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Elements of C^*-algebras commuting with their Moore–Penrose inverse

by

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Abstract. We give new necessary and sufficient conditions for an element of a C^*-algebra to commute with its Moore–Penrose inverse. We then study conditions which ensure that this property is preserved under multiplication. As a special case of our results we recover a recent theorem of Hartwig and Katz on EP matrices.

1. Introduction. The novelty of our approach to the study of Moore–Penrose inverse in C^*-algebras is considering it in terms of the Drazin inverse. For elements of a C^*-algebra that commute with their Moore–Penrose inverse, the Moore–Penrose inverse in fact coincides with the Drazin inverse. Proofs found in the literature may resort to special constructions, often very ingenious. Many of these arguments can now be presented more systematically relying on standard properties of the Drazin inverse and on properties of spectral idempotents.

We retain the notation of [10]. In particular, $\mathfrak{A}$ is a unital C^*-algebra with unit e; next, $\mathfrak{A}^{-1}$, $\mathrm{QN}(\mathfrak{A})$ and $\mathfrak{A}^\dagger$ denote the sets of all invertible, quasinilpotent and regular elements of $\mathfrak{A}$, respectively. An element $a \in \mathfrak{A}$ is *quasipolar* if 0 is an isolated—possibly removable—singularity of the resolvent of a, and *polar* if 0 is at most a pole of the resolvent. By $\sigma(a)$ we denote the spectrum of $a \in \mathfrak{A}$.

The set of all quasipolar elements of $\mathfrak{A}$ will be denoted by $\mathfrak{A}^{\mathrm{D}}$. Observe that $\mathfrak{A}^{\mathrm{D}} \cap \mathfrak{A}^\dagger \supset \mathfrak{A}^{-1}$. We write $L(H)$ for the C^*-algebra of all bounded linear operators on a Hilbert space H.

PROPOSITION 1.1 [9, Theorem 4.2]. *Let $a \in \mathfrak{A}$. Then the following conditions are equivalent*:

(i) $a \in \mathfrak{A}^{\mathrm{D}}$.

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(ii) *There exists (a unique) $x \in \mathfrak{A}$ such that*

$$(1.1)\qquad ax = xa, \quad xax = x, \quad axa = a + u, \; u \in \mathrm{QN}(\mathfrak{A}).$$

(iii) *There exists (a unique) $p = p^2 \in \mathfrak{A}$ such that $ap = pa \in \mathrm{QN}(\mathfrak{A})$ and $a + p \in \mathfrak{A}^{-1}$.*

An element x in (ii) is the *Drazin inverse* of a, written $x = a^{\mathrm{D}}$. (The original definition of this inverse [4] required that u be nilpotent.) The idempotent p satisfying (iii) is the *spectral idempotent* of $a \in \mathfrak{A}^{\mathrm{D}}$ at 0, written $p = a^\pi$. We recall that a is polar if and only if $a^k a^\pi = 0$ for some nonnegative integer k; if $aa^\pi = 0$, then a is *simply polar*. It is known [9] that

$$(1.2)\qquad a^\pi = e - a^{\mathrm{D}}a \quad \text{and} \quad a^{\mathrm{D}} = (a + a^\pi)^{-1}(e - a^\pi).$$

We observe that if a is quasipolar, then so is a^*, and $(a^*)^\pi = (a^\pi)^*$.

PROPOSITION 1.2 [5, Theorem 6], [10, Theorem 2.8]. *Let $a \in \mathfrak{A}$. Then $a \in \mathfrak{A}^\dagger$ if and only if there exists (a unique) element $x \in \mathfrak{A}$ satisfying the equations*

$$(1.3)\qquad xax = x, \quad axa = a, \quad (ax)^* = ax, \quad (xa)^* = xa;$$

x is called the Moore–Penrose inverse *of a, written $x = a^\dagger$.* (The original definition of this inverse was given in [13] for matrices.)

The two inverses are related by the following result.

PROPOSITION 1.3 [10, Theorem 2.5]. *Let $a \in \mathfrak{A}$. Then $a \in \mathfrak{A}^\dagger$ if and only if a^*a (respectively aa^*) is simply polar, in which case*

$$(1.4)\qquad a^\dagger = (a^*a)^{\mathrm{D}}a^* = a^*(aa^*)^{\mathrm{D}}.$$

If a is regular, then so is a^*, and $(a^*)^\dagger = (a^\dagger)^*$.

For future use we need the following two lemmas and some notation. Following [6], for any $a \in \mathfrak{A}$ we define the nullspace ideals

$$a^{-1}(0) = \{x \in \mathfrak{A} : ax = 0\}, \quad a_{-1}(0) = \{x \in \mathfrak{A} : xa = 0\}.$$

LEMMA 1.4 (see also [5, Theorem 9]). *Let $a \in \mathfrak{A}$ be simply polar with the spectral idempotent a^π at 0. Then*

$$a^{-1}(0) = a^\pi\mathfrak{A}, \quad a\mathfrak{A} = (a^\pi)^{-1}(0), \quad a_{-1}(0) = \mathfrak{A}a^\pi, \quad \mathfrak{A}a = a^\pi_{-1}(0),$$

and $\mathfrak{A} = a^{-1}(0) \oplus a\mathfrak{A} = a_{-1}(0) \oplus \mathfrak{A}a$ with $a\mathfrak{A}$ and $\mathfrak{A}a$ closed.

Proof. We prove only the results for $a^{-1}(0)$ and $a\mathfrak{A}$, the rest follows by symmetry.

Let $ax = 0$. Then $a^\pi x = (e - a^{\mathrm{D}}a)x = x$, and $a^{-1}(0) \subset a^\pi\mathfrak{A}$. If $x = a^\pi y$, then $ax = aa^\pi y = 0$, and $a^\pi\mathfrak{A} \subset a^{-1}(0)$.

Let $x = au$ for some $u \in \mathfrak{A}$. Then $a^\pi x = a^\pi au = 0$, and $a\mathfrak{A} \subset (a^\pi)^{-1}(0)$. Let $a^\pi x = 0$. Then $(e - aa^{\mathrm{D}})x = 0$, and $x = aa^{\mathrm{D}}x$; hence $(a^\pi)^{-1}(0) \subset a\mathfrak{A}$.

Since a^π is an idempotent, $\mathfrak{A} = a^\pi\mathfrak{A} \oplus (a^\pi)^{-1}(0)$.

LEMMA 1.5. *Let $a \in \mathfrak{A}^\dagger$. Then*

$$(1.5)\qquad a^\dagger = (a^*a + (a^*a)^\pi)^{-1}a^* = a^*(aa^* + (aa^*)^\pi)^{-1},$$

$$(1.6)\qquad a^*\mathfrak{A}^{-1} = a^\dagger\mathfrak{A}^{-1} \quad \text{and} \quad \mathfrak{A}^{-1}a^* = \mathfrak{A}^{-1}a^\dagger,$$

$$(1.7)\qquad a^{*-1}(0) = (a^\dagger)^{-1}(0) \quad \text{and} \quad a^*_{-1}(0) = a^\dagger_{-1}(0).$$

Proof. First we observe that, for any $a \in \mathfrak{A}^\dagger$, $a(a^*a)^\pi = 0$ and $(a^*a)^\pi a^* = 0$. The first equation follows from $a(a^*a)^\pi = a(e - a^\dagger a) = 0$, the second is obtained by taking adjoints. By (1.2) and (1.4),

$$a^\dagger = (a^*a)^{\mathrm{D}}a^* = (a^*a + (a^*a)^\pi)^{-1}(e - (a^*a)^\pi)a^* = (a^*a + (a^*a)^\pi)^{-1}a^*.$$

The second part of (1.5) is proved similarly. This proves (1.6), and (1.7) follows.

Condition (1.6) appears in the proof of [6, Theorem 10] as equation (10.6).

2. Elements commuting with their Moore–Penrose inverse.

In this paper we are concerned with the elements $a \in \mathfrak{A}^\dagger$ satisfying $a^\dagger a = aa^\dagger$. Matrices with this property are called EP_r or EP *matrices* [1, 7]. We give a new characterization—in terms of spectral idempotents—of the elements of a C^*-algebra which commute with their Moore–Penrose inverse.

THEOREM 2.1. *Let $a \in \mathfrak{A}^\dagger$. Then $a^\dagger a = aa^\dagger$ if and only if a is simply polar with a selfadjoint spectral idempotent at 0. In this case*

$$(2.1)\qquad a^\pi = (a^*)^\pi = (a^*a)^\pi = (aa^*)^\pi.$$

Proof. From Proposition 1.1 we deduce that if a is simply polar, then a^* is also simply polar with $(a^*)^\pi = (a^\pi)^*$. If $p = a^\pi$ is selfadjoint, then $(a^*a)p = p(a^*a) = 0$ and

$$a^*a + p = (a^* + p)(a + p) \in \mathfrak{A}^{-1}.$$

By Proposition 1.1 again, a^*a is simply polar and $(a^*a)^\pi = a^\pi$. The equality $(aa^*)^\pi = a^\pi$ follows by symmetry. By Proposition 1.3, (1.4) and the first part of (1.2), we have $a \in \mathfrak{A}^\dagger$ and

$$a^\dagger a = (a^*a)^{\mathrm{D}}a^*a = e - (a^*a)^\pi = e - (aa^*)^\pi = aa^*(aa^*)^{\mathrm{D}} = aa^\dagger.$$

Conversely, let $a \in \mathfrak{A}^\dagger$ and let $a^\dagger a = aa^\dagger$. Since $a^\dagger aa^\dagger = a^\dagger$ and $aa^\dagger a = a + u$ with $u = 0$, a^{D} exists and $a^{\mathrm{D}} = a^\dagger$. Further, $aa^\pi = a(e - a^{\mathrm{D}}a) = a(e - a^\dagger a) = 0$, and a is simply polar. Finally,

$$(a^\pi)^* = (e - a^{\mathrm{D}}a)^* = (e - a^\dagger a)^* = e - (a^\dagger a)^* = e - a^\dagger a = a^\pi.$$

The following corollary can be deduced from Theorem 2.1 and its proof. The equivalence of (i) and (ii) is [10, Proposition 2.2]. We omit the proof.

Corollary 2.2. *Let $a \in \mathfrak{A}$. Then the following conditions are equivalent*:

(i) $a \in \mathfrak{A}^\dagger$ *and* $a^\dagger a = aa^\dagger$;
(ii) $a \in \mathfrak{A}^\dagger \cap \mathfrak{A}^{\mathrm{D}}$ *and* $a^\dagger = a^{\mathrm{D}}$;
(iii) *a is simply polar and* $(a^*)^\pi = a^\pi$;
(iv) *a is simply polar and* $a^\pi = (a^*a)^\pi$ (*respectively* $a^\pi = (aa^*)^\pi$);
(v) $a \in \mathfrak{A}^\dagger$ *and* $(a^*a)^\pi = (aa^*)^\pi$.

We now show that the commuting Moore–Penrose inverse can be expressed in terms of the holomorphic calculus.

Corollary 2.3. *Let $a \in \mathfrak{A}^\dagger$. Then $a^\dagger a = aa^\dagger$ if and only if*

$$a^\dagger = f(a) \tag{2.2}$$

for some function f holomorphic in a neighbourhood of $\sigma(a)$.

Proof. Let $a^\dagger a = aa^\dagger$. By the preceding corollary, $a^\dagger = a^{\mathrm{D}}$. According to [9, Theorem 4.4], $a^{\mathrm{D}} = f(a)$, where f is holomorphic in a neighbourhood of $\sigma(a)$, and $f(\lambda) = 0$ in a neighbourhood of 0, $f(\lambda) = \lambda^{-1}$ in a neighbourhood of $\sigma(a)\backslash\{0\}$.

Conversely, if $a^\dagger = f(a)$ for some function holomorphic in a neighbourhood of $\sigma(a)$, then by a property of the holomorphic calculus, $a^\dagger$ commutes with a.

Corollary 2.3 yields a result of Wong [14, Theorem 2], who showed that a matrix A commutes with $A^\dagger$ if and only if $A^\dagger = f(A)$ for some polynomial f.

It is interesting to observe that if $a^\dagger$ commutes with $a \in \mathfrak{A}$, then it double commutes with a, that is,

$$ax = xa \Rightarrow a^\dagger x = xa^\dagger, \quad x \in \mathfrak{A}.$$

This follows from the holomorphic calculus representation (2.2) of $a^\dagger$.

3. Further conditions. Previously Brock [2] characterized the bounded linear operators A on a Hilbert space satisfying $A^\dagger A = AA^\dagger$, and Harte and Mbekhta [6] generalized this characterization to C^*-algebras. We extend their results in the following theorem.

Theorem 3.1. *If $a \in \mathfrak{A}^\dagger$, then the following are equivalent.*

(i) $aa^\dagger = a^\dagger a$;
(ii) $a^2a^\dagger = a = a^\dagger a^2$;
(iii) $(a^*a)^\pi a = 0 = a(aa^*)^\pi$;
(iv) $a^{-1}(0) = a^{*-1}(0)$;
(v) $a_{-1}(0) = a^*_{-1}(0)$;
(vi) $a\mathfrak{A} = a^*\mathfrak{A}$;

(vii) $\mathfrak{A}a = \mathfrak{A}a^*$;
(viii) $a\mathfrak{A}^{-1} = a^*\mathfrak{A}^{-1}$;
(ix) $\mathfrak{A}^{-1}a = \mathfrak{A}^{-1}a^*$;
(x) $a \in a^\dagger\mathfrak{A} \cap \mathfrak{A}a^\dagger$;
(xi) $a \in a^\dagger\mathfrak{A}^{-1} \cap \mathfrak{A}^{-1}a^\dagger$.

Proof. (i), (ii) and (iii) are easily seen to be equivalent. (Recall that $(a^*a)^\pi = e - a^\dagger a$ and $(aa^*)^\pi = e - aa^\dagger$.)

We prove that (i) implies (iv)–(xi). Suppose that (i) holds. According to Theorem 2.1, a is simply polar and the spectral idempotent a^π is selfadjoint. By Lemma 1.4,

$$a^{-1}(0) = a^\pi\mathfrak{A} = (a^\pi)^*\mathfrak{A} = (a^*)^\pi\mathfrak{A} = a^{*-1}(0)$$

and

$$a\mathfrak{A} = (a^\pi)^{-1}(0) = (a^\pi)^{*-1}(0) = ((a^*)^\pi)^{-1}(0) = a^*\mathfrak{A}.$$

The equalities $a_{-1}(0) = a^*_{-1}(0)$ and $\mathfrak{A}a = \mathfrak{A}a^*$ follow by taking adjoints. This proves (iv), (v), (vi) and (vii). To prove (xi) we write $a = a^2a^\dagger = (a + a^\pi)^2a^\dagger$ and observe that $a + a^\pi \in \mathfrak{A}^{-1}$ by Proposition 1.1; similarly $a = a^\dagger(a + a^\pi)^2$. Condition (x) follows from (xi), and (viii) and (ix) follow from (xi) and (1.6).

Conversely, we show that any of the conditions (iv)–(xi) implies (ii). We note that (viii) and (ix) are equivalent (take adjoints), and together they imply (xi); (xi) in turn implies (x). Conditions (vi) and (vii) are equivalent (adjoints), and together they also imply (x). From (x) we deduce (ii): Indeed, if $a = ua^\dagger$, then $a - a^2a^\dagger = u(a^\dagger - a^\dagger aa^\dagger) = 0$; if $a = a^\dagger v$, then $a - a^\dagger a^2 = (a^\dagger - a^\dagger aa^\dagger)v = 0$. Conditions (iv) and (v) are equivalent (take adjoints) and together they imply (ii): $e - aa^\dagger \in (a^\dagger)^{-1}(0) = (a^*)^{-1}(0) = a^{-1}(0)$ by (1.7) and (iv), and similarly $e - a^\dagger a \in a_{-1}(0)$ by (1.7) and (v).

We have recovered [6, Theorem 10] which gives the equivalence of (i), (iv), (v), (viii) and (ix) of the preceding theorem.

In [2], Brock proved the equivalence of the following conditions for operators on a Hilbert space. We write $N(A)$ for the nullspace of $A \in L(H)$, and $R(A)$ for the range of A.

Corollary 3.2 (Brock [2]). *Let $A \in L(H)$ be a closed range operator on a Hilbert space H. Then the following conditions are equivalent*:

(i) $A^\dagger A = AA^\dagger$;
(ii) $H = N(A) \oplus^\perp R(A)$;
(iii) $N(A) = N(A^*)$;
(iv) $A^* = PA$ *for some* $P \in L(H)^{-1}$.

Proof. Let $\mathfrak{A} = L(H)$, the full algebra of bounded linear operators on H. Brock's condition (i)$\Leftrightarrow$(ii) follows from Theorem 2.1 (A simply polar with a selfadjoint spectral projection). The equivalence (i)$\Leftrightarrow$(iv) follows from Theorem 3.1(ix). To recover (i)$\Leftrightarrow$(iii), we observe that, for any pair of bounded linear operators A, B on H,

$$[N(L_A) \subset N(L_B)] \Leftrightarrow [N(A) \subset N(B)] \tag{3.1}$$

(where $L_T : U \mapsto TU$ on $L(H)$), and apply Theorem 3.1(iv).

COROLLARY 3.3. *Let $A \in L(H)$ be an upper semi-Fredholm operator on a Hilbert space H. Then the following conditions are equivalent*:

(i) $A^\dagger A = AA^\dagger$;
(ii) $(A^*A)^\pi A = 0$;
(iii) $A(AA^*)^\pi = 0$.

Proof. Recall that an operator $A \in L(H)$ is upper semi-Fredholm if $R(A)$ is closed and $N(A)$ finite-dimensional. Closed range operators on a Hilbert space are Moore–Penrose invertible.

If $A^\dagger A = AA^\dagger$, then (ii) and (iii) hold by Theorem 3.1(iii).

Conversely, if (ii) (which is the first half of condition (iii) in Theorem 3.1) is satisfied, then $A^*(A^*A)^\pi = 0$, which means that

$$N(A) = N(A^*A) = R((A^*A)^\pi) \subset N(A^*).$$

Since $N(A)$ has a finite dimension equal to that of $N(A^*)$, we have $N(A) = N(A^*)$. This implies that $A^\dagger A = AA^\dagger$ by Corollary 3.2.

Finally, (iii) is condition (ii) with A^* in place of A, and hence, by the preceding argument, $(A^*)^\dagger$ commutes with A^*, which then implies (i).

From this corollary we recover the result of Marek and Žitný [12, p. 143], who proved the foregoing criterion under the assumption that A is a linear operator on a finite-dimensional Hilbert space. Their result follows since linear operators on a finite-dimensional space are Fredholm.

4. The product of elements commuting with their Moore–Penrose inverse. By $\mathfrak{A}^\dagger_{\rm com}$ we denote the class of all elements $a \in \mathfrak{A}^\dagger$ such that $a^\dagger a = aa^\dagger$. It is well known that $\mathfrak{A}^\dagger_{\rm com}$ is not closed under multiplication of elements in $\mathfrak{A}$; this is evident even for finite matrices.

EXAMPLE 4.1. In the C^*-algebra $\mathfrak{A} = \mathbb{C}^{3\times 3}$ let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \text{so} \quad AB = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A, B \in \mathfrak{A}^\dagger_{\rm com}$ as $A^\dagger = A$ and $B^\dagger = B^{-1}$. But the matrices $(AB)^\dagger$, $(AB)(AB)^\dagger$ and $(AB)^\dagger(AB)$ are equal to

$$\frac{1}{10}\begin{bmatrix} 0 & 2 & 0 \\ 5 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{5}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix},$$

respectively, and $(AB)(AB)^\dagger \neq (AB)^\dagger(AB)$.

Adding commutativity makes a difference. The following result is known for matrices [8].

THEOREM 4.2. *Suppose that $a, b \in \mathfrak{A}^\dagger_{\rm com}$ and $ab = ba$. Then $ab \in \mathfrak{A}^\dagger_{\rm com}$ and $(ab)^\dagger = a^\dagger b^\dagger = b^\dagger a^\dagger$.*

Proof. By Corollary 2.2, $a^\dagger = a^{\rm D}$ and $b^\dagger = b^{\rm D}$. The set $\{a, b, a^{\rm D}, b^{\rm D}\}$ is commutative. By [9, Theorem 5.5], ab is Drazin invertible with $(ab)^{\rm D} = a^{\rm D}b^{\rm D} = b^{\rm D}a^{\rm D}$. We verify that ab is simply polar with a selfadjoint spectral idempotent.

We have

$$\begin{aligned}(ab)(ab)^\pi &= ab(e - (ab)(ab)^{\rm D}) = ab(e - aa^{\rm D}bb^{\rm D}) \\ &= ab - (a^2a^{\rm D})(b^2b^{\rm D}) = ab - ab = 0,\end{aligned}$$

which shows that ab is simply polar. By Theorem 2.1, the spectral idempotents $a^\pi = e - a^{\rm D}a$ and $b^\pi = e - b^{\rm D}b$ are selfadjoint; hence $a^{\rm D}a$ and $b^{\rm D}b$ are selfadjoint, and

$$((ab)(ab)^{\rm D})^* = (aa^{\rm D}bb^{\rm D})^* = (bb^{\rm D})^*(aa^{\rm D})^* = bb^{\rm D}aa^{\rm D} = (ab)(ab)^{\rm D}.$$

Therefore $(ab)^\pi = e - (ab)(ab)^{\rm D}$ is selfadjoint, $ab \in \mathfrak{A}^\dagger_{\rm com}$ by Theorem 2.1, and $(ab)^\dagger = (ab)^{\rm D} = a^{\rm D}b^{\rm D} = a^\dagger b^\dagger$.

Hartwig and Katz [7, Theorem 1] recently gave necessary and sufficient conditions for the product of two EP matrices to be an EP matrix. The following theorem, the main result of this section, generalizes their result to C^*-algebras.

THEOREM 4.3. *Let $a, b \in \mathfrak{A}^\dagger_{\rm com}$ and let $a^{-1}(0)$, $b^{-1}(0)$ be finite-dimensional vector subspaces of $\mathfrak{A}$. Then the following conditions are equivalent*:

(i) $ab \in \mathfrak{A}^\dagger_{\rm com}$;
(ii) $(ab)a^\pi = 0$ *and* $b^\pi(ab) = 0$;
(iii) $a^{-1}(0) \subset (ab)^{-1}(0)$ *and* $b_{-1}(0) \subset (ab)_{-1}(0)$;
(iv) $(ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0)$ *and* $(ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0)$.

Proof. (i)$\Rightarrow$(ii). By Theorem 3.1(vii), there is $c \in \mathfrak{A}$ such that $ab = c(b^*a^*)$. Since a^π is selfadjoint, $(ab)a^\pi = cb^*a^*a^\pi = cb^*(a^\pi a)^* = 0$. From $ab \in \mathfrak{A}^\dagger_{\rm com}$ it follows that $b^*a^* \in \mathfrak{A}^\dagger_{\rm com}$; hence $(b^*a^*)b^\pi = 0$ by the foregoing argument. Hence $b^\pi(ab) = 0$.

(ii)⇒(iii). Since $(ab)a^\pi = 0$ by assumption and since $a^\pi\mathfrak{A} = a^{-1}(0)$ by Lemma 1.4, we have $a^{-1}(0) \subset (ab)^{-1}(0)$. Similarly, $b^\pi(ab) = 0$ and $\mathfrak{A}b^\pi = b_{-1}(0)$ imply $b_{-1}(0) \subset (ab)_{-1}(0)$.

(iii)⇒(iv). We note that $b^{-1}(0) \subset (ab)^{-1}(0)$. Write

$$X = (ab)^{-1}(0), \quad U = a^{-1}(0), \quad V = b^{-1}(0), \quad Y = b\mathfrak{A}.$$

Define $f : X/V \to \mathfrak{A}$ by $f(x + V) = bx$. The map f is linear and injective with $f(X/V) = U \cap Y$. Therefore X/V and $U \cap Y$ are isomorphic as vector spaces. Since $U \cap Y$ and V are finite-dimensional, so is X. By Lemma 1.4, $\mathfrak{A} = b\mathfrak{A} \oplus b^{-1}(0) = Y \oplus V$. Then

$$(U \cap Y) \oplus (U \cap V) \subset U \quad \text{and} \quad (U \cap Y) \oplus V \subset U + V \subset X,$$

taking into account the hypothesis $U = a^{-1}(0) \subset (ab)^{-1}(0) = X$. Then

$$\dim((U \cap Y) \oplus V) = \dim(U \cap Y) + \dim V = \dim(X/V) + \dim V = \dim X.$$

We conclude that $(U \cap Y) \oplus V = U + V = X$, that is, $(ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0)$. We note that $b_{-1}(0) \subset (ab)_{-1}(0)$ implies $b^{*-1}(0) \subset (ab)^{*-1}(0)$. Applying the foregoing result to b^*, a^*, b^*a^* in place of a, b, ab, we obtain $(b^*a^*)^{-1}(0) = a^{*-1}(0)+b^{*-1}(0)$. This, in turn, implies $(ab)_{-1}(0) = a_{-1}(0)+b_{-1}(0)$.

(iv)⇒(i). By Theorem 3.1, $a^{-1}(0) = a^{*-1}(0)$ and $b^{-1}(0) = b^{*-1}(0)$. Further, $(ab)_{-1}(0) = a_{-1}(0)+b_{-1}(0)$ is equivalent to $(ab)^{*-1}(0) = a^{*-1}(0)+b^{*-1}(0)$. Hence

$$(ab)^{*-1}(0) = a^{-1}(0) + b^{-1}(0) = (ab)^{-1}(0),$$

and $ab \in \mathfrak{A}^\dagger_{\mathrm{com}}$ follows on another application of Theorem 3.1.

The preceding theorem can be modified for the C^*-algebra $\mathfrak{A} = L(H)$ to obtain a result for bounded linear operators on a Hilbert space.

COROLLARY 4.4. *Let A, B be bounded linear upper semi-Fredholm operators on a Hilbert space H satisfying $A^\dagger A = AA^\dagger$ and $B^\dagger B = BB^\dagger$. Then the following conditions are equivalent:*

(i) $(AB)^\dagger AB = AB(AB)^\dagger$;
(ii) $(AB)A^\pi = 0$ *and* $B^\pi(AB) = 0$;
(iii) $N(A) \subset N(AB)$ *and* $R(AB) \subset R(B)$;
(iv) $N(AB) = N(A) + N(B)$ *and* $R(AB) = R(A) \cap R(B)$.

Proof. Theorem 4.3 cannot be directly applied in the setting of Hilbert space operators since the spaces $N(L_A)$, $N(L_B)$ (where $L_T : U \mapsto TU$) may be infinite-dimensional even though $N(A)$ and $N(B)$ are finite-dimensional.

Instead we retrace the proof of that theorem replacing formally

$$\begin{aligned} b\mathfrak{A} \quad &\text{by} \quad R(B), \\ a^{-1}(0),\ b^{-1}(0),\ (ab)^{-1}(0) \quad &\text{by} \quad N(A),\ N(B),\ N(AB), \\ a_{-1}(0),\ b_{-1}(0),\ (ab)_{-1}(0) \quad &\text{by} \quad N(A^*),\ N(B^*),\ N(B^*A^*), \end{aligned}$$

respectively. To this end we note that

$$b_{-1}(0) \subset (ab)_{-1}(0) \Leftrightarrow b^{*-1}(0) \subset (b^*a^*)^{-1}(0),$$
$$(ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0) \Leftrightarrow (b^*a^*)^{-1}(0) = a^{*-1}(0) + b^{*-1}(0).$$

Also, for any bounded linear operators A, B on H we have

$$N(B^*) \subset N(B^*A^*) \Leftrightarrow R(AB) \subset R(B),$$
$$N(B^*A^*) = N(A^*) + N(B^*) \Leftrightarrow R(AB) = R(A) \cap R(B).$$

We can then check that the proof of Theorem 4.3 appropriately modified yields the required result.

When the preceding corollary is specialized to matrices we get the product theorem of Hartwig and Katz [7], which answered a problem that was open for over 25 years (see [1]). For a proof of the Hartwig–Katz theorem it is enough to observe that matrices are Fredholm operators on a finite-dimensional Hilbert space, and to verify that

$$N(A) \subset N(AB) \Leftrightarrow \mathrm{RS}(AB) \subset \mathrm{RS}(A),$$
$$N(AB) = N(A) + N(B) \Leftrightarrow \mathrm{RS}(AB) = \mathrm{RS}(A) \cap \mathrm{RS}(B),$$

where $\mathrm{RS}(A)$ is the row space of the matrix A.

COROLLARY 4.5 [7, Theorem 1]. *Let A, B be* EP *matrices. Then the following are equivalent:*

(i) $R(AB) = R(A) \cap R(B)$ *and* $\mathrm{RS}(AB) = \mathrm{RS}(A) \cap \mathrm{RS}(B)$;
(ii) $R(AB) \subset R(B)$ *and* $\mathrm{RS}(AB) \subset \mathrm{RS}(A)$;
(iii) AB *is* EP.

A simple direct proof of the Hartwig–Katz theorem for matrices was given in [11].

Added in proof (March 2000). Recently Djordjević [15, Theorem 1] proved that, for closed range EP operators on a Hilbert space ($R(A^*) = R(A)$), conditions (i) and (iv) of Corollary 4.4 are equivalent. Lešnjak [16, Example] showed that, for general EP operators on a Hilbert space, condition (iii) of Corollary 4.4 need not imply condition (iv).

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On operator bands

by

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Abstract. A multiplicative semigroup of idempotent operators is called an operator band. We prove that for each $K > 1$ there exists an irreducible operator band on the Hilbert space l^2 which is norm-bounded by K. This implies that there exists an irreducible operator band on a Banach space such that each member has operator norm equal to 1.

Given a positive integer r, we introduce a notion of weak r-transitivity of a set of bounded operators on a Banach space. We construct an operator band on l^2 that is weakly r-transitive and is not weakly $(r+1)$-transitive.

We also study operator bands $\mathcal{S}$ satisfying a polynomial identity $p(A,B) = 0$ for all non-zero $A, B \in \mathcal{S}$, where p is a given polynomial in two non-commuting variables. It turns out that the polynomial $p(A,B) = (AB - BA)^2$ has a special role in these considerations.

1. Introduction. Let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on a (real or complex) Banach space X. A subset $\mathcal{S}$ of $\mathcal{B}(X)$ is said to be *irreducible* if the only closed subspaces of X invariant under all members of $\mathcal{S}$ are $\{0\}$ and X. Otherwise, $\mathcal{S}$ is called *reducible*. A set $\mathcal{S}$ of $\mathcal{B}(X)$ is said to be *triangularizable* if there is a chain of closed subspaces that are invariant under every member of $\mathcal{S}$ and this chain is maximal in the lattice of all closed subspaces of X.

An operator A on a vector space V is called *idempotent* if $A^2 = A$. A semigroup $\mathcal{S}$ of idempotents on V is called an *operator band.* If V is a Banach space, we also assume that all operators in $\mathcal{S}$ are bounded. Reducibility of operator bands on Hilbert spaces has recently been studied in [2], [5], and [6]. In [2] an irreducible operator band on the Hilbert space l^2 has been constructed. After having such an example it is natural to ask about the existence of irreducible operator bands with some additional properties. Sections 2 and 3 are devoted to this question. In Section 2 we construct an irreducible operator band on l^2 which is norm-bounded. This implies that

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