On operator bands

by

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Abstract. A multiplicative semigroup of idempotent operators is called an operator band. We prove that for each $K > 1$ there exists an irreducible operator band on the Hilbert space $l^2$ which is norm-bounded by $K$. This implies that there exists an irreducible operator band on a Banach space such that each member has operator norm equal to 1.

Given a positive integer $r$, we introduce a notion of weak $r$-transitivity of a set of bounded operators on a Banach space. We construct an operator band on $l^2$ that is weakly $r$-transitive and is not weakly $(r+1)$-transitive.

We also study operator bands $S$ satisfying a polynomial identity $p(A,B) = 0$ for all non-zero $A,B \in S$, where $p$ is a given polynomial in two non-commuting variables. It turns out that the polynomial $p(A,B) = (AB - BA)^2$ has a special role in these considerations.

1. Introduction. Let $B(X)$ denote the algebra of all bounded linear operators on a (real or complex) Banach space $X$. A subset $S$ of $B(X)$ is said to be irreducible if the only closed subspaces of $X$ invariant under all members of $S$ are $\{0\}$ and $X$. Otherwise, $S$ is called reducible. A set $S$ of $B(X)$ is said to be triangularizable if there is a chain of closed subspaces that are invariant under every member of $S$ and this chain is maximal in the lattice of all closed subspaces of $X$.

An operator $A$ on a vector space $V$ is called idempotent if $A^2 = A$. A semigroup $S$ of idempotents on $V$ is called an operator band. If $V$ is a Banach space, we also assume that all operators in $S$ are bounded. Reducibility of operator bands on Hilbert spaces has recently been studied in [2], [4], and [6]. In [2] an irreducible operator band on the Hilbert space $l^2$ has been constructed. After having such an example it is natural to ask about the existence of irreducible operator bands with some additional properties. Sections 2 and 3 are devoted to this question. In Section 2 we construct an irreducible operator band on $l^2$ which is norm-bounded. This implies that

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there exist an irreducible operator band $S$ on $l^2$ and an equivalent norm on $l^2$ with respect to which each member of $S$ has operator norm equal to 1. In Section 3 we introduce a notion of weak $r$-transitivity of a set of bounded operators on a Banach space, where $r$ is a given positive integer. We construct an operator band on $l^2$ that is weakly $r$-transitive and is not weakly $(r + 1)$-transitive.

In [6] it is shown that every operator band $S$ on a Hilbert space satisfying $(AB - BA)^2 = 0$ for all $A, B \in S$ is triangularizable. This result motivates the study of operator bands $S$ satisfying a polynomial identity $p(A, B) = 0$ for all non-zero $A, B \in S$, where $p$ is a given polynomial in two non-commuting variables. The results of Section 4 show that the polynomial $p(A, B) = (AB - BA)^2$ has a special role in these considerations.

A reference for what follows is [7]. It should be noted that the definitions and remarks below are not needed to understand Theorems 2.2 and 2.3 and their proofs.

Define a relation $\leq$ on an operator band $S$ by

$$A \leq B \iff ABA = A.$$ 

Then $\leq$ is a pre-order on $S$ (it is reflexive and transitive). This (in fact, each) pre-order determines an equivalence relation $\sim$ on $S$ by

$$A \sim B \iff A \leq B \text{ and } B \leq A.$$ 

Let $C_A$ denote the equivalence class of $A \in S$. Then $C_A$ is a subband of $S$. We refer to the equivalence classes as components of $S$. Define the multiplication on the set $S/\sim$ of all components of $S$ by

$$C_A C_B = C_{AB}.$$ 

This operation is well defined and $S/\sim$ is an abelian band under it. The band pre-order on $S/\sim$ is a partial order. We denote it by $\leq$. It is easy to see that

$$C_A \leq C_B \iff A \leq B.$$ 

An ideal of a semigroup $S$ is a subset of $S$ which is closed under right and left multiplications by elements of $S$. An ideal generated by one element of $S$ is said to be a principal ideal. A principal-ideal band is a band with identity in which every ideal is principal. Principal-ideal matrix bands have been studied in [3].

2. Norm-bounded irreducible operator bands. In [2] an irreducible operator band on the Hilbert space $l^2$ has been constructed. Essentially, this construction is based upon the following operators on $l^2$.

Given $k \times k$ matrices $A$ and $B$, let $P_{AB}$ be the $3k \times 3k$ matrix

$$P_{AB} = \begin{bmatrix} A & B & I \\ A & -B & A \\ I & B & I \end{bmatrix},$$

where $I$ is the identity matrix of order $k$. Let $T_{AB}$ be the infinite block-diagonal matrix

$$T_{AB} = \text{diag}(D_0, D_1, D_2, \ldots),$$

where the block $D_i$ is equal to $P_{AB}$ if the number $i$ is representable in the ternary system by 0’s and 1’s only, and $D_i$ equals the identity matrix of order $3k$ otherwise. We regard $T_{AB}$ as an operator on $l^2$. One readily shows that

$$(1) \quad \|T_{AB}\| \leq 4\|AB\| + 2\|A\| + 2\|B\| + 1 \leq (2\|A\|^2 + 1)(2\|B\|^2 + 1).$$

It is easy to see that $T_{AB} = T_{C,D} = T_{A,D}$ for all $k \times k$ matrices $A, B, C$ and $D$. In particular, $T_{AB}^3 = T_{AB}$.

After the publication of [2], M. D. Choi posed the question of whether there exists an irreducible operator band on a Hilbert space that is also norm bounded. This problem can be reformulated in the following way:

**Problem 2.1.** Let $S$ be an operator band on a Hilbert space such that for some $K \geq 1$ we have $\|S\| \leq K$ for all $S \in S$. Is $S$ necessarily reducible?

If we also assume that $K = 1$, then every member of $S$ is Hermitian (see e.g. [1, Proposition 3.3]). In this case we have $ST = (ST)^* = T^* = TS$ for all $S, T \in S$, so that $S$ is a commutative band, and hence reducible. However, for $K > 1$ the following result holds.

**Theorem 2.2.** Let $K > 1$. Then there exists an operator band $S$ on the Hilbert space $l^2$ such that

$$(a) \quad \|S\| \leq K \text{ for all } S \in S,

(b) \quad \text{the semigroup } \mathbb{R}^+ S = \{\lambda S : \lambda > 0, S \in S\} \text{ is weakly dense in } B(l^2),

and so $S$ is irreducible.

**Proof.** Let $d := \sqrt{(K - 1)/2}$ and $c := d/K$. For each positive integer $n$ let $S_n$ denote the set of all operators $T_{A,B}$ as $A$ and $B$ range over all $3^n \times 3^n$ matrices with norm at most $c$, and let $T_n$ denote the set of all operators $T_{A,B}$ as $A$ and $B$ range over all $3^n \times 3^n$ matrices with norm at least $d$. It is obvious that $S_n$ and $T_n$ are both operator bands satisfying $S_n \subset T_n$. Furthermore, by (1) we have $\|T_{A,B}\| \leq 2c^2 + 1 = K$ for all $T_{A,B} \in T_n$.

We shall prove that $S_n T_m \subseteq T_n$ and $T_m S_n \subseteq T_n$ for all positive integers $m$ and $n$ with $m < n$. Pick $T_{A,B} \in S_n$ and $S \in T_m$. Then there exists a $3^n \times 3^n$ matrix $M$ with norm at most $K$ such that

$$S = \text{diag}(C_0, C_1, C_2, \ldots),$$

where $C_i$ are matrices in $S_n$.
where the block $C_i$ is equal to $M$ if the number $i$ is representable in the ternary system by 0’s and 1’s only, and $C_i$ equals the identity matrix of order $3^n$ otherwise. From

$$\begin{bmatrix}
A & [B & -B & I] \\
A & [M & 0 & 0] \\
I & [0 & M & 0] \\
I & [0 & 0 & I]
\end{bmatrix} = \begin{bmatrix}
A & [BM & -BM & I] \\
I & [I]
\end{bmatrix}$$

it follows that $T_{A,B,S} = T_{A,B,M}$. Since $\|BM\| \leq \|B\|\|M\| \leq cK = d$, we conclude that $T_{A,B,M} \in T_n$. This completes the proof of the inclusion $S_n T_n \subseteq T_n$. The proof of the other inclusion is similar.

Now let $S$ be the semigroup generated by the union $\bigcup_{n=1}^{\infty} S_n$. We claim that $S$ is an operator band and that (a) holds. To this end, pick $S \in S$. Then $S$ is a finite product of some members of $\bigcup_{n=1}^{\infty} S_n$. Let $p$ be the smallest integer such that these members belong to the finite union $\bigcup_{n=1}^{p} S_n$. Using the above inclusions and the facts that $S_n$ and $T_n$ are semigroups, we easily conclude that $S \in T_p$. Therefore $S$ is an idempotent with norm at most $K$.

In order to prove (b) we consider $T \in B(l^2)$ and $x, y \in l^2$. There is no loss of generality in assuming that $\|T\| \leq \delta^2$. For each $n \in \mathbb{N}$ there exists a $3^n \times 3^n$ matrix $A_n$ with norm at most $c$ such that the operators $T_n := T_{A_n,cf} \in S_n \subseteq S$ have the same upper-left $3^n \times 3^n$ corner. The rest of the proof goes along the lines of the last part of the proof from [2]. The weak density of $B^+ S$ also implies that $S$ is irreducible.

Theorem 2.3. There exist an irreducible operator band $S$ on $l^2$ and an equivalent norm on $l^2$ with respect to which each member of $S$ has operator norm equal to 1.

Proof. It is well known and easily shown that for each bounded semigroup $S$ of operators on a Banach space containing the identity, we can define an equivalent norm on the Banach space by

$$\|x\|' = \sup\{\|Sx\| : S \in S\},$$

with respect to which every member of $S$ has norm at most 1. Now take for $S$ any operator band obtained in Theorem 2.2, and adjoin the identity to it.

Theorem 2.2 (and therefore Theorem 2.3 as well) can be improved as follows.

Theorem 2.4. Let $K > 1$. Then there exists an operator band $P$ on $l^2$ such that

(a) $P$ is norm-bounded by $K$,
(b) $R^+ P$ is weakly dense in $B(l^2)$ (and so $P$ is irreducible),
(c) $P$ is a principal-ideal band with countably many elements, and each component of $P$ is finite.

Proof. Let us use the notation from the proof of Theorem 2.2. For each positive integer $n$ let $\{W_{nk}\}_{k \in \mathbb{N}}$ be a sequence of $3^n \times 3^n$ matrices that is dense in the ball of all $3^n \times 3^n$ matrices of norm at most $c$. Furthermore, let $P_n$ be the natural embedding of $C^3$ into $l^2$. Define the double sequence $\{Z_{nk}\}_{k \in \mathbb{N}}$ of bounded operators on $l^2$ by $Z_{nk} = P_n W_{nk} P_n^*$. Let $\{A_n\}_{n \in \mathbb{N}}$ be a renumeration of the terms of the sequence

$$Z_{11}, Z_{12}, Z_{21}, Z_{22}, Z_{31}, Z_{32}, Z_{33}, Z_{34}, \ldots$$

Define an increasing sequence $\{P_n\}_{n \in \mathbb{N}}$ of semigroups inductively. Let $P_1$ be the operator band on $l^2$ generated by $T_{I, A_1, cf}$ and the identity, and let $P_n$ be the semigroup generated by $P_{n-1}$ and the operators $T_{P_1 A_1 P_{n-1}, cf}, T_{P_1 A_2 P_{n-1}, cf}, \ldots , T_{P_1 A_n P_{n-1}, cf}$. Since $S_n T_n \subseteq T_n$ and $T_n S_n \subseteq T_n$ for all positive integers $m$ and $n$ with $m < n$, we conclude that $P_n \subseteq \{I \cup T_1 \cup \ldots \cup T_n\}$, and so the semigroup $P_n$ is an operator band. By the famous theorem of Green and Rees [4] every finitely generated band is finite, so that $P_n$ has finitely many elements. Note that for each $n \in \mathbb{N},$

$$T_{P_1 A_1 P_n, cf} \sim T_{P_1 A_2 P_n, cf} \sim \ldots \sim T_{P_1 A_n P_n, cf},$$

and

$$T_{P_1 A_1 P_n, cf} \cdot T_{P_1 A_2 P_n, cf} \sim T_{P_1 A_3 P_n, cf} \sim T_{P_1 A_4 P_n, cf} \sim T_{P_1 A_5 P_n, cf} \cdot T_{P_1 A_6 P_n, cf},$$

for all $n > m$ and $i, j \in \mathbb{N}$. (See the proof of the inclusions $S_n T_m \subseteq T_n$ and $T_m S_n \subseteq T_n$ for $n > m$. It follows that $P_n$ gains only one component in addition to those which make up $P_{n-1}$, and this component is the smallest one (with respect to $\leq$) of $P_n$. In particular, $P_n$ is a principal-ideal band. Then $P = \bigcup_{n \in \mathbb{N}} P_n$ is a countable principal-ideal band with finite components, and so (c) holds. Since $P \setminus \{I\}$ is the union of all $T_n$, $P$ is norm-bounded by $K$.

For each positive integer $n$ define $Q_n = P_n P_n^*$. For the proof of (b) it is enough to show that each $T \in B(l^2)$ satisfying $Q_n T Q_n = T$ for some $m$ is in the weak closure of $B^+ P$, because the set of such operators is weakly dense in $B(l^2)$. Fix $x, y \in l^2$, and $1 > \epsilon > 0$. We may assume that $\|T\| \leq \delta^2$. Then there exists $j \geq m$ such that for each $n \geq j$ the operator $T_n := T_{P_1 A_j P_n, cf} \in P_n$ satisfies the estimate $\|Q_n (T - T_n) Q_n\| \leq \epsilon$. (Note that $Q_n T_n Q_n = c A_j$.) Decompose $l^2$ into the direct sum of the range and the kernel of $Q_n$. Then the matrix of $T - T_n$ is of the form

$$\begin{bmatrix}
E_n & U_n \\
V_n & W_n
\end{bmatrix},$$

where the norm of the $3^n \times 3^n$ matrix $E_n$ is at most $\epsilon$. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with respect to the same decomposition of the space $l^2$, we
have
\[(T - T_n)x, y\| \leq \|E_n\| \|x\| \|y\| + \|U_n\| \|x_2\| \|y_1\|
+ \|V_n\| \|x_1\| \|y_2\| + \|W_n\| \|x_2\| \|y_2\|.
\]

Note that by (1),
\[
\max\{\|U_n\|, \|V_n\|, \|W_n\|\} \leq \|T - T_n\| \leq \|T\| + \|T_n\|
\leq \|T\| + \sqrt{2(2\|A_j\| + 1)(2\|A_j\|^2 + 1)}.
\]

Since \(Q_nTQ_n - cA_j = Q_n(T - T_n)Q_n\| \leq \epsilon\), we have \(\|cA_j\| \leq \|Q_n TQ_n\| + \epsilon \leq \|T\| + 1\). It follows that there exists a constant \(L\) not depending on \(n\) (depending on \(\|T\|\) and \(c\) only) such that \(\max\{\|U_n\|, \|V_n\|, \|W_n\|\} \leq L\). Hence
\[
\|(T - T_n)x, y\| \leq \epsilon \|x\| \|y\| + L(\|\|x_1\| \|x_2\| + \|x_1\| \|y_2\| + \|x_2\| \|y_2\|).
\]

If \(n\) tends to infinity, then \(\max\{\|x_2\|, \|y_2\|\}\) is arbitrarily small, which implies that \(T\) is in the weak closure of \(P\). This completes the proof of (b). ■

3. Weakly-transitive operator bands. Let \(r\) be a positive integer, and let \(X\) be a Banach space. We say that a subset \(S\) of \(B(X)\) is weakly \(r\)-transitive if for each linearly independent set \(\{x_1, \ldots, x_r\}\) of \(X\) and for each weak neighborhood \(V\) of \(0\) in \(X\) there exists \(S_n \in S\) such that \(S_nx_i - y_i \in V\) for each \(i = 1, \ldots, r\). It is easy to see that a subset \(S\) of \(B(X)\) is weakly \(r\)-transitive if for each linearly independent subset \(\{x_1, \ldots, x_r\}\) in \(X\) and for each sequence \(\{y_1, \ldots, y_r\}\) of \(X\) there is a sequence \(\{S_n\}_{n \in \mathbb{N}}\) in \(S\) such that for each \(i = 1, \ldots, r\) the sequence \(\{S_nx_i\}_{n \in \mathbb{N}}\) converges weakly to \(y_i\). It is not difficult to see that a subset of \(B(X)\) is weakly dense if and only if it is weakly \(r\)-transitive for every positive integer \(r\). Furthermore, every weakly 1-transitive subset of \(B(X)\) is irreducible. On the other hand, not every bounded irreducible operator band on \(l^2\) is weakly 1-transitive, as can be quickly checked.

Let \(r\) be a fixed positive integer. In view of the above remarks every weakly dense subset of \(B(X)\) is weakly \(r\)-transitive. The converse assertion is not true, even within the class of operator bands. Moreover, the following theorem holds.

**Theorem 3.1.** Let \(r\) be a positive integer. Then there exists a principal-ideal operator band \(R\) on \(l^2\) that is weakly \(r\)-transitive and is not weakly \((r + 1)\)-transitive.

**Proof.** For each positive integer \(n\) satisfying \(3^n > r\), let \(R_n\) denote the set of all operators \(T_{A,B}\) as \(A\) and \(B\) range over all \(3^n \times 3^n\) matrices such that the rank of \(A\) is at most \(r\). Denote by \(\mathcal{R}\) the union of the identity and all \(R_n\). It is easy to verify that \(\mathcal{R}\) is a principal-ideal operator band.

To prove that \(\mathcal{R}\) is weakly \(r\)-transitive, choose linearly independent vectors \(x_1, \ldots, x_r \in l^2\), and choose any vectors \(y_1, \ldots, y_r \in l^2\). Then there exists an operator \(R \in B(l^2)\) of rank at most \(r\) such that \(Rx_i = y_i\) for all \(i = 1, \ldots, r\). For each \(n \in \mathbb{N}\) satisfying \(3^n > r\) there exists an operator \(R_n := T_{A_n,B_n} \in R_n\) such that the rank of \(R - R_n\) is of the form
\[
\begin{bmatrix}
0 & U_n \\
V_n & W_n
\end{bmatrix},
\]

where the 0 is an \(3^n \times 3^n\) zero matrix. Choose any vector \(z \in l^2\), and write \(z = (z_1, z_2)\) (1 = 1, \ldots, \(r\)) and \(z = (z^{(1)}, z^{(2)})\) with respect to the above decomposition of the space \(l^2\). Then, for any \(i = 1, \ldots, r\),
\[
\|(R - R_n)x_i, z\| \leq \|U_n\| \|x^{(2)}_i\| \|z^{(1)}\|
+ \|V_n\| \|x^{(1)}_i\| \|z^{(2)}\| + \|W_n\| \|x^{(2)}_i\| \|z^{(2)}\|.
\]

Note that
\[
\max\{\|U_n\|, \|V_n\|, \|W_n\|\} \leq \|R - R_n\| \leq \|R\| + \|R_n\|
\leq \|R\| + \sqrt{3(2\|A_n\|^2 + 1)}
\leq \|R\| + \sqrt{3(2\|A_n\|^2 + 1)}.
\]

If we let \(M := \|R\| + \sqrt{3(2\|A_n\|^2 + 1)}\), we have
\[
\|(R - R_n)x_i, z\| \leq M(\|z\| \max_{1 \leq j \leq r} \|x^{(2)}_j\| + \|z^{(2)}\| \max_{1 \leq j \leq r} \|x^{(2)}_j\|)
\]
for any \(i = 1, \ldots, r\). If \(n\) is sufficiently large, then \(\max\{\|x^{(1)}_i\|, \|x^{(2)}_i\|, \|z^{(2)}\|\}\) is arbitrarily small. Since \(Rx_i = y_i\), it follows that the semigroup \(\mathcal{R}\) is weakly \(r\)-transitive.

Next we demonstrate that \(\mathcal{R}\) is not weakly \((r + 1)\)-transitive. Let \(e_1, e_2, \ldots\) be the standard ortho-basis vectors of \(l^2\). Define \(x_1 = e_i\) for \(i = 1, \ldots, r + 1\), \(y_1 = e_i\), \(y_2 = e_i\), and \(y_i = e_i\) for \(3 \leq i \leq r + 1\) (provided \(r \geq 2\)). Suppose that \(\mathcal{R}\) is weakly \((r + 1)\)-transitive. Then for each \(n \in \mathbb{N}\) there exists \(T_{A_n,B_n} \in \mathcal{R}\) such that
\[
\|(T_{A_n,B_n}x_i - y_i, z_\delta)\| \leq 2^{-n}
\]
for all \(i, k = 1, \ldots, r + 1\). Denote by \(P\) the natural embedding of \(C^{r+1}\) into \(l^2\), and by \(\tilde{e}_1, \ldots, \tilde{e}_{r+1}\) the standard ortho-basis vectors of \(C^{r+1}\). We then conclude that the sequence \(\{T_{A_n,B_n}P\}_{n \in \mathbb{N}}\) converges in the operator norm to the operator \(J\) on \(C^{r+1}\) defined by \(J\tilde{e}_1 = \tilde{e}_1, J\tilde{e}_2 = \tilde{e}_1, J\tilde{e}_3 = \tilde{e}_3\) for \(3 \leq i \leq r + 1\). In particular, there exists an integer \(n\) such that the operator \(P^* T_{A_n,B_n}P\) on \(C^{r+1}\) is invertible. But the matrix of this operator
is a principal $(r + 1) \times (r + 1)$ submatrix of the matrix $A_nB_n$ which has rank at most $r$. This contradiction completes the proof.

4. Operator bands satisfying a polynomial identity. Throughout the section, $V$ denotes a real or complex vector space. In [6] the following results on commutators in operator bands have been shown.

Theorem 4.1. Let $S$ be an operator band on $V$. Then $(AB - BA)^2 = 0$ for all $A, B \in S$.

Theorem 4.2. Let $S$ be an operator band on a Banach space satisfying $(AB - BA)^2 = 0$ for all $A, B \in S$. Then $S$ is triangularizable.

We remark that the preceding theorem is proved in [6] in a Hilbert space setting. However, it is clear from that proof that the theorem is true in a Banach space setting as well. Note also that every operator band on a vector space is algebraically triangularizable (see [6]).

The above results motivate the consideration of operator bands $S$ on $V$ satisfying a polynomial identity $p(A, B) = 0$ for all $A, B \in S \setminus \{0\}$, where $p$ is a given polynomial in two non-commuting variables. We first observe that the subband generated by $A$ and $B$ has at most 6 elements: $A, B, AB, BA, ABA,$ and $BAB$. Hence, we may assume with no loss of generality that the polynomial $p$ has the form

$$p(A, B) = s_1A + s_2B + t_1AB + t_2BA + u_1ABA + u_2BAB,$$

where at least one of the scalars $s_1, s_2, t_1, t_2, u_1,$ and $u_2$ is non-zero. We shall now state the main result of this section.

Theorem 4.3. Let $p$ be as above, and let $S$ be an operator band on $V$ with more than one non-zero component such that $p(A, B) = 0$ for all $A, B \in S \setminus \{0\}$. Then $(AB - BA)^2 = 0$ for all $A, B \in S$. In a Banach space setting this implies that $S$ is triangularizable.

The following proposition covers the special case of Theorem 4.3 when $s_1 = s_2 = 0$.

Proposition 4.4. Assume that at least one of scalars $t_1, t_2, u_1,$ and $u_2$ is non-zero. Let $S$ be an operator band on a vector space such that

$$(2) \quad t_1AB + t_2BA + u_1ABA + u_2BAB = 0$$

for all $A, B \in S$. Then $(AB - BA)^2 = 0$ for all $A, B \in S$.

Proof. We may assume that $S$ is non-zero. Putting $A = B \neq 0$ in (2) we obtain

$$t_1 + t_2 + u_1 + u_2 = 0.$$
PROPOSITION 4.5. Assume that at least one of the scalars \( s_1, s_2, t_1, \) and \( t_2 \) is non-zero. Let \( S \) be an operator band with exactly one non-zero component such that
\[
(3) \quad s_1 A + s_2 B + t_1 AB + t_2 BA = 0
\]
for all non-zero \( A, B \in S \). Then \((AB - BA)^2 = 0\) for all \( A, B \in S \).

Proof. If \( A, B \in S \setminus \{0\} \), then \( A = ABA \) and \( B = BAB \), so that
(3) implies \( t_1 AB + t_2 BA + s_1 ABA + s_2 BAB = 0 \). Hence Proposition 4.4 completes the proof. \( \square \)

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