition 3.3]). In this case we have $ST=(ST)^*=T^*S^*=TS$ for all $S,T\in\mathcal{S}$, so that $\mathcal{S}$ is a commutative band, and hence reducible. However, for $K>1$ the following result holds.

THEOREM 2.2. *Let $K>1$. Then there exists an operator band $\mathcal{S}$ on the Hilbert space l^2 such that*

(a) $\|S\|\leq K$ *for all* $S\in\mathcal{S}$,
(b) *the semigroup* $\mathbb{R}^+\mathcal{S}:=\{\lambda S:\lambda>0,\ S\in\mathcal{S}\}$ *is weakly dense in* $\mathcal{B}(l^2)$, *and so $\mathcal{S}$ is irreducible.*

Proof. Let $d:=\sqrt{(K-1)/2}$ and $c:=d/K$. For each positive integer n let $\mathcal{S}_n$ denote the set of all operators $T_{A,B}$ as A and B range over all $3^n\times 3^n$ matrices with norm at most c, and let $\mathcal{T}_n$ denote the set of all operators $T_{A,B}$ as A and B range over all $3^n\times 3^n$ matrices with norm at most d. It is obvious that $\mathcal{S}_n$ and $\mathcal{T}_n$ are both operator bands satisfying $\mathcal{S}_n\subset\mathcal{T}_n$. Furthermore, by (1) we have $\|T_{A,B}\|\leq 2d^2+1=K$ for all $T_{A,B}\in\mathcal{T}_n$.

We shall prove that $\mathcal{S}_n\mathcal{T}_m\subseteq\mathcal{T}_n$ and $\mathcal{T}_m\mathcal{S}_n\subseteq\mathcal{T}_n$ for all positive integers m and n with $m<n$. Pick $T_{A,B}\in\mathcal{S}_n$ and $S\in\mathcal{T}_m$. Then there exists a $3^n\times 3^n$ matrix M with norm at most K such that

$$S=\operatorname{diag}\{C_0,C_1,C_2,\ldots\},$$

where the block C_i is equal to M if the number i is representable in the ternary system by 0's and 1's only, and C_i equals the identity matrix of order 3^n otherwise. From

$$\begin{bmatrix} A \\ A \\ I \end{bmatrix} [B \quad -B \quad I] \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A \\ A \\ I \end{bmatrix} [BM \quad -BM \quad I]$$

it follows that $T_{A,B}S = T_{A,BM}$. Since $\|BM\| \le \|B\|\,\|M\| \le cK = d$, we conclude that $T_{A,BM} \in \mathcal{T}_n$. This completes the proof of the inclusion $\mathcal{S}_n\mathcal{T}_m \subseteq \mathcal{T}_n$. The proof of the other inclusion is similar.

Now let $\mathcal{S}$ be the semigroup generated by the union $\bigcup_{n=1}^{\infty} \mathcal{S}_n$. We claim that $\mathcal{S}$ is an operator band and that (a) holds. To this end, pick $S \in \mathcal{S}$. Then S is a finite product of some members of $\bigcup_{n=1}^{\infty} \mathcal{S}_n$. Let p be the smallest integer such that these members belong to the finite union $\bigcup_{n=1}^{p} \mathcal{S}_n$. Using the above inclusions and the facts that $\mathcal{S}_n$ and $\mathcal{T}_n$ are semigroups, we easily conclude that $S \in \mathcal{T}_p$. Therefore S is an idempotent with norm at most K.

In order to prove (b) we consider $T \in \mathcal{B}(l^2)$ and $x, y \in l^2$. There is no loss of generality in assuming that $\|T\| \le c^2$. For each $n \in \mathbb{N}$ there exists a $3^n \times 3^n$ matrix A_n with norm at most c such that the operators T and $T_n := T_{A_n,cI} \in \mathcal{S}_n \subseteq \mathcal{S}$ have the same upper-left $3^n \times 3^n$ corner. The rest of the proof goes along the lines of the last part of the proof from [2]. The weak density of $\mathbb{R}^+\mathcal{S}$ also implies that $\mathcal{S}$ is irreducible. ∎

In the case of Banach spaces we have the following:

THEOREM 2.3. *There exist an irreducible operator band $\mathcal{S}$ on l^2 and an equivalent norm on l^2 with respect to which each member of $\mathcal{S}$ has operator norm equal to* 1.

Proof. It is well known and easily shown that for each bounded semigroup $\mathcal{S}$ of operators on a Banach space containing the identity, we can define an equivalent norm on the Banach space by

$$\|x\|' = \sup\{\|Sx\| : S \in \mathcal{S}\},$$

with respect to which every member of $\mathcal{S}$ has norm at most 1. Now take for $\mathcal{S}$ any operator band obtained in Theorem 2.2, and adjoin the identity to it. ∎

Theorem 2.2 (and therefore Theorem 2.3 as well) can be improved as follows.

THEOREM 2.4. *Let $K > 1$. Then there exists an operator band $\mathcal{P}$ on l^2 such that*

(a) *$\mathcal{P}$ is norm-bounded by K,*
(b) *$\mathbb{R}^+\mathcal{P}$ is weakly dense in $\mathcal{B}(l^2)$ (and so $\mathcal{P}$ is irreducible),*

(c) *$\mathcal{P}$ is a principal-ideal band with countably many elements, and each component of $\mathcal{P}$ is finite.*

Proof. Let us use the notation from the proof of Theorem 2.2. For each positive integer n let $\{W_{nk}\}_{k\in\mathbb{N}}$ be a sequence of $3^n \times 3^n$ matrices that is dense in the ball of all $3^n \times 3^n$ matrices of norm at most c. Furthermore, let P_n be the natural embedding of $\mathbb{C}^{3^n}$ into l^2. Define the double sequence $\{Z_{nk}\}_{n,k\in\mathbb{N}}$ of bounded operators on l^2 by $Z_{nk} = P_n W_{nk} P_n^*$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a renumbering of the terms of the sequence

$$Z_{11}, Z_{12}, Z_{21}, Z_{13}, Z_{22}, Z_{31}, Z_{14}, Z_{23}, Z_{32}, Z_{41}, \ldots$$

Define an increasing sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of semigroups inductively. Let $\mathcal{P}_1$ be the operator band on l^2 generated by $T_{P_1^*A_1P_1,cI}$ and the identity, and let $\mathcal{P}_n$ be the semigroup generated by $\mathcal{P}_{n-1}$ and the operators $T_{P_n^*A_1P_n,cI}$, $T_{P_n^*A_2P_n,cI}, \ldots, T_{P_n^*A_nP_n,cI}$. Since $\mathcal{S}_n\mathcal{T}_m \subseteq \mathcal{T}_n$ and $\mathcal{T}_m\mathcal{S}_n \subseteq \mathcal{T}_n$ for all positive integers m and n with $m < n$, we conclude that $\mathcal{P}_n \subseteq \{I\} \cup \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n$, and so the semigroup $\mathcal{P}_n$ is an operator band. By the famous theorem of Green and Rees [4] every finitely generated band is finite, so that $\mathcal{P}_n$ has finitely many elements. Note that for each $n \in \mathbb{N}$,

$$T_{P_n^*A_1P_n,cI} \sim T_{P_n^*A_2P_n,cI} \sim \ldots \sim T_{P_n^*A_nP_n,cI},$$

and

$$T_{P_n^*A_iP_n,cI} \cdot T_{P_m^*A_jP_m,cI} \sim T_{P_n^*A_iP_n,cI} \sim T_{P_m^*A_jP_m,cI} \cdot T_{P_n^*A_iP_n,cI}$$

for all $n > m$ and $i, j \in \mathbb{N}$. (See the proof of the inclusions $\mathcal{S}_n\mathcal{T}_m \subseteq \mathcal{T}_n$ and $\mathcal{T}_m\mathcal{S}_n \subseteq \mathcal{T}_n$ for $n > m$.) It follows that $\mathcal{P}_n$ gains only one component in addition to those which make up $\mathcal{P}_{n-1}$, and this component is the smallest one (with respect to $\le$) of $\mathcal{P}_n$. In particular, $\mathcal{P}_n$ is a principal-ideal band. Then $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ is a countable principal-ideal band with finite components, and so (c) holds. Since $\mathcal{P} \setminus \{I\}$ is contained the union of all $\mathcal{T}_n$, $\mathcal{P}$ is norm-bounded by K.

For each positive integer n define $Q_n = P_n P_n^*$. For the proof of (b) it is enough to show that each $T \in \mathcal{B}(l^2)$ satisfying $Q_m T Q_m = T$ for some m is in the weak closure of $\mathbb{R}^+\mathcal{P}$, because the set of such operators is weakly dense in $\mathcal{B}(l^2)$. Fix $x, y \in l^2$, and $1 > \varepsilon > 0$. We may assume that $\|T\| \le c^2$. Then there exists $j \ge m$ such that for each $n \ge j$ the operator $T_n := T_{P_n^*A_jP_n,cI} \in \mathcal{P}_n$ satisfies the estimate $\|Q_n(T - T_n)Q_n\| \le \varepsilon$. (Note that $Q_nT_nQ_n = cA_j$.) Decompose l^2 into the direct sum of the range and the kernel of Q_n. Then the matrix of $T - T_n$ is of the form

$$\begin{bmatrix} E_n & U_n \\ V_n & W_n \end{bmatrix},$$

where the norm of the $3^n \times 3^n$ matrix E_n is at most ε. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with respect to the same decomposition of the space l^2, we have

$$|\langle (T-T_n)x,y\rangle| \le \|E_n\|\,\|x_1\|\,\|y_1\| + \|U_n\|\,\|x_2\|\,\|y_1\| + \|V_n\|\,\|x_1\|\,\|y_2\| + \|W_n\|\,\|x_2\|\,\|y_2\|.$$

Note that by (1),

$$\max\{\|U_n\|,\|V_n\|,\|W_n\|\} \le \|T-T_n\| \le \|T\|+\|T_n\| \le \|T\| + \sqrt{(2\|P_n^*A_jP_n\|^2+1)(2c^2+1)} \le \|T\| + \sqrt{(2\|A_j\|^2+1)(2c^2+1)}.$$

Since $\|Q_nTQ_n - cA_j\| = \|Q_n(T-T_n)Q_n\| \le \varepsilon$, we have $\|cA_j\| \le \|Q_nTQ_n\| + \varepsilon < \|T\|+1$. It follows that there exists a constant L not depending on n (depending on $\|T\|$ and c only) such that $\max\{\|U_n\|,\|V_n\|,\|W_n\|\} \le L$. Hence

$$|\langle (T-T_n)x,y\rangle| \le \varepsilon\|x\|\,\|y\| + L(\|y\|\,\|x_2\| + \|x\|\,\|y_2\| + \|x_2\|\,\|y_2\|).$$

If n tends to infinity, then $\max\{\|x_2\|,\|y_2\|\}$ is arbitrarily small, which implies that T is in the weak closure of $\mathcal{P}$. This completes the proof of (b). ∎

3. Weakly-transitive operator bands. Let r be a positive integer, and let X be a Banach space. We say that a subset $\mathcal{S}$ of $\mathcal{B}(X)$ is *weakly r-transitive* if for each linearly independent set $\{x_1,\ldots,x_r\}$ in X, for each subset $\{y_1,\ldots,y_r\}$ of X and for each weak neighborhood V of $0\in X$ there exists $S\in\mathcal{S}$ such that $Sx_i - y_i \in V$ for each $i=1,\ldots,r$. It is easy to see that a subset $\mathcal{S}$ of $\mathcal{B}(X)$ is weakly r-transitive if for each linearly independent set $\{x_1,\ldots,x_r\}$ in X and for each subset $\{y_1,\ldots,y_r\}$ of X there is a sequence $\{S_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that for each $i=1,\ldots,r$ the sequence $\{S_nx_i\}_{n\in\mathbb{N}}$ converges weakly to y_i. It is not difficult to see that a subset of $\mathcal{B}(X)$ is weakly dense if and only if it is weakly r-transitive for every positive integer r. Furthermore, every weakly 1-transitive subset of $\mathcal{B}(X)$ is irreducible. On the other hand, not every bounded irreducible operator band on l^2 is weakly 1-transitive, as can be quickly checked.

Let r be a fixed positive integer. In view of the above remarks every weakly dense subset of $\mathcal{B}(X)$ is weakly r-transitive. The converse assertion is not true, even within the class of operator bands. Moreover, the following theorem holds.

THEOREM 3.1. *Let r be a positive integer. Then there exists a principal-ideal operator band $\mathcal{R}$ on l^2 that is weakly r-transitive and is not weakly $(r+1)$-transitive.*

Proof. For each positive integer n satisfying $3^n > r$, let $\mathcal{R}_n$ denote the set of all operators $T_{A,B}$ as A and B range over all $3^n\times 3^n$ matrices such

that the rank of A is at most r. Denote by $\mathcal{R}$ the union of the identity and all $\mathcal{R}_n$. It is easy to verify that $\mathcal{R}$ is a principal-ideal operator band. To prove that $\mathcal{R}$ is weakly r-transitive, choose linearly independent vectors $x_1,\ldots,x_r\in l^2$, and choose any vectors $y_1,\ldots,y_r\in l^2$. Then there exists an operator $R\in\mathcal{B}(l^2)$ of rank at most r such that $Rx_i=y_i$ for all $i=1,\ldots,r$. For each $n\in\mathbb{N}$ satisfying $3^n>r$ there exists an operator $R_n := T_{A_n,I}\in\mathcal{R}_n$ such that the matrix of $R-R_n$ is of the form

$$\begin{bmatrix} 0 & U_n \\ V_n & W_n \end{bmatrix},$$

where the 0 is the $3^n\times 3^n$ zero matrix. Choose any vector $z\in l^2$, and write $x_i = (x_i^{(1)}, x_i^{(2)})$ $(i=1,\ldots,r)$ and $z=(z^{(1)},z^{(2)})$ with respect to the above decomposition of the space l^2. Then, for any $i=1,\ldots,r$,

$$|\langle (R-R_n)x_i,z\rangle| \le \|U_n\|\,\|x_i^{(2)}\|\,\|z^{(1)}\| + \|V_n\|\,\|x_i^{(1)}\|\,\|z^{(2)}\| + \|W_n\|\,\|x_i^{(2)}\|\,\|z^{(2)}\|.$$

Note that

$$\max\{\|U_n\|,\|V_n\|,\|W_n\|\} \le \|R-R_n\| \le \|R\|+\|R_n\| \le \|R\| + \sqrt{3(2\|A_n\|^2+1)} \le \|R\| + \sqrt{3(2\|R\|^2+1)}.$$

If we let $M := \|R\| + \sqrt{3(2\|R\|^2+1)}$, we have

$$|\langle (R-R_n)x_i,z\rangle| \le M(\|z\| \max_{1\le j\le r}\|x_j^{(2)}\| + 2\|z^{(2)}\| \max_{1\le j\le r}\|x_j\|)$$

for any $i=1,\ldots,r$. If n is sufficiently large, then $\max\{\|x_1^{(2)}\|,\ldots,\|x_r^{(2)}\|, \|z^{(2)}\|\}$ is arbitrarily small. Since $Rx_i=y_i$, it follows that the semigroup $\mathcal{R}$ is weakly r-transitive.

Next we demonstrate that $\mathcal{R}$ is not weakly $(r+1)$-transitive. Let $e_1,e_2,\ldots$ be the standard ortho-basis vectors of l^2. Define $x_i=z_i=e_i$ for $i=1,\ldots,r+1$, $y_1=e_2$, $y_2=e_1$, and $y_i=e_i$ for $3\le i\le r+1$ (provided $r\ge 2$). Suppose that $\mathcal{R}$ is weakly $(r+1)$-transitive. Then for each $n\in\mathbb{N}$ there exists $T_{A_n,B_n}\in\mathcal{R}$ such that

$$|\langle T_{A_n,B_n}x_i - y_i, z_k\rangle| \le 2^{-n}$$

for all $i,k=1,\ldots,r+1$. Denote by P the natural embedding of $\mathbb{C}^{r+1}$ into l^2, and by $\widehat{e}_1,\ldots,\widehat{e}_{r+1}$ the standard ortho-basis vectors of $\mathbb{C}^{r+1}$. We then conclude that the sequence $\{P^*T_{A_n,B_n}P\}_{n\in\mathbb{N}}$ converges in the operator norm to the operator J on $\mathbb{C}^{r+1}$ defined by $J\widehat{e}_1=\widehat{e}_2$, $J\widehat{e}_2=\widehat{e}_1$, and $J\widehat{e}_i=\widehat{e}_i$ for $3\le i\le r+1$. In particular, there exists an integer n such that the operator $P^*T_{A_n,B_n}P$ on $\mathbb{C}^{r+1}$ is invertible. But the matrix of this operator is a principal $(r+1)\times(r+1)$ submatrix of the matrix A_nB_n which has rank at most r. This contradiction completes the proof. ∎

4. Operator bands satisfying a polynomial identity. Throughout the section, V denotes a real or complex vector space. In [6] the following results on commutators in operator bands have been shown.

THEOREM 4.1. *Let $\mathcal{S}$ be an operator band on V. Then $(AB-BA)^3=0$ for all $A,B\in\mathcal{S}$.*

THEOREM 4.2. *Let $\mathcal{S}$ be an operator band on a Banach space satisfying $(AB-BA)^2=0$ for all $A,B\in\mathcal{S}$. Then $\mathcal{S}$ is triangularizable.*

We remark that the preceding theorem is proved in [6] in a Hilbert space setting. However, it is clear from that proof that the theorem is true in a Banach space setting as well. Note also that every operator band on a vector space is algebraically triangularizable (see [6]).

The above results motivate the consideration of operator bands $\mathcal{S}$ on V satisfying a polynomial identity $p(A,B)=0$ for all $A,B\in\mathcal{S}\setminus\{0\}$, where p is a given polynomial in two non-commuting variables. We first observe that the subband generated by A and B has at most 6 elements: A, B, AB, BA, ABA, and BAB. Hence, we may assume with no loss of generality that the polynomial p has the form

$$p(A,B)=s_1A+s_2B+t_1AB+t_2BA+u_1ABA+u_2BAB,$$

where at least one of the scalars s_1, s_2, t_1, t_2, u_1, and u_2 is non-zero. We shall now state the main result of this section.

THEOREM 4.3. *Let p be as above, and let $\mathcal{S}$ be an operator band on V with more than one non-zero component such that $p(A,B)=0$ for all $A,B\in\mathcal{S}\setminus\{0\}$. Then $(AB-BA)^2=0$ for all $A,B\in\mathcal{S}$. In a Banach space setting this implies that $\mathcal{S}$ is triangularizable.*

The following proposition covers the special case of Theorem 4.3 when $s_1=s_2=0$.

PROPOSITION 4.4. *Assume that at least one of scalars t_1, t_2, u_1, and u_2 is non-zero. Let $\mathcal{S}$ be an operator band on a vector space such that*

$$t_1AB+t_2BA+u_1ABA+u_2BAB=0 \tag{2}$$

for all $A,B\in\mathcal{S}$. Then $(AB-BA)^2=0$ for all $A,B\in\mathcal{S}$.

Proof. We may assume that $\mathcal{S}$ is non-zero. Putting $A=B\neq 0$ in (2) we obtain

$$t_1+t_2+u_1+u_2=0.$$

We now multiply (2) on the right by A to get

$$(t_1+u_1)ABA+(t_2+u_2)BA=0,$$

and so

$$(t_1+u_1)(ABA-BA)=0.$$

If $t_1+u_1\neq 0$, we have $ABA=BA$ for all $A,B\in\mathcal{S}$. It follows that $BAB=AB$ by interchanging the roles of A and B. Thus

$$(AB-BA)^2=AB-ABA-BAB+BA=0$$

for all $A,B\in\mathcal{S}$.

Assume now that $t_1+u_1=0$, and so $t_2+u_2=0$ as well. Multiplying (2) on the left by B we obtain

$$(t_1+u_2)BAB+(t_2+u_1)BA=0,$$

and hence

$$(t_1+u_2)(BAB-BA)=0.$$

If $t_1+u_2\neq 0$, we conclude (just as before) that $(AB-BA)^2=0$. So, we must consider only the case when $t_1+u_1=0$, $t_2+u_2=0$, and $t_1+u_2=0$. Then we have $t_1=t_2=-u_1=-u_2$, so that (2) becomes

$$AB+BA-ABA-BAB=0$$

for all $A,B\in\mathcal{S}$. It follows that $(AB-BA)^2=0$ for all $A,B\in\mathcal{S}$. ∎

Proof of Theorem 4.3. Let A_0 and B_0 be non-zero elements of $\mathcal{S}$ which come from two distinct components. Then $A_0B_0\preceq A_0$ and $A_0B_0\preceq B_0$. Moreover, either $A_0B_0\not\sim A_0$ or $A_0B_0\not\sim B_0$. Since $A_0B_0\sim B_0A_0$, there is no loss of generality in assuming that $A_0B_0\not\sim A_0$. Define $C_0=A_0B_0$. We then have

$$s_1A_0+s_2C_0+t_1A_0C_0+t_2C_0A_0+u_1A_0C_0A_0+u_2C_0A_0C_0=0,$$

or equivalently,

$$s_1A_0+(s_2+t_1+u_2)C_0+(t_2+u_1)C_0A_0=0.$$

Multiplying by A_0 on the right, we get

$$s_1A_0+(s_2+t_1+u_2+t_2+u_1)C_0A_0=0.$$

It follows that $s_1=0$, and by symmetry also $s_2=0$. Hence Proposition 4.4 can be applied to complete the proof the theorem. ∎

Consider now the case when $\mathcal{S}$ has exactly one non-zero component. If $A,B\in\mathcal{S}\setminus\{0\}$, then $ABA=A$ and $BAB=B$, so that there is no loss of generality in assuming that the coefficients u_1 and u_2 of the polynomial p are zero.

PROPOSITION 4.5. *Assume that at least one of the scalars s_1, s_2, t_1, and t_2 is non-zero. Let $\mathcal{S}$ be an operator band with exactly one non-zero component such that*

$$s_1 A + s_2 B + t_1 AB + t_2 BA = 0 \tag{3}$$

for all non-zero $A, B \in \mathcal{S}$. Then $(AB - BA)^2 = 0$ for all $A, B \in \mathcal{S}$.

Proof. If $A, B \in \mathcal{S} \setminus \{0\}$, then $A = ABA$ and $B = BAB$, so that (3) implies $t_1 AB + t_2 BA + s_1 ABA + s_2 BAB = 0$. Hence Proposition 4.4 completes the proof. ∎

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