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Two-parameter Hardy–Littlewood inequality and its variants

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Abstract. Let s^* denote the maximal function associated with the rectangular partial sums $s_{mn}(x,y)$ of a given double function series with coefficients c_{jk}. The following generalized Hardy–Littlewood inequality is investigated:

$$\|s^*\|_{p,\mu} \le C_{p,\alpha,\beta}\Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}|c_{jk}|^p\Big\}^{1/p},$$

where $\overline{\xi} = \max(\xi, 1)$, $0 < p < \infty$, and μ is a suitable positive Borel measure. We give sufficient conditions on c_{jk} and μ under which the above Hardy–Littlewood inequality holds. Several variants of this inequality are also examined. As a consequence, the $\|\cdot\|_{p,\mu}$-convergence property of $s_{mn}(x,y)$ is established. These results generalize the work of Askey–Wainger [1], Balashov [2], Boas [3], Chen [5], [6], [8], [9], Marzug [15], Móricz [16]–[18], [19], Móricz–Schipp–Wade [20], Ram–Bhatia [22], Stechkin [24], Weisz [26]–[28], and Young [30].

1. Introduction. Let $\Phi = \{\phi_n\}_{n=0}^{\infty}$ be a uniformly bounded family of Borel measurable functions defined on a finite interval $I \subset \mathbb{R}$. We assume that the following condition is satisfied by some $t_0 \in I$:

$$\sup_{n\ge 0; t\in I} |(t-t_0)D_n(t)| < \infty, \tag{1.1}$$

where $D_n(t) = \sum_{j=0}^{n}\phi_j(t)$. Then

$$s_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\phi_j(x)\phi_k(y) \quad (m,n\ge 0;\ x,y\in I)$$

are known as the *rectangular partial sums* of the double function series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\phi_j(x)\phi_k(y) \quad (x,y\in I). \tag{1.2}$$

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We are interested in finding conditions on the coefficients c_{jk}, the value of p, and the positive Borel measure μ under which $s_{mn}(x,y)$ converges in $L^p(I^2, d\mu)$ as $\min(m,n) \to \infty$. Here $L^p(I^2,d\mu)$ denotes the space of all f for which $\|f\|_{p,\mu} < \infty$, where

$$\|f\|_{p,\mu} = \Big(\iint_{I^2} |f(x,y)|^p \, d\mu\Big)^{1/p} \quad (0 < p < \infty),$$
$$\|f\|_{\infty,\mu} = \operatorname*{ess\,sup}_{(x,y)\in I^2} |f(x,y)|.$$

For $d\mu = dxdy$, we write $\|f\|_p$ instead of $\|f\|_{p,\mu}$, where $dxdy$ is the Lebesgue measure on I^2.

The above-mentioned problem has drawn attention of mathematicians for a long time. It is closely related to the magnitude problem for $\|s^*\|_{p,\mu}$, where

$$s^*(x,y) = \sup_{m,n\ge 0} |s_{mn}(x,y)| \quad (x,y \in I).$$

In [7], the first author and G.-B. Chen discussed the case $p = \infty$. Their results generalize the work of Chaundy–Jolliffe [4], Jolliffe [12], Nurcombe [21], and Xie–Zhou [29].

In this paper, we focus our attention on the case $0 < p < \infty$. Associated with the sequence $\{c_{jk} : j,k \ge 0\}$, we introduce the following four types of numbers:

$$c_{jk}^{00} = \sum_{u=0}^{j}\sum_{v=0}^{k} |c_{uv}|, \qquad c_{jk}^{10} = \Big\{\bar{j}\sum_{u=j}^{\infty}\Big\}\Big\{\sum_{v=0}^{k}\Big\}|\Delta_{10}c_{uv}|,$$
$$c_{jk}^{01} = \Big\{\sum_{u=0}^{j}\Big\}\Big\{\bar{k}\sum_{v=k}^{\infty}\Big\}|\Delta_{01}c_{uv}|, \quad c_{jk}^{11} = \Big\{\bar{j}\sum_{u=j}^{\infty}\Big\}\Big\{\bar{k}\sum_{v=k}^{\infty}\Big\}|\Delta_{11}c_{uv}|.$$

Here $\bar{\xi} = \max(\xi, 1)$ and the finite differences $\Delta_{\alpha\beta}c_{jk}$ are defined by

$$\Delta_{\alpha\beta}c_{jk} = \sum_{s=0}^{\alpha}\sum_{t=0}^{\beta}(-1)^{s+t}\binom{\alpha}{s}\binom{\beta}{t}c_{j+s,k+t}.$$

Denote by $\mathbb{N}_0$ the set of all nonnegative integers. Set $\Omega_\infty = \{t_0\}$ and $\Omega_j = \{t \in I : 1/(j+1) < |t - t_0| \le 1/j\}$ for $j \in \mathbb{N}_0$. Corresponding to μ, let $\mu^\#$ denote the measure on $\mathbb{N}_0 \times \mathbb{N}_0$ with

$$\mu^\#(\{(j,k)\}) = \mu(\Omega_j \times \Omega_k) \quad (j,k \ge 0),$$

and $\|\cdot\|_{p,\mu^\#}$ is defined by

$$\|\{d_{jk}\}\|_{p,\mu^\#} = \Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |d_{jk}|^p \mu^\#(\{(j,k)\})\Big)^{1/p} \quad (0 < p < \infty).$$

In Theorem 2.1, we shall investigate the validity of the following inequality:

$$\|s^*\|_{p,\mu} \le C_{p,\mu} \sum_{0\le\gamma,\delta\le 1} \|\{c_{jk}^{\gamma\delta}\}\|_{p,\mu^\#}, \tag{1.3}$$

where $C_{p,\mu}$ is a constant. We prove that (1.3) is true for all $0 < p < \infty$ provided the following two conditions are satisfied:

$$c_{jk} \to 0 \quad \text{as } \max(j,k) \to \infty, \tag{1.4}$$
$$\mu(I \times \Omega_\infty) = \mu(\Omega_\infty \times I) = 0. \tag{1.5}$$

Our result is the L_p-version of [7, Lemma 2.1].

For $v_0 \ge 0$, we introduce a new measure $\mu^\#_{v_0}$ on $\mathbb{N}_0 \times \mathbb{N}_0$, defined by

$$\mu^\#_{v_0}(\{(j,k)\}) = \begin{cases} \mu(\Omega_j \times \Omega_k) & (j,k \ge v_0); \\ \mu(\Omega_j \times I) & (j \ge v_0;\ 0 \le k < v_0); \\ \mu(I \times \Omega_k) & (0 \le j < v_0;\ k \ge v_0); \\ \mu(I^2) & (0 \le j,k < v_0). \end{cases}$$

In Theorem 3.4, we transform (1.3) into the form

$$\|s^*\|_{p,\mu} \le C_{p,\mu,v_0}\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}, \tag{1.6}$$

where $\|\cdot\|_{p,\mu^\#,v_0}$ is obtained from $\|\cdot\|_{p,\mu^\#}$ by changing $\mu^\#$ to $\mu^\#_{v_0}$. We shall verify that (1.6) is true for $1 \le p < \infty$ provided (1.4)–(1.5) and the following two conditions are satisfied for some constant C_μ:

$$\mu(\Omega_j^\infty \times \Omega_k) \le C_\mu\{\bar{j}\mu(\Omega_j \times \Omega_k)\} \quad (j \ge v_0;\ 0 \le k < \infty), \tag{1.7}$$
$$\mu(\Omega_j \times \Omega_k^\infty) \le C_\mu\{\bar{k}\mu(\Omega_j \times \Omega_k)\} \quad (0 \le j < \infty;\ k \ge v_0), \tag{1.8}$$

where $\Omega_\gamma^\delta \equiv \bigcup_{k=\gamma}^{\delta} \Omega_k$. The measure $\mu^\#_{v_0}$ and the number $\|\cdot\|_{p,\mu^\#,v_0}$ are introduced in order to overcome the problem that $\Omega_j = \emptyset$ for small $j \in \mathbb{N}_0$. This problem arises for some function systems, such as the bounded Vilenkin system and the Paley–Walsh system. For these two systems, we see $\Omega_0 = \emptyset$ (cf. the paragraph after Corollary 3.5).

Conditions (1.5) and (1.7)–(1.8) are satisfied by the measure $d\mu = |x - t_0|^\alpha |y - t_0|^\beta dxdy$, where $\alpha, \beta > -1$ and v_0 is the smallest nonnegative integer with $\sup_{t\in I}|t - t_0| > 1/(v_0 + 1)$. For such μ and $1 \le p < \infty$, we shall prove that (1.6) reduces to

$$\|s^*\|_{p,\mu}^p \le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}|\Delta_{11}c_{uv}|\Big\}^p. \tag{1.9}$$

Our result generalizes Askey–Wainger [1] and Ram–Bhatia [22] (cf. Corollary 3.5). Let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of z, respectively. Consider the case

$$\operatorname{Re}\Delta_{11}c_{jk} \ge 0, \quad \operatorname{Im}\Delta_{11}c_{jk} \ge 0 \quad (j,k \ge 0). \tag{1.10}$$

Then (1.9) can be further reduced to

$$\|s^*\|_{p,\mu} \le C_{p,\alpha,\beta}\Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}|c_{jk}|^p\Big\}^{1/p} \tag{1.11}$$

(cf. Corollary 3.6). This extends Móricz [16]–[18], Stechkin [24], and Weisz [26]–[28] from the case $\alpha=\beta=0$ to the range $\alpha,\beta>-1$, and generalizes Chen [8], [9], Marzug [15]. The particular case $\alpha=\beta=0$ of inequality (1.11) is known as the two-parameter Hardy–Littlewood inequality.

Related to the work of [6], we consider $d\mu=|\phi(x)\psi(y)|dxdy$, where ϕ and ψ are Borel measurable functions defined on I. Assume that θ and ϑ are positive functions defined on $[1,\infty)$ such that the following two inequalities hold for some constants C_ϕ and C_ψ:

$$\int_{I\setminus\Omega_{j+1}^{\infty}}\left|\frac{\phi(x)}{x-t_0}\right|dx \le C_\phi\theta(\bar{j}) \qquad (j\ge 0), \tag{1.12}$$

$$\int_{I\setminus\Omega_{k+1}^{\infty}}\left|\frac{\psi(y)}{y-t_0}\right|dy \le C_\psi\vartheta(\bar{k}) \qquad (k\ge 0). \tag{1.13}$$

In Corollary 3.7, we shall prove that (1.6) with $p=1$ reduces to

$$\|s^*\|_{1,\mu} \le C_{\phi\psi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(\bar{j})\vartheta(\bar{k})|\Delta_{11}c_{jk}|. \tag{1.14}$$

This generalizes Balashov [2], Boas [3], Chen [5], [6], Marzug [15], Móricz [17], [19], Móricz–Schipp–Wade [20], and Young [30].

In §4, we extend the theory developed in §2–§3 to the trigonometric system $\{e^{int}\}_{n=-\infty}^{\infty}$. Following the proofs given in §2–§4, we also see that the whole theory can be extended to any dimension without difficulty.

Throughout this paper, the symbol $C_{p,\mu}$ and its variants denote positive constants depending on the parameters concerned. The constants are not necessarily the same at each occurrence. If no ambiguity can arise, we also use $\{c_{jk}\}$ to denote both $\{c_{jk}: j,k\ge 0\}$ and $\{c_{jk}: -\infty<j,k<\infty\}$.

2. Inequality (1.3). Let $0<p<\infty$ and $\gamma,\delta=0,1$. Denote by $S_{p,\mu\#}^{\gamma\delta}$ the space of all sequences $\{c_{jk}: j,k\ge 0\}$ satisfying (1.4) and $\|\{c_{jk}^{\gamma\delta}\}\|_{p,\mu\#}<\infty$. Condition (1.4) implies inequalities of the type $|c_{jk}|\le\sum_{i=j}^{\infty}|\Delta_{10}c_{ik}|$, etc. We should keep this fact in mind whenever we use (1.4). Set

$$\chi_{mn}(j,k)=\begin{cases}1 & \text{if } |j|>m \text{ or } |k|>n,\\ 0 & \text{otherwise,}\end{cases}$$

and $c_{jk}(m,n)=\chi_{mn}(j,k)c_{jk}$. Define the sums $c_{jk}^{\gamma\delta}(m,n)$ from $c_{jk}^{\gamma\delta}$ by changing c_{jk} to $c_{jk}(m,n)$. The first main result of this paper reads as follows. It is the L_p-version of [7, Lemma 2.1].

THEOREM 2.1. *Assume that $0<p<\infty$ and μ satisfies (1.5). Then there exists $C_{p,\mu}<\infty$ such that (1.3) holds for all $\{c_{jk}\}\in\bigcap_{0\le\gamma,\delta\le 1}S_{p,\mu\#}^{\gamma\delta}$. Moreover, for such $\{c_{jk}\}$, s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n)\to\infty$, and*

$$\|f\|_{p,\mu}\le C_{p,\mu}\sum_{0\le\gamma,\delta\le 1}\|\{c_{jk}^{\gamma\delta}\}\|_{p,\mu\#}, \tag{2.1}$$

$$\|s_{mn}-f\|_{p,\mu}\le C_{p,\mu}\sum_{0\le\gamma,\delta\le 1}\|\{c_{jk}^{\gamma\delta}(m,n)\}\|_{p,\mu\#}. \tag{2.2}$$

Proof. Let $M=[1/|x-t_0|]$ and $N=[1/|y-t_0|]$, where $[\cdot]$ means the greatest integral part. Then for $m,n\ge 0$, we have

$$|s_{mn}(x,y)|\le\Sigma_{11}+\Sigma_{12}+\Sigma_{21}+\Sigma_{22},$$

where

$$\Sigma_{\alpha\beta}=\Big|\sum_{j=0}^{m}\sum_{k=0}^{n}\chi_M^\alpha(j)\chi_N^\beta(k)c_{jk}\phi_j(x)\phi_k(y)\Big|,$$

$\chi_M^1=\chi_{[0,M]}$, and $\chi_M^2=\chi_{\mathbb{R}\setminus[0,M]}$. Obviously, $\Sigma_{11}\le\lambda^2\sum_{j=0}^{M}\sum_{k=0}^{N}|c_{jk}|=\lambda^2c_{MN}^{00}$, where $\lambda=\sup_n\|\phi_n\|_\infty$. Set $\tau=\sup_{n\ge 0;t\in I}|(t-t_0)D_n(t)|$. Then $|D_k(y)|\le 2\tau\overline{N}$ for all k. By (1.4) and summation by parts, we get

$$\begin{aligned}\Sigma_{12}&\le\Big|\sum_{j=0}^{m}\sum_{k=0}^{n}\chi_M^1(j)\Delta_{01}(\chi_N^2(k)c_{jk})\phi_j(x)D_k(y)\Big|\\&\quad+\Big|\sum_{j=0}^{m}\chi_M^1(j)\chi_N^2(n+1)c_{j,n+1}\phi_j(x)D_n(y)\Big|\\&\le 2\lambda\tau\overline{N}\sum_{j=0}^{M}\sum_{k=0}^{\infty}|\Delta_{01}(\chi_N^2(k)c_{jk})|\le 4\lambda\tau\overline{N}\sum_{j=0}^{M}\sum_{k=N}^{\infty}|\Delta_{01}c_{jk}|=4\lambda\tau c_{MN}^{01}.\end{aligned}$$

Analogously, $\Sigma_{21}\le 4\lambda\tau c_{MN}^{10}$. We have $|D_j(x)|\le 2\tau\overline{M}$ and $|D_k(y)|\le 2\tau\overline{N}$ for all $j,k\ge 0$. The double summation by parts implies

$$\Sigma_{22}\le 4\tau^2\overline{M}\overline{N}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}(\chi_M^2(j)\chi_N^2(k)c_{jk})|\le 16\tau^2c_{MN}^{11}.$$

Putting these together yields

$$\begin{aligned}|s^*(x,y)|&\le\lambda^2c_{MN}^{00}+4\lambda\tau c_{MN}^{01}+4\lambda\tau c_{MN}^{10}+16\tau^2c_{MN}^{11}\\&=J_1(x,y)+J_2(x,y)+J_3(x,y)+J_4(x,y),\quad\text{say.}\end{aligned} \tag{2.3}$$

For $j,k \geq 0$ and $(x,y) \in \Omega_j \times \Omega_k$, we have $M = j$ and $N = k$, and so $J_1(x,y) = \lambda^2 c_{jk}^{00}$. By (1.5), we get

$$\|J_1\|_{p,\mu}^p = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \iint_{\Omega_j \times \Omega_k} |J_1(x,y)|^p \, d\mu = \lambda^{2p} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |c_{jk}^{00}|^p \mu(\Omega_j \times \Omega_k) = \lambda^{2p} \|\{c_{jk}^{00}\}\|_{p,\mu^\#}^p. \tag{2.4}$$

Similarly, we have

$$\|J_2\|_{p,\mu}^p \leq (4\lambda\tau)^p \|\{c_{jk}^{01}\}\|_{p,\mu^\#}^p, \tag{2.5}$$

$$\|J_3\|_{p,\mu}^p \leq (4\lambda\tau)^p \|\{c_{jk}^{10}\}\|_{p,\mu^\#}^p, \tag{2.6}$$

$$\|J_4\|_{p,\mu}^p \leq (16\tau^2)^p \|\{c_{jk}^{11}\}\|_{p,\mu^\#}^p. \tag{2.7}$$

Putting (2.3)–(2.7) together yields (1.3). Denote by $s_{mn}^*(x,y)$ the maximal function associated with the rectangular partial sums of the double function series $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}(m,n)\phi_j(x)\phi_k(y)$. Then the previous result implies $\|s_{mn}^*\|_{p,\mu} \leq C_{p,\mu} \sum_{0\leq\gamma,\delta\leq 1} \|\{c_{jk}^{\gamma\delta}(m,n)\}\|_{p,\mu^\#}$. For $M > \max(m,n)$, we have

$$\begin{aligned}\|s_{mn} - s_{MM}\|_{p,\mu} &= \Big\|\sum_{j=0}^{M}\sum_{k=0}^{M} c_{jk}(m,n)\phi_j(x)\phi_k(y)\Big\|_{p,\mu} \\ &\leq \|s_{mn}^*\|_{p,\mu} \leq C_{p,\mu} \sum_{0\leq\gamma,\delta\leq 1} \|\{c_{jk}^{\gamma\delta}(m,n)\}\|_{p,\mu^\#} \\ &\to 0 \quad \text{as } \min(m,n) \to \infty.\end{aligned}$$

Hence, $\{s_{MM}\}_{M=1}^{\infty}$ forms a Cauchy sequence in $L^p(I^2, d\mu)$. Let f be its limit in $L^p(I^2, d\mu)$. Then

$$\|f\|_{p,\mu} \leq \|s^*\|_{p,\mu} \leq C_{p,\mu} \sum_{0\leq\gamma,\delta\leq 1} \|\{c_{jk}^{\gamma\delta}\}\|_{p,\mu^\#},$$

$$\|s_{mn} - f\|_{p,\mu} = \lim_{M\to\infty} \|s_{mn} - s_{MM}\|_{p,\mu} \leq C_{p,\mu} \sum_{0\leq\gamma,\delta\leq 1} \|\{c_{jk}^{\gamma\delta}(m,n)\}\|_{p,\mu^\#}. \ \blacksquare$$

3. Variants of (1.3). Let $S_{p,\mu^\#,v_0}^{11}$ (respectively, $S_{p,\alpha,\beta}^{11}, S_{p,\alpha,\beta}^{\mathrm{HL}}, S_{\theta,\vartheta}^{11}$) be the space of all sequences $\{c_{jk} : j,k \geq 0\}$ for which (1.4) is satisfied and the right-hand side of (1.6) (respectively, (1.9), (1.11), (1.14)) is finite. Denote by S_+^{11} the set of all $\{c_{jk} : j,k \geq 0\}$ subject to (1.4) and (1.10). The purpose of this section is to investigate the validity of the following inclusion relations:

$$S_+^{11} \cap S_{p,\alpha,\beta}^{\mathrm{HL}} \subset S_{p,\alpha,\beta}^{11} \subset S_{p,\mu^\#,v_0}^{11} \subset \bigcap_{0\leq\gamma,\delta\leq 1} S_{p,\mu^\#}^{\gamma\delta},$$

$$S_{\theta,\vartheta}^{11} \subset S_{1,\mu^\#,v_0}^{11},$$

and then, to derive (1.6), (1.9), (1.11), (1.14) from Theorem 2.1.

As defined in [14], we say that a nonnegative sequence $\{a_n\}_{n=1}^{\infty}$ is *quasi-decreasing* if there exists $\gamma > 0$ such that

$$a_{n+k} \leq \gamma a_n \qquad (1 \leq k \leq n). \tag{3.1}$$

Analogously, $\{a_{mn} : m,n \geq 1\}$ is said to be quasi-decreasing if $a_{mn} \geq 0$ for all $m,n \geq 1$ and there exists $\gamma > 0$ such that

$$a_{m+j,n} \leq \gamma a_{mn} \qquad (1 \leq j \leq m;\ n \geq 1), \tag{3.2}$$

$$a_{m,n+k} \leq \gamma a_{mn} \qquad (m \geq 1;\ 1 \leq k \leq n). \tag{3.3}$$

In the following, $(1,n)$ denotes the interval $\{1 < x < n\}$, $\overline{A}_n$ denotes the closure of A_n in $\mathbb{R}$, and $k \in \overline{A}_n$ means that k runs over all positive integers in $\overline{A}_n$.

Lemma 3.1. *Let $\lambda_n \geq 0$ $(n = 1,2,\ldots)$ and $\{A_n\}_{n=1}^{\infty}$ be $\{(1,n)\}_{n=1}^{\infty}$ or $\{(n,\infty)\}_{n=1}^{\infty}$. Assume there exists $0 \leq \alpha < \infty$ such that*

$$\sum_{k\in\overline{A}_n} \lambda_k \leq \alpha n \lambda_n \qquad (n = 1,2,\ldots) \tag{3.4}$$

Then for $1 \leq p < \infty$, there exists $C_{p,\alpha}$ $(= \alpha^p p^p) < \infty$ such that

$$\sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k\in(1,\infty)\setminus A_n} a_k\Big)^p \leq C_{p,\alpha} \sum_{n=1}^{\infty} \lambda_n (n a_n)^p \tag{3.5}$$

for all nonnegative sequences $\{a_n\}_{n=1}^{\infty}$. Inequality (3.5) *remains true for $0 < p < 1$ provided that the $C_{p,\alpha}$ is replaced by $C_{p,\alpha,\gamma}$ and $\{a_n\}_{n=1}^{\infty}$ is further assumed to be quasi-decreasing, where γ is defined by* (3.1).

Lemma 3.1 is a generalization of an inequality of Hardy and Littlewood. It can be proved directly from [13] and [14, Theorem B]. Applying it twice, we can easily obtain the following extension of [18, Lemma 1]. Móricz's result corresponds to the special case $\lambda_{mn} = m^{-c_1} n^{-c_2}$.

Lemma 3.2. *Let $\lambda_{mn} \geq 0$ $(m,n = 1,2,\ldots)$ and any of $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be of the form $\{(1,k)\}_{k=1}^{\infty}$ or $\{(k,\infty)\}_{k=1}^{\infty}$. Assume there exists $0 \leq \alpha < \infty$ such that*

$$\sum_{j\in\overline{A}_m} \lambda_{jn} \leq \alpha m \lambda_{mn} \qquad (m,n \geq 1), \tag{3.6}$$

$$\sum_{k\in\overline{B}_n} \lambda_{mk} \leq \alpha n \lambda_{mn} \qquad (m,n \geq 1). \tag{3.7}$$

Then for $1 \leq p < \infty$, there exists $C_{p,\alpha}$ $(= \alpha^{2p} p^{2p}) < \infty$ such that

$$(3.8)\qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big(\sum_{j\in\overline{(1,\infty)\setminus A_m}}\ \sum_{k\in\overline{(1,\infty)\setminus B_n}} a_{jk}\Big)^p \le C_{p,\alpha}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}(mna_{mn})^p$$

for all nonnegative sequences $\{a_{mn} : m,n \ge 1\}$. *Inequality* (3.8) *remains true for* $0 < p < 1$ *provided that the* $C_{p,\alpha}$ *is replaced by* $C_{p,\alpha,\gamma}$ *and* $\{a_{mn} : m,n \ge 1\}$ *is further assumed to be quasi-decreasing, where* γ *is defined by* (3.2) *and* (3.3).

THEOREM 3.3. *Let* $1 \le p < \infty$ *and* $v_0 \ge 0$. *Assume that* μ *satisfies* (1.5) *and* (1.7)–(1.8). *Then* $S^{11}_{p,\mu^\#,v_0} \subset \bigcap_{0\le\alpha,\beta\le1} S^{\alpha\beta}_{p,\mu^\#}$. *Moreover, there exists* $C_{p,\mu,v_0} < \infty$ *such that for* $\alpha,\beta = 0,1$,

$$(3.9)\qquad \|\{c^{\alpha\beta}_{jk}\}\|_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|_{p,\mu^\#,v_0} \quad (\{c_{jk}\}\in S^{11}_{p,\mu^\#,v_0}).$$

Inequality (3.9) *remains true for* $0 < p < 1$ *provided that the* C_{p,μ,v_0} *is replaced by* $C_{p,\mu,v_0,\gamma}$ *and the* c_{jk} *is further assumed to satisfy conditions* (3.10)–(3.13) *below for some* $\gamma > 0$:

$$(3.10)\qquad |c_{m+j,k}| \le \gamma|c_{mk}| \qquad (m \ge v_0;\ k \ge 0;\ 1 \le j \le m - v_0 + 1),$$

$$(3.11)\qquad |c_{j,n+k}| \le \gamma|c_{jn}| \qquad (j \ge 0;\ n \ge v_0;\ 1 \le k \le n - v_0 + 1),$$

$$(3.12)\qquad |\Delta_{10}c_{j,n+k}| \le \gamma|\Delta_{10}c_{jn}| \qquad (j \ge 0;\ n \ge v_0;\ 1 \le k \le n - v_0 + 1),$$

$$(3.13)\qquad |\Delta_{01}c_{m+j,k}| \le \gamma|\Delta_{01}c_{mk}| \qquad (m \ge v_0;\ k \ge 0;\ 1 \le j \le m - v_0 + 1).$$

Proof. First, consider $1 \le p < \infty$ and $\alpha = \beta = 0$. Set

$$d_{jk} = c_{jk}\chi_{[v_0,\infty)\times[v_0,\infty)}(j,k), \qquad f_{jk} = c_{jk}\chi_{[v_0,\infty)\times[0,v_0)}(j,k),$$
$$g_{jk} = c_{jk}\chi_{[0,v_0)\times[v_0,\infty)}(j,k), \qquad h_{jk} = c_{jk}\chi_{[0,v_0)\times[0,v_0)}(j,k).$$

Then

$$\|\{c^{00}_{jk}\}\|_{p,\mu^\#} \le \|\{d^{00}_{jk}\}\|_{p,\mu^\#} + \|\{f^{00}_{jk}\}\|_{p,\mu^\#} + \|\{g^{00}_{jk}\}\|_{p,\mu^\#} + \|\{h^{00}_{jk}\}\|_{p,\mu^\#}.$$

Let $\lambda_{mn} = \mu(\Omega_{m+v_0-1}\times\Omega_{n+v_0-1})$ and $a_{mn} = |d_{m+v_0-1,n+v_0-1}|$. By (1.7) and (1.8), we get

$$\sum_{j=m}^{\infty}\lambda_{jn} = \mu(\Omega^{\infty}_{m+v_0-1}\times\Omega_{n+v_0-1}) \le C_\mu(1+v_0)(m\lambda_{mn}),$$

$$\sum_{k=n}^{\infty}\lambda_{mk} = \mu(\Omega_{m+v_0-1}\times\Omega^{\infty}_{n+v_0-1}) \le C_\mu(1+v_0)(n\lambda_{mn}).$$

Applying Lemma 3.2 to this case yields

$$(3.14)\qquad \|\{d^{00}_{jk}\}\|^p_{p,\mu^\#} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big(\sum_{u=1}^{m}\sum_{v=1}^{n}a_{uv}\Big)^p \le C_{p,\mu,v_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}(mna_{mn})^p$$
$$\le C_{p,\mu,v_0}\|\{d^{11}_{jk}\}\|^p_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

Set $\lambda_m = \mu(\Omega_{m+v_0-1}\times I)$ and $a_m = \sum_{v=0}^{v_0-1}|f_{m+v_0-1,v}|$. Then (1.5) and (1.7) ensure the validity of (3.4) with $\alpha = C_\mu(1+v_0)$. Hence, Lemma 3.1 implies

$$(3.15)\qquad \|\{f^{00}_{jk}\}\|^p_{p,\mu^\#} \le \sum_{m=1}^{\infty}\lambda_m\Big(\sum_{u=1}^{m}a_u\Big)^p \le C_{p,\mu,v_0}\sum_{m=1}^{\infty}\lambda_m(ma_m)^p$$
$$\le C_{p,\mu,v_0}\|\{f^{11}_{jk}\}\|^p_{p,\mu^\#,v_0} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

Similarly, we have

$$(3.16)\qquad \|\{g^{00}_{jk}\}\|^p_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{g^{11}_{jk}\}\|^p_{p,\mu^\#,v_0} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

From the definition of h_{jk}, we get

$$(3.17)\qquad \|\{h^{00}_{jk}\}\|^p_{p,\mu^\#} \le \Big(\sum_{j=0}^{v_0-1}\sum_{k=0}^{v_0-1}|c_{jk}|\Big)^p\mu(I^2) \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

Putting (3.14)–(3.17) together gives (3.9) for the case $\alpha = \beta = 0$.

Now, change the definitions of λ_{mn} and a_{mn} to $\lambda_{jn} = \mu(\Omega_j\times\Omega_{n+v_0-1})$ and $a_{jn} = \bar{j}\sum_{u=j}^{\infty}|\Delta_{10}d_{u,n+v_0-1}|$. Then (1.4), (1.8), and Lemma 3.1 together give

$$(3.18)\qquad \|\{d^{10}_{jk}\}\|^p_{p,\mu^\#} = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{jn}\Big(\sum_{v=1}^{n}a_{jv}\Big)^p \le C_{p,\mu,v_0}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{jn}(na_{jn})^p$$
$$\le C_{p,\mu,v_0}\|\{d^{11}_{jk}\}\|^p_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

The same argument also implies

$$(3.19)\qquad \|\{g^{10}_{jk}\}\|^p_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{g^{11}_{jk}\}\|^p_{p,\mu^\#} \le C_{p,\mu,v_0}\|\{c^{11}_{jk}\}\|^p_{p,\mu^\#,v_0}.$$

By the definition of f_{jk} and (1.4)–(1.5), we obtain

$$(3.20)\qquad \|\{f^{10}_{jk}\}\|^p_{p,\mu^\#} \le \sum_{j=0}^{\infty}\Big(\Big\{\bar{j}\sum_{u=j}^{\infty}\Big\}\Big\{\sum_{v=0}^{v_0-1}\Big\}|\Delta_{10}f_{uv}|\Big)^p\sum_{k=0}^{\infty}\mu(\Omega_j\times\Omega_k)$$
$$= \sum_{j=0}^{\infty}\Big(\Big\{\bar{j}\sum_{u=j}^{\infty}\Big\}\Big\{\sum_{v=0}^{v_0-1}\Big\}|\Delta_{10}f_{uv}|\Big)^p\mu(\Omega_j\times I)$$

$$\le C_{p,\mu,v_0}\sum_{j=0}^{\infty}\sum_{v=0}^{v_0-1}\Big(\bar{j}\sum_{u=j}^{\infty}|\Delta_{10}f_{uv}|\Big)^p\mu(\Omega_j\times I)$$
$$\le C_{p,\mu,v_0}\|\{f_{jk}^{11}\}\|_{p,\mu^\#,v_0}^p \le C_{p,\mu,v_0}\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}^p.$$

Analogously, we have

$$\|\{h_{jk}^{10}\}\|_{p,\mu^\#}^p \le C_{p,\mu,v_0}\|\{h_{jk}^{11}\}\|_{p,\mu^\#,v_0}^p \le C_{p,\mu,v_0}\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}^p. \tag{3.21}$$

Thus, (3.9) with $\alpha=1$ and $\beta=0$ follows from (3.18)–(3.21). We can prove (3.9) with $\alpha=0$ and $\beta=1$ in a similar way. As for $\alpha=\beta=1$, it can be derived from the definitions of $\|\cdot\|_{p,\mu^\#}$ and $\|\cdot\|_{p,\mu^\#,v_0}$. From (3.10)–(3.13), we see that the sequences $\{a_{mn}: m,n\ge 1\}$, $\{a_{jn}\}_{n=1}^{\infty}$, and $\{a_m\}_{m=1}^{\infty}$ are quasi-decreasing, where $0\le j<\infty$. Therefore, the above proof still works for $0<p<1$. We leave it to the readers. ∎

THEOREM 3.4. *Let $1\le p<\infty$ and $v_0\ge 0$. Assume that μ satisfies (1.5) and (1.7)–(1.8). Then there exists $C_{p,\mu,v_0}<\infty$ such that inequality (1.6) holds for all $\{c_{jk}\}\in S_{p,\mu^\#,v_0}^{11}$. Moreover, for $\{c_{jk}\}\in S_{p,\mu^\#,v_0}^{11}$, s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n)\to\infty$, and*

$$\|f\|_{p,\mu}\le C_{p,\mu,v_0}\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}, \tag{3.22}$$
$$\|s_{mn}-f\|_{p,\mu}\le C_{p,\mu,v_0}\|\{c_{jk}^{11}(m,n)\}\|_{p,\mu^\#,v_0}. \tag{3.23}$$

These conclusions remain true for $0<p<1$ provided that (3.23) is replaced by (2.2), the C_{p,μ,v_0} in (1.6) and (3.22) is replaced by $C_{p,\mu,v_0,\gamma}$, and the c_{jk} is further assumed to satisfy (3.10)–(3.13) for some $\gamma>0$.

Proof. Consider the case $1\le p<\infty$. For $\{c_{jk}\}\in S_{p,\mu^\#,v_0}^{11}$, we have $\{c_{jk}(m,n): j,k\ge 0\}\in S_{p,\mu^\#,v_0}^{11}$. Theorem 3.3 tells us that

$$\|\{c_{jk}^{\alpha\beta}\}\|_{p,\mu^\#}\le C_{p,\mu,v_0}\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}\quad(\alpha,\beta=0,1),$$
$$\|\{c_{jk}^{\alpha\beta}(m,n)\}\|_{p,\mu^\#}\le C_{p,\mu,v_0}\|\{c_{jk}^{11}(m,n)\}\|_{p,\mu^\#,v_0}\quad(\alpha,\beta=0,1;m,n\ge 0).$$

Thus, Theorem 3.4 with $1\le p<\infty$ follows directly from Theorem 2.1. As indicated in Theorem 3.3, (3.9) is still true if $0<p<1$ and $\{c_{jk}\}$ satisfies (3.10)–(3.13). Hence, the desired results for $0<p<1$ still come from Theorem 2.1. ∎

Obviously, $S_{p,\mu^\#,v_0}^{11}=S_{p,\mu^\#}^{11}$ for $v_0=0$, and (1.6) with $v_0=0$ reduces to the following form:

$$\|s^*\|_{p,\mu}\le C_{p,\mu}\|\{c_{jk}^{11}\}\|_{p,\mu^\#}. \tag{3.24}$$

This indicates that the special case $v_0=0$ of Theorem 3.4 improves Theorem 2.1. In what follows, we focus our attention on the case $d\mu=|\phi(x)\psi(y)|dxdy\ne 0$. In this case, condition (1.5) is automatically satisfied.

Moreover, (1.7) and (1.8) are equivalent to the following two conditions:

$$\int_{\Omega_j^\infty}|\phi(x)|\,dx\le C_\phi\int_{\Omega_j}\left|\frac{\phi(x)}{x-t_0}\right|dx\quad(j\ge v_0), \tag{3.25}$$
$$\int_{\Omega_k^\infty}|\psi(y)|\,dy\le C_\psi\int_{\Omega_k}\left|\frac{\psi(y)}{y-t_0}\right|dy\quad(k\ge v_0). \tag{3.26}$$

These two conditions are satisfied in many cases. For them, (1.6) can take other forms. The first example is $d\mu=|x-t_0|^\alpha|y-t_0|^\beta dxdy$, where $\alpha,\beta>-1$. This corresponds to $\phi(x)=|x-t_0|^\alpha$ and $\psi(y)=|y-t_0|^\beta$. An elementary calculation shows that (3.25) and (3.26) hold for all $j,k\ge v_0$, where v_0 is the smallest nonnegative integer with

$$\sup_{t\in I}|t-t_0|>\frac{1}{v_0+1}. \tag{3.27}$$

For $j,k\ge 0$, we have $\mu_{v_0}^\#(\{(j,k)\})\le C_\mu(\bar{j})^{-\alpha-2}(\bar{k})^{-\beta-2}$, which implies

$$\|\{c_{jk}^{11}\}\|_{p,\mu^\#,v_0}^p \le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}|\Delta_{11}c_{uv}|\Big\}^p. \tag{3.28}$$

Therefore, $S_{p,\alpha,\beta}^{11}\subset S_{p,\mu^\#,v_0}^{11}$. For such μ and v_0, we also have

$$\|\{c_{jk}^{11}(m,n)\}\|_{p,\mu^\#,v_0}^p\le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\times\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}\chi_{mn}(u,v)|\Delta_{11}c_{uv}|\Big\}^p. \tag{3.29}$$

By Theorem 3.4, we obtain

COROLLARY 3.5. *Let $1\le p<\infty$ and $d\mu=|x-t_0|^\alpha|y-t_0|^\beta dxdy$, where $\alpha,\beta>-1$. Then there exists $C_{p,\alpha,\beta}<\infty$ such that inequality (1.9) holds for all $\{c_{jk}\}\in S_{p,\alpha,\beta}^{11}$. Moreover, for $\{c_{jk}\}\in S_{p,\alpha,\beta}^{11}$, s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n)\to\infty$, and*

$$\|f\|_{p,\mu}^p\le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}|\Delta_{11}c_{uv}|\Big\}^p, \tag{3.30}$$

$$\|s_{mn}-f\|_{p,\mu}^p\le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\times\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}\chi_{mn}(u,v)|\Delta_{11}c_{uv}|\Big\}^p. \tag{3.31}$$

These conclusions remain true for $0<p<1$ *provided that* (3.31) *is replaced by* (2.2), *the* $C_{p,\alpha,\beta}$ *in* (1.9) *and* (3.30) *is replaced by* $C_{p,\alpha,\beta,\gamma}$, *and the* c_{jk} *is further assumed to satisfy* (3.10)–(3.13) *for some* $\gamma>0$ *and some* $v_0\ge 0$.

Consider the following cases of $\phi_n(t)$: $\cos nt$, $\sin nt$, e^{int}, $\omega_n(t)$, and $P_n(t)$, where $\{\omega_n\}_{n=0}^{\infty}$ denotes the Paley–Walsh system, or more generally, the bounded Vilenkin system, and $\{P_n\}_{n=0}^{\infty}$ represents the Legendre system. As proved in [11], [23], [25], and [31], these families are uniformly bounded and satisfy (1.1). The corresponding I, t_0, and v_0 are

$$\begin{array}{llll} I=[-\pi,\pi], & t_0=0, & v_0=0 & \text{(trigonometric system)},\\ I=[0,1), & t_0=0, & v_0=1 & \text{(bounded Vilenkin system)},\\ I=[0,1), & t_0=0, & v_0=1 & \text{(Paley–Walsh system)},\\ I=[-1,1], & t_0=1, & v_0=0 & \text{(Legendre system)}, \end{array}$$

where t_0 is subject to (1.1) and v_0 is the smallest nonnegative integer defined by (3.27). Hence, the theory developed so far can be applied to any of the above systems. In particular, Corollary 3.5 works for these systems. It extends [1] and [22] from the cosine system to any of them. Corollary 3.5 generalizes [1, Theorem 2] and [22, Theorem 2]. It extends them from the range $-1<\alpha,\beta<p-1$ to $\alpha,\beta>-1$. It also expands the range $1\le p<\infty$ to $0<p<\infty$.

For $\{c_{jk}\}\in S_+^{11}$, we have $\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}|\Delta_{11}c_{uv}|\le 2|c_{jk}|$, and so

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\Big\{\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}|\Delta_{11}c_{uv}|\Big\}^p \le 2^p\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}|c_{jk}|^p. \tag{3.32}$$

This implies that $S_+^{11}\cap S_{p,\alpha,\beta}^{\mathrm{HL}}\subset S_{p,\alpha,\beta}^{11}$. In this case, (1.9) reduces to (1.11). Moreover, any $\{c_{jk}\}$ in S_+^{11} satisfies (3.10)–(3.13) with $\gamma=2$ and $v_0=0$, and

$$\sum_{u=j}^{\infty}\sum_{v=k}^{\infty}\chi_{mn}(u,v)|\Delta_{11}c_{uv}|\le\begin{cases}2|c_{jk}| & \text{if } j>m \text{ or } k>n;\\ 2|c_{m+1,k}|+2|c_{j,n+1}| & \text{if } j\le m \text{ and } k\le n.\end{cases}$$

By Corollary 3.5, we obtain

COROLLARY 3.6. *Let* $0<p<\infty$ *and* $d\mu=|x-t_0|^{\alpha}|y-t_0|^{\beta}dxdy$, *where* $\alpha,\beta>-1$. *Then there exists* $C_{p,\alpha,\beta}<\infty$ *such that inequality* (1.11) *holds for all* $\{c_{jk}\}\in S_+^{11}\cap S_{p,\alpha,\beta}^{\mathrm{HL}}$. *Moreover, for* $\{c_{jk}\}\in S_+^{11}\cap S_{p,\alpha,\beta}^{\mathrm{HL}}$, s_{mn} *converges in* $L^p(I^2,d\mu)$ *to some function* f *as* $\min(m,n)\to\infty$, *and*

$$\|f\|_{p,\mu}\le C_{p,\alpha,\beta}\Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}|c_{jk}|^p\Big\}^{1/p}. \tag{3.33}$$

For $1\le p<\infty$, *we also have*

$$\begin{aligned}\|s_{mn}-f\|_{p,\mu}^p\le C_{p,\alpha,\beta}\Big\{&\Big(\sum_{j=0}^{m}(\bar{j})^{p-\alpha-2}\Big)\sum_{k=0}^{n}(\bar{k})^{p-\beta-2}|c_{m+1,k}|^p\\ &+\Big(\sum_{k=0}^{n}(\bar{k})^{p-\beta-2}\Big)\sum_{j=0}^{m}(\bar{j})^{p-\alpha-2}|c_{j,n+1}|^p\\ &+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\chi_{mn}(j,k)|c_{jk}|^p\Big\}.\end{aligned} \tag{3.34}$$

Corollary 3.6 generalizes [8], [9], [15]–[18], [24], [26]–[28]. This corollary extends [16]–[18], [24], [26]–[28] from the case $\alpha=\beta=0$ to the range $\alpha,\beta>-1$. As indicated in [10], the condition $\{c_{jk}\}\in S_+^{11}\cap S_{p,\alpha,\beta}^{\mathrm{HL}}$ in Corollary 3.6 cannot be weakened to those $\{c_{jk}\}$ in $S_{p,\alpha,\beta}^{\mathrm{HL}}$ with the following condition:

$$(m,n)\ge(m^*,n^*)\Rightarrow c_{mn}\le c_{m^*,n^*}. \tag{3.35}$$

Here $(m,n)\ge(m^*,n^*)$ means $m\ge m^*$ and $n\ge n^*$.

The second example we investigate is $d\mu=|\phi(x)\psi(y)|dxdy$, where (ϕ,θ) and (ψ,ϑ) satisfy (1.12)–(1.13) and (3.25)–(3.26) for some $v_0\ge 0$. The inequalities (1.12)–(1.13) describe a certain concept related to the definition of type I given in [6]. By the Fubini theorem, we get

$$\begin{aligned}\|\{c_{jk}^{11}\}\|_{1,\mu^{\#},v_0}&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}c_{jk}|\Big\{\sum_{u=0}^{j}\sum_{v=0}^{k}\overline{u}\,\overline{v}\mu_{v_0}^{\#}(\{(u,v)\})\Big\}\\ &=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}c_{jk}|(\Sigma_{jk}^{11}+\Sigma_{jk}^{12}+\Sigma_{jk}^{21}+\Sigma_{jk}^{22}),\end{aligned}$$

where

$$\Sigma_{jk}^{\alpha\beta}=\sum_{u=0}^{j}\sum_{v=0}^{k}\chi_{\alpha}(u)\chi_{\beta}(v)\overline{u}\,\overline{v}\mu_{v_0}^{\#}(\{(u,v)\}),$$

$\chi_1(u)=\chi_{[0,v_0)}(u)$ and $\chi_2(u)=\chi_{[v_0,\infty)}(u)$. Conditions (1.12)–(1.13) and (3.25)–(3.26) imply $\phi,\psi\in L^1(I,dx)$ and so there exist constants $C_{\phi,v_0},C_{\psi,v_0}$ such that

$$\int_I|\phi(x)|\,dx\le C_{\phi,v_0}\theta(\bar{j})\quad(0\le j<v_0),$$

$$\int_I|\psi(y)|\,dy\le C_{\psi,v_0}\vartheta(\bar{k})\quad(0\le k<v_0).$$

Putting these and (1.12)–(1.13) together, we get $\Sigma_{jk}^{\gamma\delta} \le C_{\phi\psi}\theta(\bar{j})\vartheta(\bar{k})$ for $\gamma,\delta = 1,2$ and $j,k \ge 0$. For example,

$$\Sigma_{jk}^{12} \le v_0^2 C_{\phi,v_0}\theta(\bar{j})\Bigg\{\int\limits_{I\setminus\Omega_{k+1}^{\infty}}\left|\frac{\psi(y)}{y-t_0}\right|dy\Bigg\} \le C_{\phi\psi}\theta(\bar{j})\vartheta(\bar{k}).$$

The above argument tells us that

(3.36) $$\|\{c_{jk}^{11}\}\|_{1,\mu^\#,v_0} \le C_{\phi\psi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(\bar{j})\vartheta(\bar{k})|\Delta_{11}c_{jk}|.$$

Therefore, $S_{\theta,\vartheta}^{11} \subset S_{1,\mu^\#,v_0}^{11}$. The proof of (3.36) also verifies

(3.37) $$\|\{c_{jk}^{11}(m,n)\}\|_{1,\mu^\#,v_0} \le C_{\phi\psi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(\bar{j})\vartheta(\bar{k})|\Delta_{11}c_{jk}(m,n)|.$$

By (3.36)–(3.37) and Theorem 3.4, we obtain

COROLLARY 3.7. *Let $d\mu = |\phi(x)\psi(y)|dxdy$, where (ϕ,θ) and (ψ,ϑ) satisfy* (1.12)–(1.13) *and* (3.25)–(3.26) *for some $v_0 \ge 0$. Then there exists $C_{\phi\psi} < \infty$ such that inequality* (1.14) *holds for all $\{c_{jk}\} \in S_{\theta,\vartheta}^{11}$. Moreover, for $\{c_{jk}\} \in S_{\theta,\vartheta}^{11}$, s_{mn} converges in $L^1(I^2,d\mu)$ to some function f as* $\min(m,n) \to \infty$, *and*

(3.38) $$\|f\|_{1,\mu} \le C_{\phi\psi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(\bar{j})\vartheta(\bar{k})|\Delta_{11}c_{jk}|,$$

(3.39) $$\|s_{mn}-f\|_{1,\mu} \le C_{\phi\psi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(\bar{j})\vartheta(\bar{k})|\Delta_{11}c_{jk}(m,n)|.$$

Corollary 3.7 has the same format as in [6]. Inequality (3.38) has implicitly appeared in the proof of [6, Theorem 1]. This corollary will apply to any of the following systems: $\phi_n(t) = \cos nt$, $\sin nt$, e^{int}, $\omega_n(t)$, and $P_n(t)$. An elementary calculation shows that the pairs (ϕ,θ) and (ψ,ϑ) involved in Corollary 3.7 can be chosen from any of (i)–(vi), stated below:

(i) $((\log 1/|t-t_0|)^{-\varepsilon}, 1)$ $(\varepsilon > 1)$;
(ii) $((\log 1/|t-t_0|)^{-1}, \overline{\log\log t})$;
(iii) $((\log 1/|t-t_0|)^{-1}(\log\log 1/|t-t_0|)^{-1}, \overline{\log\log\log t})$;
(iv) $((\log 1/|t-t_0|)^{-\varepsilon}, \{\overline{\log t}\}^{1-\varepsilon})$ $(0 < \varepsilon < 1)$;
(v) $(1, \overline{\log t})$;
(vi) $(|t-t_0|^{-\alpha}, t^{\alpha})$ $(0 < \alpha < 1)$.

The logarithm functions appearing in (i)–(v) have the original value whenever they are well defined; otherwise, they are defined as 0. As indicated in [6], Corollary 3.7 generalizes [2], [3], [5], [6], [15], [19], [20], and [30]. Consider $d\mu = dxdy$. By (v), (1.14) becomes

(3.40) $$\|s^*\|_1 \le C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\overline{\log j})(\overline{\log k})|\Delta_{11}c_{jk}|.$$

If $\Delta_{11}c_{jk} \ge 0$ for all $j,k \ge 0$, then by double summation by parts, we can change (3.40) to

(3.41) $$\|s^*\|_1 \le C\bigg\{c_{00} + \sum_{j=1}^{\infty}\frac{c_{j0}}{j} + \sum_{k=1}^{\infty}\frac{c_{0k}}{k} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{c_{jk}}{jk}\bigg\}.$$

Therefore, Corollary 3.7 includes [17, Theorems 1 & 2] as special cases.

4. The trigonometric system $\{e^{int}\}_{n=-\infty}^{\infty}$. Inspecting the proofs given in §2–§3, we find that the results established there can be extended to the system $\{e^{int}\}_{n=-\infty}^{\infty}$ without difficulty. We summarize them below. Let $I = [-\pi,\pi]$. Consider the double trigonometric series

(4.1) $$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}c_{jk}e^{i(jx+ky)} \quad (x,y \in I).$$

Set $\Psi_{0+}(t) = \Psi_{0-}(t) = 1/2$, $\Psi_j(t) = 1/2 + (e^{it} + e^{2it} + \ldots + e^{ijt})$ for $j \ge 1$, and $\Psi_{-j}(t) = \Psi_j(-t)$ for $j \ge 1$. Then we have

(4.2) $$\sup_{\text{all } j;\, t\in I}|t\Psi_j(t)| \le \pi < \infty.$$

This corresponds to (1.1) with $t_0 = 0$. The difference is to replace $D_n(t)$ by $\Psi_j(t)$. Define the rectangular partial sums $s_{mn}(x,y)$ of series (4.1) and the associated maximal function s^* by

$$s_{mn}(x,y) = \sum_{|j|\le m}\sum_{|k|\le n}c_{jk}e^{i(jx+ky)} \quad (m,n \ge 0;\ x,y \in I),$$

and

$$s^*(x,y) = \sup_{m,n\ge 0}|s_{mn}(x,y)| \quad (x,y \in I).$$

With the help of [6, Lemma 2], a modified proof of Theorem 2.1 will lead us to

(4.3) $$|s^*(x,y)| \le \widetilde{c}_{MN}^{00} + 4\pi\widetilde{c}_{MN}^{01} + 4\pi\widetilde{c}_{MN}^{10} + 16\pi^2\widetilde{c}_{MN}^{11},$$

where $M = [1/|x|]$, $N = [1/|y|]$, and

$$\widetilde{c}_{jk}^{00} = \sum_{|u|\le j} \sum_{|v|\le k} |c_{uv}|, \qquad \widetilde{c}_{jk}^{10} = \Big\{\bar{j} \sum_{|u|=j}^{\infty}\Big\}\Big\{\sum_{|v|\le k}\Big\}|\Delta_{10}^* c_{uv}|,$$

$$\widetilde{c}_{jk}^{01} = \Big\{\sum_{|u|\le j}\Big\}\Big\{\bar{k}\sum_{|v|=k}^{\infty}\Big\}|\Delta_{01}^* c_{uv}|, \quad \widetilde{c}_{jk}^{11} = \Big\{\bar{j}\sum_{|u|=j}^{\infty}\Big\}\Big\{\bar{k}\sum_{|v|=k}^{\infty}\Big\}|\Delta_{11}^* c_{uv}|.$$

The finite differences $\Delta_{\gamma\delta}^* c_{uv}$ are defined by the formulas

$$\Delta_{10}^* c_{uv} = c_{uv} - c_{\tau(u),v}, \qquad \Delta_{01}^* c_{uv} = c_{uv} - c_{u,\tau(v)},$$
$$\Delta_{11}^* c_{uv} = \Delta_{10}^*\Delta_{01}^* c_{uv} = \Delta_{01}^*\Delta_{10}^* c_{uv}.$$

Here $|0+| = |0-| = 0$, $c_{0+,v} = c_{0-,v} = c_{0v}, c_{u,0+} = c_{u,0-} = c_{u0}$, and the function $\tau(u)$ is defined by $\tau(0+) = 1, \tau(0-) = -1, \tau(u) = u+1$ for $u \ge 1$, and $\tau(u) = u-1$ for $u \le -1$. Set $c_{jk}(m,n) = \chi_{mn}(j,k)c_{jk}$, where $\chi_{mn}(j,k)$ is defined in §2. Define the sums $\widetilde{c}_{jk}^{\gamma\delta}(m,n)$ from $\widetilde{c}_{jk}^{\gamma\delta}$ by changing c_{jk} to $c_{jk}(m,n)$. Instead of (1.4), we consider

$$c_{jk} \to 0 \quad \text{as } \max(|j|,|k|) \to \infty. \tag{4.4}$$

THEOREM 4.1. *Assume that $0 < p < \infty$ and μ satisfies (1.5). If the c_{jk} satisfy (4.4) and $\|\{\widetilde{c}_{jk}^{\gamma\delta}\}\|_{p,\mu^\#} < \infty$ for all $\gamma,\delta = 0,1$, then s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n) \to \infty$. Moreover,*

$$\|f\|_{p,\mu} \le \|s^*\|_{p,\mu} \le C_{p,\mu} \sum_{0\le\gamma,\delta\le1} \|\{\widetilde{c}_{jk}^{\gamma\delta}\}\|_{p,\mu^\#}, \tag{4.5}$$

$$\|s_{mn} - f\|_{p,\mu} \le C_{p,\mu} \sum_{0\le\gamma,\delta\le1} \|\{\widetilde{c}_{jk}^{\gamma\delta}(m,n)\}\|_{p,\mu^\#}. \tag{4.6}$$

THEOREM 4.2. *Let $1 \le p < \infty$ and $v_0 \ge 0$. Assume that μ satisfies (1.5) and (1.7)–(1.8). If the c_{jk} satisfy (4.4) and $\|\{\widetilde{c}_{jk}^{11}\}\|_{p,\mu^\#,v_0} < \infty$, then s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n) \to \infty$. Moreover,*

$$\|f\|_{p,\mu} \le \|s^*\|_{p,\mu} \le C_{p,\mu,v_0}\|\{\widetilde{c}_{jk}^{11}\}\|_{p,\mu^\#,v_0}, \tag{4.7}$$

$$\|s_{mn} - f\|_{p,\mu} \le C_{p,\mu,v_0}\|\{\widetilde{c}_{jk}^{11}(m,n)\}\|_{p,\mu^\#,v_0}. \tag{4.8}$$

These conclusions remain true for $0 < p < 1$ provided that (4.8) is replaced by (4.6), the C_{p,μ,v_0} in (4.7) is replaced by $C_{p,\mu,v_0,\gamma}$, and $\{c_{jk}\}$ is further assumed to satisfy the condition: (3.10)–(3.13) are satisfactory by replacing $\{c_{jk}\}$ with $\{c_{\varepsilon j,\delta k} : j,k \ge 0\}$ for $\varepsilon,\delta = \pm1$.

COROLLARY 4.3. *Let $1 \le p < \infty$ and $d\mu = |x|^\alpha|y|^\beta dxdy$, where $\alpha,\beta > -1$. If the c_{jk} satisfies (4.4) and the last term in (4.9) is finite, then s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n) \to \infty$. Moreover,*

$$\begin{aligned}\|f\|_{p,\mu}^p &\le \|s^*\|_{p,\mu}^p \\ &\le C_{p,\alpha,\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2}\Big\{\sum_{|u|=j}^{\infty}\sum_{|v|=k}^{\infty}|\Delta_{11}^* c_{uv}|\Big\}^p,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\|s_{mn} - f\|_{p,\mu}^p \le C_{p,\alpha,\beta}&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j})^{p-\alpha-2}(\bar{k})^{p-\beta-2} \\ &\times\Big\{\sum_{|u|=j}^{\infty}\sum_{|v|=k}^{\infty}\chi_{mn}(|u|,|v|)|\Delta_{11}^* c_{uv}|\Big\}^p.\end{aligned} \tag{4.10}$$

These conclusions remain true for $0 < p < 1$ provided that (4.10) is replaced by (4.6), the $C_{p,\alpha,\beta}$ in (4.9) is replaced by $C_{p,\alpha,\beta,\gamma}$, and $\{c_{jk}\}$ is further assumed to satisfy the condition: (3.10)–(3.13) are satisfied upon replacing $\{c_{jk}\}$ with $\{c_{\varepsilon j,\delta k} : j,k \ge 0\}$ for $\varepsilon,\delta = \pm1$.

COROLLARY 4.4. *Let $0 < p < \infty$ and $d\mu = |x|^\alpha|y|^\beta dxdy$, where $\alpha,\beta > -1$. Assume that the c_{jk} satisfy (4.4) and*

$$\operatorname{Re}\Delta_{11}^* c_{jk} \ge 0, \quad \operatorname{Im}\Delta_{11}^* c_{jk} \ge 0 \quad (j,k = 0\pm,\pm1,\pm2,\ldots). \tag{4.11}$$

If the last term in (4.12) is finite, then s_{mn} converges in $L^p(I^2,d\mu)$ to some function f as $\min(m,n) \to \infty$, and

$$\begin{aligned}\|f\|_{p,\mu} &\le \|s^*\|_{p,\mu} \\ &\le C_{p,\alpha,\beta}\Big\{\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(|\bar{j}|)^{p-\alpha-2}(|\bar{k}|)^{p-\beta-2}|c_{jk}|^p\Big\}^{1/p}.\end{aligned} \tag{4.12}$$

For $1 \le p < \infty$, we also have

$$\begin{aligned}\|s_{mn} - f\|_{p,\mu}^p & \\ \le C_{p,\alpha,\beta}\Big\{&\Big(\sum_{j=0}^{m}(\bar{j})^{p-\alpha-2}\Big)\sum_{|u|=m+1}\sum_{|k|\le n}(|\bar{k}|)^{p-\beta-2}|c_{uk}|^p \\ &+\Big(\sum_{k=0}^{n}(\bar{k})^{p-\beta-2}\Big)\sum_{|j|\le m}\sum_{|v|=n+1}(|\bar{j}|)^{p-\alpha-2}|c_{jv}|^p \\ &+\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(|\bar{j}|)^{p-\alpha-2}(|\bar{k}|)^{p-\beta-2}\chi_{mn}(j,k)|c_{jk}|^p\Big\}.\end{aligned} \tag{4.13}$$

COROLLARY 4.5. *Let $d\mu = |\phi(x)\psi(y)|dxdy$, where (ϕ,θ) and (ψ,ϑ) satisfy (1.12)–(1.13) and (3.25)–(3.26) for some $v_0 \ge 0$. If the c_{jk} satisfy (4.4) and the last term in (4.14) is finite, then s_{mn} converges in $L^1(I^2,d\mu)$ to* *some function* f *as* $\min(m,n)\to\infty$. *Moreover*,

$$(4.14)\qquad \|f\|_{1,\mu}\le\|s^*\|_{1,\mu}\le C_{\phi\psi}\sum_{|j|=0}^{\infty}\sum_{|k|=0}^{\infty}\theta(|\bar{j}|)\vartheta(|\bar{k}|)|\Delta_{11}^{*}c_{jk}|,$$

$$(4.15)\qquad \|s_{mn}-f\|_{1,\mu}\le C_{\phi\psi}\sum_{|j|=0}^{\infty}\sum_{|k|=0}^{\infty}\theta(|\bar{j}|)\vartheta(|\bar{k}|)|\Delta_{11}^{*}c_{jk}(m,n)|.$$

Inspecting the proofs given in §2–§3, we find that the theory developed here can be extended to any dimension without difficulty. We leave it to the reader.

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References

[1] R. Askey and S. Wainger, *Integrability theorems for Fourier series*, Duke Math. J. 33 (1966), 223–228.

[2] L. A. Balashov, *Series with respect to the Walsh system with monotone coefficients*, Sibirsk. Mat. Zh. 12 (1971), 25–39 (in Russian).

[3] R. P. Boas, *Integrability Theorems for Trigonometric Transforms*, Springer, Berlin, 1967.

[4] T. W. Chaundy and A. E. Jolliffe, *The uniform convergence of a certain class of trigonometric series*, Proc. London Math. Soc. (2) 15 (1916), 214–216.

[5] C.-P. Chen, *Integrability and L-convergence of multiple trigonometric series*, Bull. Austral. Math. Soc. 49 (1994), 333–339.

[6] —, *Weighted integrability and L^1-convergence of multiple trigonometric series*, Studia Math. 108 (1994), 177–190.

[7] C.-P. Chen and G.-B. Chen, *Uniform convergence of double trigonometric series*, ibid. 118 (1996), 245–259.

[8] Y.-M. Chen, *On the integrability of functions defined by trigonometric series*, Math. Z. 66 (1956), 9–12.

[9] —, *Some asymptotic properties of Fourier constants and integrability theorems*, ibid. 68 (1957), 227–244.

[10] M. I. D'yachenko, *On the convergence of double trigonometric series and Fourier series with monotone coefficients*, Math. USSR-Sb. 57 (1987), 57–75.

[11] S. Fridli and P. Simon, *On the Dirichlet kernels and a Hardy space with respect to the Vilenkin system*, Acta Math. Hungar. 45 (1985), 223–234.

[12] A. E. Jolliffe, *On certain trigonometric series which have a necessary and sufficient condition for uniform convergence*, Proc. Cambridge Philos. Soc. 19 (1921), 191–195.

[13] L. Leindler, *Generalization of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged) 31 (1970), 279–285.

[14] —, *Some inequalities of Hardy–Littlewood type*, Anal. Math. 20 (1994), 95–106.

[15] M. M. H. Marzug, *Integrability theorem of multiple trigonometric series*, J. Math. Anal. Appl. 157 (1991), 337–345.

[16] F. Móricz, *On Walsh series with coefficients tending monotonically to zero*, Acta Math. Acad. Sci. Hungar. 38 (1981), 183–189.

[17] —, *On the maximum of the rectangular partial sums of double trigonometric series with nonnegative coefficients*, Anal. Math. 15 (1989), 283–290.

[18] —, *On double cosine, sine, and Walsh series with monotone coefficients*, Proc. Amer. Math. Soc. 109 (1990), 417–425

[19] —, *On the integrability and L^1-convergence of double trigonometric series*, Studia Math. 98 (1991), 203–225.

[20] F. Móricz, F. Schipp and W. R. Wade, *On the integrability of double Walsh series with special coefficients*, Michigan Math. J. 37 (1990), 191–201.

[21] J. R. Nurcombe, *On the uniform convergence of sine series with quasimonotone coefficients*, J. Math. Anal. Appl. 166 (1992), 577–581.

[22] B. Ram and S. S. Bhatia, *On weighted integrability of double cosine series*, ibid. 208 (1997), 510–519.

[23] F. Schipp, P. Simon and W. R. Wade, *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, IOP Publishing and Akadémiai Kiadó, Budapest, 1990.

[24] S. B. Stechkin, *On power series and trigonometric series with monotone coefficients*, Uspekhi Mat. Nauk 18 (1963), no. 1, 173–180 (in Russian).

[25] G. Szegő, *Orthogonal Polynomials*, 4th ed., Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975.

[26] F. Weisz, *Inequalities relative to two-parameter Vilenkin–Fourier coefficients*, Studia Math. 99 (1991), 221–233.

[27] —, *Martingale Hardy Spaces and Their Applications in Fourier-Analysis*, Lecture Notes in Math. 1568, Springer, Berlin, 1994.

[28] —, *Two-parameter Hardy–Littlewood inequalities*, Studia Math. 118 (1996), 175–184.

[29] T. F. Xie and S. P. Zhou, *The uniform convergence of certain trigonometric series*, J. Math. Anal. Appl. 181 (1994), 171–180.

[30] W. H. Young, *On the Fourier series of bounded functions*, Proc. London Math. Soc. 12 (1913), 41–70.

[31] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1968.

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