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The L^p solvability of the Dirichlet problems for parabolic equations

by

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Abstract. For two general second order parabolic equations in divergence form in $\mathrm{Lip}(1,1/2)$ cylinders, we give a criterion for the preservation of L^p solvability of the Dirichlet problems.

1. Introduction. The purpose of this article is to study the solvability of the L^p Dirichlet problem for second order parabolic divergence form operators with time dependent coefficients in a $\mathrm{Lip}(1,1/2)$ cylinder Ω. The operators L we consider are of the form

$$Lu = \operatorname{div}(A(x,t)\nabla u) - \partial_t u \quad \text{in } \Omega_T \in \mathbb{R}^{n+1}$$

where Ω_T is a finite cylinder having lateral boundary S_T and parabolic boundary $\partial_{\mathrm{p}}\Omega_T$, and the matrix $A(x,t)$ is assumed to be symmetric, bounded, measurable and to satisfy the ellipticity condition; that is, there exists a positive constant λ such that for all $(x,t) \in \mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \le \lambda|\xi|^2.$$

It is well known that if $f \in C(\partial_{\mathrm{p}}\Omega_T)$ is given, then the classical Dirichlet problem

$$Lu = 0 \quad \text{in } \Omega_T, \qquad u|_{\partial_{\mathrm{p}}\Omega_T} = f \in C(\partial_{\mathrm{p}}\Omega_T),$$

is solvable. The solvability of the L^p Dirichlet problem for L is related to the $D(p, S_T)$ property. If there exists a $p \in (1,\infty)$ such that the solution function u satisfies

$$\|N(u)\|_{L^p(S_T,d\sigma)} \le C\|f\|_{L^p(S_T,d\sigma)}, \qquad f \in C(S_T),$$

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where N denotes the non-tangential maximal operator, then we say that $D(p, S_T)$ holds for L.

From similar arguments to those used in [N] and [K], we know that if $D(p, S_T)$ holds for L, then given $f \in L^p(d\sigma, S_T)$ there exists a unique u with $Lu = 0$ in Ω_T such that $\lim_{t\to 0^+} u(x,t) = 0$ uniformly on compact subsets of $\Omega \cap \{t = 0\}$, and for $d\sigma$-a.e. $(Q,s) \in S_T$,

$$\lim_{(x,t)\in\Gamma_\alpha(Q,s)\cap\Omega_T,\,(x,t)\to(Q,s)} u(x,t) = f(Q,s)$$

where $\Gamma_\alpha(Q,s)$ is an appropriate parabolic non-tangential cone at (Q,s), and

$$\|N_\alpha(u)\|_{L^p(S_T,d\sigma)} \le C\|f\|_{L^p(S_T,d\sigma)}.$$

This shows that if $D(p, S_T)$ holds for L, then we can, for data in $L^p(d\sigma, S_T)$, uniquely solve the Dirichlet problem for L in Ω_T with an $L^p(d\sigma, S_T)$-estimate for the non-tangential maximal function.

Recently, Nyström [N] obtained the parabolic analog of the elliptic perturbation theory developed in [FKP]. Let $L_0 = \operatorname{div}(A_0(x,t)\nabla) - \partial_t$ and $L_1 = \operatorname{div}(A_1(x,t)\nabla) - \partial_t$ be two parabolic operators of the type described above with caloric measures ω_0 and ω_1. Define

$$a(x,t) = \sup_{O_{\delta(x,t)/4}(x,t)\cap\{t>0\}} |A_1(x,t) - A_0(x,t)|,$$

where $\delta(x,t) = \delta(x,t,S_T)$ is the parabolic distance from (x,t) to the lateral boundary S_T and $O_{\delta(x,t)/4}(x,t)$ is a parabolic cylinder of size $\delta(x,t)$ centered at (x,t). It has been proved in [N, Theorem 6.4] that if $D(p, S_T)$ holds for the operator L_0 and there exists a constant C such that

$$\sup_{0<r<r_0} \int_{\Delta_r} \frac{1}{\sigma(\Delta_r)} \bigg(\int_{\Gamma_{\gamma,r}(Q,s)} \frac{a(x,t)^2}{\delta(x,t,S_T)^{n+1}}\, dx\, dt \bigg) d\sigma(Q,s) \le C \tag{1.1}$$

for any surface cubes Δ_r in the lateral boundary S_T, then $D(q, S_T)$ holds for the operator L_1 with some other q.

In this paper, we will give a criterion depending on p for preservation of L^p solvability of the Dirichlet problem for data in $L^p(d\sigma, S_T)$, for the same p.

2. Notation and preliminaries. We retain the notations used in the introduction. In particular, we let Ω be a $\mathrm{Lip}(1,1/2)$ cylinder with constants M and r_0, whose boundary $\partial\Omega$ may be covered by a finite collection of congruent coordinate cylinders Z_i, $i = 1,\dots,N$. For each i, $Z_i = \{(x,t) = (x', x_n, t) : |x_j| < r_0,\ j = 1,\dots,n-1,\ |x_n| < 2nMr_0,\ t \in \mathbb{R}\}$ in some coordinate system on $\mathbb{R}^{n+1}$ (depending on i). We let $2Z_i$ denote the concentric double of Z_i. There exists a function $\varphi : \mathbb{R}^n \to (-Mr_0, Mr_0)$ satisfying the $\mathrm{Lip}(1,1/2)$ condition $|\varphi(x',t)-\varphi(y',s)| \le M(|x'-y'|^2+|t-s|)^{1/2}$ such that

$$2Z \cap \partial\Omega = \{(x', x_n, t) : x_n = \varphi(x',t)\} \cap 2Z,$$
$$2Z \cap \Omega = \{(x', x_n, t) : x_n > \varphi(x',t)\} \cap 2Z.$$

We denote by S the lateral boundary of Ω and by $\partial_p\Omega$ the parabolic boundary of Ω. In this paper, we consider the finite cylinder $\Omega_T = \Omega \cap \{0 < t < T\}$ with lateral boundary $S_T = S \cap \{0 < t < T\}$. Let $\delta(x,t,y,s) = |x-y| + |t-s|^{1/2}$ be the parabolic metric, and let $\delta(x,t)$ denote the parabolic distance from (x,t) to S_T. For $(Q,s) \in S_T$ and $0 < r < r_0$ we define

$$\Psi_r(Q,s) = \{(x', x_n, t) : |x_i - q_i| < r,\ |x_n - q_n| < 2nMr,\ |s-t| < r^2\},$$
$$\overline{A}_r(Q,s) = (Q', Q_n + 8nMr, t + 4r^2),$$
$$\underline{A}_r(Q,s) = (Q', Q_n + 8nMr, t - 4r^2).$$

We also denote by $\widetilde{\Psi}_r(Q,s) = \Psi_r(Q,s) \cap \Omega_T$ a Carleson region, by $\Box_r(Q,s) = \partial_p\Omega_T \cap \overline{\Psi}_r(Q,s)$ a parabolic surface cube with size r and center at $(Q,s) \in \partial_p\Omega_T$, and by $\Delta_r(Q,s) = S_T \cap \overline{\Psi}_r(Q,s)$ the lateral surface cube with center at $(Q,s) \in S_T$.

We remark that a $\mathrm{Lip}(1,1/2)$ domain is L-regular, that is, for any $\phi \in C(\partial\Omega)$ there exists a unique $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ such that $Lu = 0$ in Ω and for any $(Q,s) \in \partial\Omega$, $\lim_{(x,t)\to(Q,s),\,(x,t)\in\Omega} u(x,t) = \phi(Q,s)$. The function u may be constructed using the Perron–Wiener–Bauer method. For each $(x,t) \in \Omega$ the *L-caloric measure* $\omega(x,t,\cdot)$ is the unique probability Borel measure on $\partial_p\Omega$ with the property that the function

$$u(x,t) = \int_{\partial_p\Omega} \phi(y,l)\, d\omega(x,t,y,l)$$

is the unique solution of the Dirichlet problem $Lu = 0$ in Ω and $u|_{\partial_p\Omega} = \phi$.

The *Green function* of L on Ω with pole at $(x,t) \in \Omega$ is denoted by $G(x,t,y,l)$ and defined as

$$G(x,t,y,l) = \Gamma(x,t,y,l) - \int_{\partial_p\Omega} \Gamma(z,\tau,y,l)\, d\omega(x,t,z,\tau),$$

where $\Gamma(x,t,y,l)$ is the fundamental solution of L. We know that the L-caloric measure ω has the *doubling property*:

$$\omega(X_0, T, \Box_{2r}(Q,s)) \le C\omega(X_0, T, \Box_r(Q,s))$$

for any $(Q,s) \in \partial_p\Omega$, $0 \le s \le T - \delta_0$, $0 < r < \delta_0/4 \ll 1$, where C is a positive constant independent of (Q,s) and r.

We shall use the following results about parabolic operators (see [A], [M], [FGS], [N] for details).

THEOREM 2.1 (Local comparison theorem). *Let $(Q,s)\in S_T$, $0<\theta<1$, and u,v be two positive solutions of $Lu=0$ in $\Psi_{(1+\theta)r}(Q,s)$ vanishing continuously on $\Psi_{(1+\theta)r}(Q,s)\cap S$. Then there exists a constant $C=C(M,n,\lambda,\theta)$ such that if $r<r_0$ and $(x,t)\in\Psi_{(1-\theta)r}(Q,s)$, then*

$$\frac{u(x,t)}{v(x,t)}\le C\frac{u(\overline{A}_{(1+\theta)r/2}(Q,s))}{u(\underline{A}_{(1+\theta)r/2}(Q,s))}.$$

This theorem could be proved by slightly adjusting the arguments in [FGS], and it is related to a version in terms of the Green function and caloric measure for L.

LEMMA 2.2. *Suppose $\overline{G}(x,t,y,s)$ denotes the Green function on $\Omega\cap\{-1<t<T\}$. Let $(Q,s)\in\partial_p\Omega_T$ and $0<t<\min\{r_0/2,\sqrt{(T-s)/4}\}$. If $(x,t)\in\Omega\cap\{s+4r^2<t<T\}$ then*

$$C^{-1}r^nG(x,t,\overline{A}_r(Q,s))\le\omega(x,t,\Box_r(Q,s))\le Cr^n\overline{G}(x,t,\underline{A}_r(Q,s))$$

with a positive constant $C=C(M,n,\lambda)$.

Using the Carleson estimate, the maximum principle and the Hölder continuity (see Remark 2.2 in [N]), we also have the following lemma.

LEMMA 2.3. *Let $(Q,s)\in\partial_p\Omega_T$, $\theta>0$, and u be a non-negative solution of $Lu=0$ in $\widetilde{\Psi}_{(1+\theta)r}(Q,s)$ vanishing continuously on $\Box_{(1+\theta)r}(Q,s)$ for $r=2^j\delta<r_0/2$, where $\delta>0$, $j\in\mathbb{N}$. Then*

$$\sup_{\widetilde{\Psi}_\delta(Q,s)}u(x,t)\le 2^{-\alpha j}u(\overline{A}_{(1+\theta)r/2}(Q,s))$$

where $\alpha=\alpha(M,n,\lambda,\theta)$ is a positive number independent of r, δ, and j.

We now recall the maximal function operators which play a key role in the study of the Dirichlet problem. Given a measure μ and a function f on S_T, the *Hardy–Littlewood maximal function* of f with respect to μ is

$$M_\mu(f)(Q,s)=\sup_{(Q,s)\in\Delta_r}\frac{1}{\mu(\Delta_r)}\int_{\Delta_r}|f|\,d\mu.$$

We also define the *parabolic interior cone* with aperture $\alpha>1$ and vertex at $(Q,s)\in\partial_p\Omega$ to be $\Gamma_\alpha(Q,s)=\{(y,t)\in\Omega:\delta(y,t,Q,s)<\alpha\delta(y,t)\}$. Then the *non-tangential maximal function* of a function u on Ω is $N_\alpha(u)(Q,s)=\sup\{|u(x,t)|:(x,t)\in\Gamma_\alpha\}$, and the *averaged non-tangential maximal function*, a variant of N, is

$$\widetilde{N}_\alpha(u)(Q,s)=\sup_{(x,t)\in\Gamma_\alpha(Q,s)}\left(\frac{1}{|O_{\delta(x,t)/4}(x,t)|}\int_{O_{\delta(x,t)/4}(x,t)}|u(y,\tau)|^2\,dy\,d\tau\right)^{1/2},$$

where $O_r(x,t)$ is the parabolic cube $\{(y,s):|x-y|<r,\ |s-t|<r^2\}$. By the same argument as in [FKP, p. 76], we observe that $N_\alpha(u)$ and $\widetilde{N}_\alpha(u)$ are equivalent when u is a solution. Finally, the *area integral* S of u is given by

$$S_\alpha^2u(Q,s)=\int_{\Gamma_\alpha(Q,s)}|\nabla u(x,t)|\delta(x,t)^{2-n}\,dx\,dt,$$

where ∇u is understood in terms of L^2 averages by the energy inequality.

REMARK 2.4. As mentioned above, if u is the solution of the classical Dirichlet problem $Lu=0$ in Ω_T and $u|_{\partial_p\Omega_T}=f\in C(\partial_p\Omega_T)$, then we can write $u(x,t)=\int_{\partial_p\Omega_T}\phi(y,l)\,d\omega(x,t,y,l)$. Using Theorem 4.3 of [N], we have

$$N_\alpha(u)(Q,s)\le CM_\omega(f)(Q,s)$$

for all $(Q,s)\in\partial_p\Omega_T$. If $f\ge0$, we also have the opposite inequality.

3. Solvability of L^p Dirichlet problems. Now we consider two parabolic operators $L_0=\operatorname{div}(A_0(x,t)\nabla)-\partial_t$ and $L_1=\operatorname{div}(A_1(x,t)\nabla)-\partial_t$ as in Section 1. Denote by ω_0 and ω_1 the associated caloric measures, and by $d\sigma$ the surface measure on $\partial_p\Omega$. The Green functions are $G_0(x,t,y,s)$ and $G_1(x,t,y,s)$. Fix $(X_0,T)\in\Omega_T$ and take $\omega_i=\omega_i(X_0,T,\cdot)$ and $G_i(y,s)=G_i(X_0,T,y,s)$ for $i=1,2$. Set $\varepsilon(x,t)=A_0(x,t)-A_1(x,t)$, and define the *disagreement function*

$$|\varepsilon(x,t)|=\sup_{i,j}|\varepsilon_{ij}(x,t)|,\qquad a(y,s)=\sup_{(x,t)\in O_{\delta(y,s)/2}(y,s)}|\varepsilon(x,t)|.$$

We now formulate our main results.

THEOREM 3.1. *Suppose $1<p<\infty$ and $D(p,S_T)$ holds for the parabolic operator L_0. If there exists a constant C such that for every surface cube $\triangle\subseteq S_T$,*

$$\left\{\int_{(Q,s)\in\triangle}\left|\int_{\Gamma_\alpha(Q,s)}\frac{a(y,l)^2}{\delta(y,l)^{n+2}}\left|\frac{G_1(y,l)}{\delta(y,l)}\right|^2dy\,dl\right|^{q/2}\frac{d\sigma(Q,s)}{\sigma(\triangle)}\right\}^{1/q}\le C\frac{\omega_1(\triangle)}{\sigma(\triangle)},\tag{3.1}$$

where $1/p+1/q=1$, then $D(p,S_T)$ holds for the parabolic operator L_1.

Proof. Let u_i be the solutions of the Dirichlet problem for L_i, $i=0,1$. We need to show the a priori estimate $\|N(u_1)\|_p\le C\|f\|_p$. By using the Riesz decomposition for the parabolic operator L_1 in Ω_T (see [Do]), one can write

$$F(x,t)=u_1(x,t)-u_0(x,t)=\int_{\Omega_T}\nabla_yG_1(x,t,y,l)\cdot\varepsilon(y,l)\cdot\nabla_yu_0(y,l)\,dy\,dl.$$

Therefore, all we have to do is to prove $\|\widetilde{N}(F)\|_p\le C\|f\|_p$. Without loss of generality, we may assume by the Harnack principle that $\varepsilon(x,t)=0$ if

$(x,t) \in \Omega_T \cap \{(y,s) : \delta(y,s) > r_0\}$, and we assume $r_0 \ll 1$. From now on, let $(Q_0, s_0) \in S_T$ and $(x,t) \in \Gamma(Q_0, s_0)$ be fixed, and break $F(z,\tau)$ into two parts when $(z,\tau) \in O_{\delta(x,t)/4}(x,t) = O(x,t)$:

$$(3.2)\quad F(z,\tau) = \int\limits_{O(x,t)} \nabla_y G_1(z,\tau,y,l)\cdot\varepsilon(y,l)\cdot\nabla_y u_0(y,l)\,dy\,dl + \int\limits_{\Omega_T\setminus O(x,t)} \nabla_y G_1(z,\tau,y,l)\cdot\varepsilon(y,l)\cdot\nabla_y u_0(y,l)\,dy\,dl = F_1(z,\tau) + F_2(z,\tau).$$

Now we show that the theorem follows from the inequality

$$(3.3)\quad \widetilde{N}(F_1)(Q_0,s_0) + N(F_2)(Q_0,s_0) \le CS(u_0)(Q_0,s_0) + C\{M[S(u_0)^p](Q_0,s_0)\}^{1/p}$$

where $M = M_\sigma$ and the constant C is independent of (Q_0, s_0). Indeed, from (3.3) one can deduce that $N(u_1)(Q,s) \le CS(u_0)(Q_0,s_0) + CN(u_0)(Q,s) + C\{M_\sigma[S(u_0)^p](Q_0,s_0)\}^{1/p}$. Note that the maximal function M satisfies a weak type $(1,1)$ estimate (cf. [Mu]), and furthermore u_0 is a solution; so we can prove that $N(u_1)$ satisfies a weak type (p,p) estimate with respect to the surface measure:

$$\sigma\{(Q,s)\in S_T : N(u_1)(Q,s) > \alpha\} \le \frac{C}{\alpha^p}\|S(u_0)\|^p_{L^p(d\sigma,S_T)} \le \frac{C}{\alpha^p}\|f\|^p_{L^p(d\sigma,S_T)}.$$

Now using Remark 2.4, we conclude that $M_{\omega_1}(f)$ satisfies the corresponding weak type (p,p) estimate with respect to the surface measure σ. But the weak type (p,p) estimate for the Hardy–Littlewood maximal operator M_{ω_1} is in fact equivalent to the strong type (p,p) estimate $\|M_{\omega_1}(f)\|_{L^p(d\sigma)} \le C\|f\|_{L^p(d\sigma)}$. Furthermore, by Remark 2.4 again we have $\|N(u_1)\|_{L^p(d\sigma)} \le C\|M_{\omega_1}(f)\|_{L^p(d\sigma)}$, which completes the proof of Theorem 3.1.

We now turn to the proof of the inequality (3.3). We use the notation established above and remark that the square functions over cones of different apertures are comparable [St]. We will complete the proof with two lemmas, 3.2 and 3.3.

LEMMA 3.2. *We have*

$$\widetilde{N}(F_1)(Q_0,s_0) \le CS(u_0)(Q_0,s_0).$$

Proof. To estimate $\widetilde{N}(F_1)(Q_0,s_0)$, the averages

$$\left(\frac{1}{|O(x,t)|}\int\limits_{O(x,t)} |F_1(z,\tau)|^2\,dz\,d\tau\right)^{1/2}$$

are used. We consider $F_1(z,\tau)$, the part of the potential near the pole of $G_1(z,\tau,y,l)$, which may be estimated by the following adaptation of the argument in [N, p. 236]. For $(z,\tau) \in O(x,t)$, let $\widetilde{G}_1(z,\tau,y,l)$ be the Green function of the domain $2O(x,t)$ for L_1, and define $K(z,\tau,y,l) = G_1(z,\tau,y,l) - \widetilde{G}_1(z,\tau,y,l)$. We write $F_1(z,\tau) = F_{11}(z,\tau) + F_{12}(z,\tau)$, where

$$F_{11}(z,\tau) = \int\limits_{O(x,t)} \nabla_y \widetilde{G}_1(z,\tau,y,l)\varepsilon(y,l)\nabla_y u_0(y,l)\,dy\,dl,$$

$$F_{12}(z,\tau) = \int\limits_{O(x,t)} \nabla_y K(z,\tau,y,l)\varepsilon(y,l)\nabla_y u_0(y,l)\,dy\,dl.$$

Now letting χ_E denote the characteristic function of the set E, we first find that F_{11} is the solution of

$$L_1F_{11}(z,\tau) = -\operatorname{div}(\varepsilon(z,\tau)\nabla_z u_0(z,\tau)\chi_{O(x,t)}(z,\tau)) \quad \text{in } 2O(x,t),$$

and $F_{11}|_{\partial_p(2O(x,t))} = 0$. Using the ellipticity and integration by parts, we have

$$\lambda \int\limits_{2O(x,t)} |\nabla F_{11}|^2\,dz\,d\tau \le \int\limits_{2O(x,t)} A_1\nabla F_{11}\cdot\nabla F_{11}\,dz\,d\tau$$
$$= \int\limits_{2O(x,t)} F_{11}\operatorname{div}(\varepsilon\nabla u_0\chi_O)\,dz\,d\tau - \frac{1}{2}\int\limits_{2O(x,t)} \frac{\partial(F_{11})^2}{\partial\tau}\,dz\,d\tau$$
$$= -\int\limits_{O(x,t)} \varepsilon\nabla F_{11}\cdot\nabla u_0\,dz\,d\tau - \frac{1}{2}\int\limits_{2O(x,t)\cap\{\tau=\tau_0=t+\delta^2/4\}} F_{11}(z,\tau_0)^2\,dz$$
$$\le -\int\limits_{O(x,t)} \varepsilon\nabla F_{11}\cdot\nabla u_0\,dz\,d\tau.$$

Hence, the Hölder inequality and Sobolev inequality show that

$$\left(\frac{1}{|O(x,t)|}\int\limits_{O(x,t)} |F_{11}(z,\tau)|^2\,dz\,d\tau\right)^{1/2} \le C\delta(x,t)\left(\frac{1}{|O(x,t)|}\int\limits_{O(x,t)} |\nabla u_0(z,\tau)|^2\,dz\,d\tau\right)^{1/2},$$

which gives $\widetilde{N}_\alpha(F_{11})(Q_0,s_0) \le CS_\alpha(u_0)(Q_0,s_0)$ for some α sufficiently large, by taking the supremum over $\Gamma_\alpha(Q_0,s_0)$.

On the other hand, we note that $L_1^*K(z,\tau,y,l) = 0$ for $(y,l)\in 2O(x,t)$ and (z,τ) fixed. Using Cauchy–Schwarz and the energy estimate on K, one can get

$$|F_{12}(z,\tau)| \le C \int\limits_{O(x,t)} |\nabla K(z,\tau,y,l)|\,|\nabla u_0(y,l)|\,dy\,dl$$
$$\le C\delta^{-1}\Big(\int\limits_{\frac{4}{3}O(x,t)} |K(x,t,y,l)|^2\,dy\,dl\Big)^{1/2}\Big(\int\limits_{O(x,t)} |\nabla u_0(y,l)|^2\,dy\,dl\Big)^{1/2}$$
$$\le C\delta^n\Big(\frac{1}{|\frac{4}{3}O(x,t)|}\int\limits_{\frac{4}{3}O(x,t)} |K(x,t,y,l)|^2\,dy\,dl\Big)^{1/2} S_\alpha(u_0)(Q_0,s_0)$$

for α sufficiently large. Then the Harnack principle and Aronson's [A] estimates on G_0 and $\widetilde{G}_0$ imply that $|F_{12}(z,\tau)| \le CS_\alpha(u_0)(Q_0,s_0)$. Now we have obtained

$$\widetilde{N}_\alpha(F_1)(Q_0,s_0) \le CS_\alpha(u_0)(Q_0,s_0) \le CS(u_0)(Q_0,s_0).$$

LEMMA 3.3. *We have*

$$N(F_2)(Q,s) \le CS(u_0)(Q,s) + C\{M[S(u_0)^p](Q,s)\}^{1/p}.$$

Proof. We shall handle F_2 pointwise by breaking up the region $\Omega_T \setminus O(x,t)$ further into dyadic ring-type regions as follows: Let $\delta = \delta(x,t)$, and for $i \in \mathbb{N}$ set

$$\Omega_i = \Psi_{2^i\delta}(Q_0,s_0) \cap \Omega_T, \quad R_i = \Omega_i \setminus \Omega_{i-1}, \quad (X_i,T_i) = \overline{A}_{2^i\delta}(Q_0,s_0),$$

and let $\triangle_i = \Omega_i \cap S_T$. For $D \subset \Omega_T$ we also define

$$F(D) = \int\limits_{D\setminus O(x,t)} \nabla_y G_1(x,t,y,l)\varepsilon(y,l)\nabla_y(y,l)\,dy\,dl.$$

Let i_0 be the largest integer such that $\Omega_{i_0} \cap O(x,t) = \emptyset$, and j_0 be the largest integer such that $2^{j_0}\delta(x,t) < r_0$. Then

$$F_2(x,t) = F(\Omega_{i_0}) + \sum_{i=i_0+1}^{j_0} F(R_i) + F(\Omega_T \setminus \Omega_{j_0}), \tag{3.4}$$

and each of these integrals can be bounded, which gives us bounds for $N(F_2)$. The essential part of the proof is the bound for $F(\Omega_{i_0})$. The additional pieces of the potential will then be estimated by similar methods, and summed up appropriately.

We first remark that Ω_{i_0} can be covered by parabolic boxes whose dimension compares with their distance from S_T and whose projection onto S_T is a dyadic surface cube contained in $\triangle_{i_0}$. Let $\Gamma_h(Q,s)$ be the truncated cone of height $h \approx \operatorname{diam}(\triangle_{i_0})$. We can write

$$|F(\Omega_{i_0})| \le C \int\limits_{\triangle_{i_0}} \int\limits_{\Gamma_h(Q,s)} |\varepsilon(y,l)|$$
$$\times |\nabla_y G_1(x,t,y,l)|\,|\nabla u_0(y,l)|\delta(y,l)^{-(n+1)}\,dy\,dl\,d\sigma(Q,s)$$
$$\le C\int\limits_{\triangle_{i_0}} \sum_{I\subset\triangle_{i_0}} r(I)^{-(n+1)}$$
$$\times \int\limits_{\Gamma_h^I(Q,s)} |\varepsilon(y,l)|\,|\nabla_y G_1(x,t,y,l)|\,|\nabla u_0(y,l)|\,dy\,dl\,d\sigma(Q,s)$$

where I's are the parabolic dyadic cubes contained in $\triangle_{i_0}$, $\Gamma_h^I(Q,s) = I^+ \cap \Gamma_h(Q,s)$, I^+ is the corresponding box in Ω_T whose projection is just the cube I and whose length $r(I^+)$ compares with the distance from S_T. Let I^* be the smallest rectangle such that $\Gamma_h^I(Q,s) \subseteq I^* \subseteq I^+$. Then the Cauchy–Schwarz inequality and the energy estimate give

$$|F(\Omega_{i_0})| \le C \int\limits_{\triangle_{i_0}} \sum_{I\subset\triangle_{i_0}} \sup_{(y,l)\in\Gamma_h^I(Q,s)} |\varepsilon(y,l)| r(I)^{-n-2}$$
$$\times \Big(\int\limits_{\frac{3}{2}I^*} |G_1(x,t,y,l)|^2\,dy\,dl\Big)^{1/2}\Big(\int\limits_{\Gamma_h^I} |\nabla u_0|^2\,dy\,dl\Big)^{1/2} d\sigma(Q,s).$$

We note that there is a fixed constant C such that $|I^*| \le C|\Gamma_h^I|$ for any cubes I, and there is a sufficiently large γ such that $3I^* \subset \Gamma_{\gamma,h}$. Let $\overline{G}(x,t,y,l)$ denote the Green function on $\Omega \cap \{-1 < t < T\}$. Then one can see from the maximum principle in adjoint variables that $\overline{G}(x,t,y,l) = G(x,t,y,l)$ if $(y,l) \in \Omega_T$. By the local comparison theorem and Lemma 2.2 we may move the pole of the Green function $G_1(x,t,y,l)$, for any $(y,l) \in I^*$, from (x,t) to (X_0,T) and conclude that

$$\frac{G_1(x,t,y,l)}{G_1(X_0,T,y,l)} = \frac{\overline{G}_1(x,t,y,l)}{\overline{G}_1(X_0,T,y,l)} \le \frac{C}{\omega_1(\triangle_{i_0})},$$

where we have used the doubling property of ω_1. Hence $|F(\Omega_{i_0})|$ is bounded by a constant multiple of

$$\frac{C}{\omega_1(\triangle_{i_0})} \int\limits_{\triangle_{i_0}} \sum_{I\subset\triangle_{i_0}} r(I)^{-(n+1)}$$
$$\times \sup_{(y,l)\in\Gamma_h^I(Q,s)} |\varepsilon(y,l)|\Big(\int\limits_{\frac{3}{2}I^*} \Big|\frac{G_1(y,l)}{\delta(y,l)}\Big|^2\,dy\,dl\Big)^{1/2}$$
$$\times \Big(\int\limits_{\Gamma_h^I(Q,s)} |\nabla u_0(y,l)|^2\,dy\,dl\Big)^{1/2} d\sigma(Q,s)$$

$$\le \frac{C}{\omega_1(\triangle_{i_0})} \int\limits_{\triangle_{i_0}} \Big(\sum_{I\subset\triangle_{i_0}} \int\limits_{\frac{3}{2}I^*} \frac{a(y,l)^2}{\delta(y,l)^{n+2}} \Big| \frac{G_1(y,l)}{\delta(y,l)} \Big|^2 \, dy\, dl \Big)^{1/2}$$
$$\times \Big(\sum_{I\subset\triangle_{i_0}} \int\limits_{\Gamma_h^I(Q,s)} \frac{|\nabla u_0(y,l)|^2}{\delta(y,l)^n} \, dy\, dl \Big)^{1/2} d\sigma(Q,s)$$
$$\le \frac{C\sigma(\triangle_{i_0})}{\omega_1(\triangle_{i_0})} \Big\{ \frac{1}{\sigma(\triangle_{i_0})} \int\limits_{\triangle_{i_0}} \Big(\int\limits_{\Gamma_{\gamma,h}} \frac{a(y,l)^2}{\delta(y,l)^{n+2}} \Big| \frac{G_1(y,l)}{\delta(y,l)} \Big|^2 \, dy\, dl \Big)^{q/2} d\sigma(Q,s) \Big\}^{1/q}$$
$$\times \Big\{ \frac{1}{\sigma(\triangle_{i_0})} \int\limits_{\triangle_{i_0}} \Big(\int\limits_{\Gamma(Q,s)} \frac{|\nabla u_0(y,l)|^2}{\delta(y,l)^n} \, dy\, dl \Big)^{p/2} d\sigma(Q,s) \Big\}^{1/p}.$$

Now we apply condition (3.1) to obtain

$$|F(\Omega_{i_0})| \le C\{M[S(u_0)^p](Q_0,s_0)\}^{1/p}. \tag{3.5}$$

We next estimate $F(R_i)$ for $i = i_0+1,\ldots,j_0$. We further subdivide $R_i\backslash O(x,t) = [(R_i\backslash O(x,t))\cap\Gamma(Q_0,s_0)]\cup[(R_i\backslash O(x,t))\backslash\Gamma(Q_0,s_0)] = R_i^1\cup R_i^2$. Then $F(R_i^1)$ may be estimated by the expression

$$C \sum_{I\subset\triangle_i} \sup_{I^+\cap R_i^1} |\varepsilon(y,l)| \Big(\int\limits_{I^+\cap R_i^1} \frac{|G_1(x,t,y,l)|^2}{r_i^2} \, dy\, dl \Big)^{1/2}$$
$$\times \Big(\int\limits_{I^+\cap R_i^1} |\nabla u_0(y,l)|\, dy\, dl \Big)^{1/2},$$

where $r_i = 2^i\delta$. At this point, we need to move the pole of the Green function from (x,t) to (X_{i+1},T_{i+1}). Using Lemma 2.3 we have

$$G_1(x,t,y,l) \le C2^{-\alpha i} G_1(X_{i-1},T_{i-1},y,l)$$

for any $(y,l)\in R_i$, if $(x,t)\in\Omega_{i-1}$. We note $G_1(x,t,y,l)\ne 0$ only if $l<t$, and apply the Harnack principle and Lemma 2.3 again to get

$$G_1(x,t,y,l) \le C2^{-\alpha i} G_1(X_{i+1},T_{i+1},y,l) \le C2^{-\alpha i} \frac{G_1(y,l)}{\omega_1(\triangle_i)}$$

for any $(y,l)\in R_i$, where the last inequality may be obtained by the comparison theorem as before. Continuing our estimate, and noting that $r_i\approx\delta(y,l)$ for $(y,l)\in R_i^1$, we have the bound

$$|F(R_i^1)| \le \frac{C2^{-\alpha i}}{\omega_1(\triangle_i)} r_i^{n+1} \Big(\int\limits_{R_i^1} \frac{a(y,l)^2}{\delta(y,l)^{n+2}} \Big| \frac{G_1(y,l)}{\delta(y,l)} \Big|^2 \, dy\, dl \Big)^{1/2} S(u_0)(Q_0,s_0).$$

Now we observe that $R_i^1 \subseteq \Gamma_\gamma(Q,s)$ for all $(Q,s)\in\triangle_i$, for a fixed aperture

γ that does not depend on i. Thus the above is bounded by

$$\frac{C2^{-\alpha i}}{\omega_1(\triangle_i)} \int\limits_{(Q,s)\in\triangle_i} \Big(\int\limits_{\Gamma_\gamma(Q,s)} \frac{a(y,l)^2}{\delta(y,l)^{n+2}} \Big| \frac{G_1(y,l)}{\delta(y,l)} \Big|^2 \, dy\, dl \Big)^{1/2} d\sigma(Q,s)$$
$$\times S(u_0)(Q_0,s_0) \le C2^{-i\alpha} S(u_0)(Q_0,s_0),$$

by Hölder's inequality and condition (3.1), where C is independent of i.

It remains to estimate $F(R_i^2)$. Without loss of generality we can assume that R_i^2 may be covered by at most K, K independent of i, domains $\Psi_{\varrho_k}(Q_k,s_k)$ where $\varrho_k\sim r_i$ and $(Q_k,s_k)\in R_i^2\cap S_T$. Applying the similar argument, plus the doubling property of the surface measure, we conclude

$$|F(\Psi_{\varrho_k}(Q_k,s_k))| \le C2^{-i\alpha}\{M[S(u_0)^p](Q_0,s_0)\}^{1/p},$$

with C independent of i and k.

Altogether, we have shown the estimate

$$|F(\Omega_{i_0})| + \sum_{i=i_0+1}^{j_0} |F(R_i)| \le CS(u_0)(Q_0,s_0) + C\{M[S(u_0)^p](Q_0,s_0)\}^{1/p} \tag{3.6}$$

for any $(Q_0,s_0)\in S_T$.

Finally, we should bound the last piece, $F(\Omega_T\setminus\Omega_{j_0})$. Note that if we let $R = [(\Omega_T\setminus\Omega_{j_0})\setminus O(x,t)]\cap\{(y,l):\delta(y,l)<r_0\}$, then the region R can be broken up into $k=1,\ldots,K$ Carleson type regions $\Psi_{r_k}(Q_k,s_k)$ of length $\simeq r_0$, where K depends only on the dimension. Then for each k, we can bound $F(\Psi_{r_k}(Q_k,s_k))$ with the familiar methods used repeatedly above, except that we require the following lemma 3.4 which will enable us to use the condition (3.1) without moving the pole of $G_1(x,t,y,l)$.

LEMMA 3.4. *If the condition* (3.1) *holds, then for* $\triangle\subset S_T\cap\partial R$, *there is a constant* C *such that*

$$\Big\{ \int\limits_{(Q,s)\in\triangle} \Big| \int\limits_{R\cap\Gamma_\alpha(Q,s)} \frac{a(y,l)^2}{\delta(y,l)^{n+2}} \Big| \frac{G_1(x,t,y,l)}{\delta(y,l)} \Big|^2 \, dy\, dl \Big|^{q/2} \frac{d\sigma(Q,s)}{\sigma(\triangle)} \Big\}^{1/q} \le C\frac{\omega_1(x,t,\triangle)}{\sigma(\triangle)}.$$

This lemma can be deduced from Lemma 2.1, Lemma 2.2 and the parabolic Harnack principle. Now we can obtain $|F(R)|\le C\{M[S(u_0)^p](Q_0,s_0)\}^{1/p}$, and this completes the proof of Lemma 3.3.

REMARK 3.5. A similar argument allows one to show that Lemmas 3.2 and 3.3 hold for any point $(Q_0,s_0)\in\partial_p\Omega_T$.

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Elements of C^*-algebras commuting with their Moore–Penrose inverse

by

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Abstract. We give new necessary and sufficient conditions for an element of a C^*-algebra to commute with its Moore–Penrose inverse. We then study conditions which ensure that this property is preserved under multiplication. As a special case of our results we recover a recent theorem of Hartwig and Katz on EP matrices.

1. Introduction. The novelty of our approach to the study of Moore–Penrose inverse in C^*-algebras is considering it in terms of the Drazin inverse. For elements of a C^*-algebra that commute with their Moore–Penrose inverse, the Moore–Penrose inverse in fact coincides with the Drazin inverse. Proofs found in the literature may resort to special constructions, often very ingenious. Many of these arguments can now be presented more systematically relying on standard properties of the Drazin inverse and on properties of spectral idempotents.

We retain the notation of [10]. In particular, $\mathfrak{A}$ is a unital C^*-algebra with unit e; next, $\mathfrak{A}^{-1}$, $\mathrm{QN}(\mathfrak{A})$ and $\mathfrak{A}^\dagger$ denote the sets of all invertible, quasinilpotent and regular elements of $\mathfrak{A}$, respectively. An element $a \in \mathfrak{A}$ is *quasipolar* if 0 is an isolated—possibly removable—singularity of the resolvent of a, and *polar* if 0 is at most a pole of the resolvent. By $\sigma(a)$ we denote the spectrum of $a \in \mathfrak{A}$.

The set of all quasipolar elements of $\mathfrak{A}$ will be denoted by $\mathfrak{A}^{\mathrm{D}}$. Observe that $\mathfrak{A}^{\mathrm{D}} \cap \mathfrak{A}^\dagger \supset \mathfrak{A}^{-1}$. We write $L(H)$ for the C^*-algebra of all bounded linear operators on a Hilbert space H.

PROPOSITION 1.1 [9, Theorem 4.2]. *Let $a \in \mathfrak{A}$. Then the following conditions are equivalent*:

(i) $a \in \mathfrak{A}^{\mathrm{D}}$.

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