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A characterization of commutative Fréchet algebras with all ideals closed

by

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Abstract. Let A be a commutative unital Fréchet algebra, i.e. a completely metrizable topological algebra. Our main result states that all ideals in A are closed if and only if A is a noetherian algebra.

A *topological algebra* is a real or complex algebra A which is a topological vector space (t.v.s.) and the multiplication $(x, y) \mapsto xy$ is jointly continuous from $A \times A$ to A. In terms of neighbourhoods of zero this means that for each such neighbourhood U there is a neighbourhood V with

$$V^2 \subset U. \tag{1}$$

A unital topological algebra A is called a *Q-algebra* if the set (group) $G(A)$ of all its invertible elements is open. It is known ([6], Lemma I.6.4, pp. 43–44) that A is a Q-algebra if and only if its unit element e has a neighbourhood consisting of invertible elements.

A *Fréchet algebra* or *F-algebra* is a topological algebra which is a Fréchet (i.e. completely metrizable) t.v.s. The topology of an F-space X can be given by means of an *F-norm*, i.e. a map $x \mapsto \|x\|$ from X to the non-negative real numbers such that

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ iff $x = 0$,
(ii) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$,
(iii) the map $(\lambda, x) \mapsto \|\lambda x\|$ is jointly continuous, $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $x \in X$.

For further information on F-spaces and F-norms the reader is referred to [2] and [8].

A topological algebra is called *multiplicatively convex* (briefly m-convex) if its topology can be given by means of a family of submultiplicative (algebra) seminorms.

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If A is a commutative complex unital m-convex F-algebra, then the maximal ideal space $\mathfrak{M}(A)$, i.e. the set of all its continuous non-zero multiplicative linear functionals provided with the Gelfand topology, is non-void. The elements of $\mathfrak{M}(A)$ can be identified with the closed maximal ideals of A. Such an algebra is a Q-algebra if and only if $\mathfrak{M}(A)$ is compact (cf. [6] or [11]), or if and only if all maximal ideals in A are closed (see [1]). For further information on topological algebras, in particular on m-convex algebras, the reader is referred to [6], [7] and [10].

In this paper we are concerned with the question of when a commutative unital algebra has all ideals closed. The starting point of the present work is a result due to Carboni and Larotonda [3]. They have constructed a family of commutative complex m-convex unital F-algebras A with the following properties:

(a) All ideals in A are principal, i.e. of the form $I = xA$, $x \in I$,
(b) A is an integral domain,
(c) the maximal ideal space $\mathfrak{M}(A)$ is compact.

Subsequently I have shown that if an m-convex F-algebra satisfies (a)–(c), then all its ideals are closed. In my talk during the 1999 Banach Algebras Conference (Claremont, CA) I posed the conjecture that a commutative complex m-convex unital F-algebra has all ideals closed if and only if it is noetherian (I was told by Jaroslav Zemánek that the question of when a topological algebra has all ideals closed was also discussed during the conference Théorie des Opérateurs et Algèbres de Banach, Rabat, April 12–14, 1999). Subsequently I have been able to prove this conjecture. Here I give a much more general result without the assumption of local convexity. The result also holds for the real scalars. Some results in this direction are already known. Grauert and Remmert [5] showed that a commutative noetherian Banach algebra is necessarily finite-dimensional. Ferreira and Tomassini [4] studied the noetherian m-convex algebras and showed, among other results, that a noetherian commutative complex unital m-convex Fréchet algebra has all ideals closed (Theorem 2.6 of [4]).

Recall that a commutative unital algebra A is said to be *noetherian* if every proper ideal I of A is finitely generated, i.e. it is of the form

$$I = x_1A + \ldots + x_nA, \quad x_i \in I. \tag{2}$$

The following result generalizes a result given in [4] for multiplicatively convex complex Fréchet algebras. The proof given here should be compared with the proof of Proposition 17 in [1] stating that a commutative Fréchet algebra has all maximal ideals closed if and only if it is a Q-algebra (many thanks to Mati Abel for calling my attention to that paper).

PROPOSITION 1. *Let A be a commutative unital real or complex F-algebra which is noetherian. Then A is a Q-algebra.*

Proof. Assume that A is not a Q-algebra. By assumption, there is a sequence (x_i) of non-invertible elements with $\lim_i x_i = e$, the unit element of A. First we show that there is a subsequence $z_k = x_{i_k}$ such that

$$\|z_k - e\| < 2^{-k} \quad \text{for all } k, \tag{3}$$

and

$$\|u_{k+1}^{(r)} - u_k^{(r)}\| < 2^{-(k+1)}, \quad 0 \le r \le k, \tag{4}$$

where

$$u_k^{(r)} = \begin{cases} z_{r+1}\ldots z_k & \text{for } k > r, \\ e & \text{for } r = k, \end{cases} \quad 0 \le r \le k. \tag{5}$$

First we choose i_1 so that $z_1 = x_{i_1}$ satisfies (3). Then (4) is also satisfied for $k = r = 0$ since in this case it coincides with (3). Suppose now that we have chosen $z_1, \ldots, z_n$, with $z_k = x_{i_k}$, so that (3) and (4) hold respectively for $k \le n$ and for $0 \le r \le k \le n-1$. We shall be done if we find i_{n+1} so that

$$\|u_n^{(r)} x_{i_{n+1}} - u_n^{(r)}\| < 2^{-(n+1)}, \quad 0 \le r \le n. \tag{6}$$

But such an i_{n+1} must exist since $\lim x_i = e$ and we have only a finite number of inequalities in (6). Thus (4) holds for $k = n+1$, as does (3) which is obtained by setting $r = n$ in (6). The existence of the sequence (z_k) follows.

Fix now an index r. Relations (4) imply that for $r \le p < q$ we have

$$\|u_p^{(r)} - u_q^{(r)}\| \le \|u_p^{(r)} - u_{p+1}^{(r)}\| + \ldots + \|u_{q-1}^{(r)} - u_q^{(r)}\| \le 2^{-(p+1)} + \ldots + 2^{-q} < 2^{-p}.$$

Thus $(u_i^{(r)})_{i \ge r}$ is a Cauchy sequence for each fixed r and so the limits

$$u^{(r)} = \lim_i u_i^{(r)}, \quad r = 0, 1, 2, \ldots,$$

exist. Moreover, (5) implies $u_k^{(p)} = z_{p+1}\ldots z_q u_k^{(q)}$ for $q > p$, and so

$$u^{(p)} = z_{p+1}\ldots z_q u^{(q)}, \quad q > p. \tag{7}$$

Observe now that

$$\lim_r u^{(r)} = e. \tag{8}$$

Indeed, we have $z_{r+1} = u_{r+1}^{(r)}$ and so for any $p > r+1$ we have, by (4),

$$\|u_p^{(r)} - z_{r+1}\| = \|u_p^{(r)} - u_{r+1}^{(r)}\| \le \|u_{r+1}^{(r)} - u_{r+2}^{(r)}\| + \ldots + \|u_{p-1}^{(r)} - u_p^{(r)}\| \le 2^{-(r+2)} + \ldots + 2^{-p} < 2^{-(r+1)}.$$

Passing with p to infinity we obtain $\|u^{(r)} - z_{r+1}\| \le 2^{-(r+1)}$, which together with (3) gives

$$\|u^{(r)} - e\| \le \|u^{(r)} - z_{r+1}\| + \|z_{r+1} - e\| < 2^{-r},$$

and (8) follows.

Since the elements z_k are non-invertible, formula (7) implies that so are all $u^{(r)}$, so that each $I_n = u^{(n)}A$ is a proper ideal in A. Formula (7) also implies

(9) $$I_n \subset I_{n+1} \quad \text{for all } n$$

and consequently $I = \bigcup_n I_n$ is a proper ideal in A. For every x in A we have $xu^{(r)} \in I$, and so, by (8), $x = \lim_r xu^{(r)} \in \overline{I}$. Thus $\overline{I} = A$, i.e. I is a dense ideal in A. Since A is noetherian, there are $y_1, \ldots, y_k \in A$ such that $I = y_1A + \ldots + y_kA$. Since all y_i are in I, there is also a (smallest) index n_0 such that $y_1, \ldots, y_k \in I_{n_0}$. Consequently,

(10) $$I = I_{n_0},$$

which implies $I_n = I_{n_0}$ for all $n \ge n_0$. In particular $I_{n_0+1} = I_{n_0}$. Thus there is a $v \in A$ with

(11) $$u^{(n_0+1)} = u^{(n_0)}v.$$

By (7) we have $u^{(n_0)} = z_{n_0+1}u^{(n_0+1)}$, which together with (11) gives $u^{(n_0+1)} = z_{n_0+1}u^{(n_0+1)}v$, or

(12) $$u^{(n_0+1)}(e - z_{n_0+1}v) = 0.$$

Since, by (10), I_{n_0+1} is dense in A, there exists a sequence $(v_i) \subset A$ with $e = \lim_i u^{(n_0+1)}v_i$. Multiplying both sides of (12) by v_i and letting i tend to infinity, we obtain $e = z_{n_0+1}v$, i.e. z_{n_0+1} is invertible, which is the desired contradiction.

LEMMA 2. *Let A be a commutative, real or complex F-algebra with all ideals closed. Then A is noetherian.*

Proof. Let I be a proper ideal in A and choose a non-zero $x_1 \in I$ so that $I_1 = x_1A \subset I$. If $I = I_1$ we are done. If not, there is an $x_2 \in I \setminus I_1$ and so $I_2 = I_1 + x_2A$ is a subideal of I. Again either $I_2 = I$ and in this case we are done, or the process can be continued. If it does not terminate, we obtain a sequence $I_1 \subset I_2 \subset \ldots \subset I$ of closed ideals with all imbeddings proper. Setting $J = \bigcup_{i\ge1} I_i$ we obtain a proper ideal in A ($J \subset I$) which is not closed as the union of an increasing sequence of closed subspaces, which is impossible. Thus $I = I_n$ for some n and the conclusion follows.

Observe that the one-sided version of the above lemma also holds true.

LEMMA 3. *Let A be a commutative real or complex topological algebra. Then for each polynomial p in n variables with $p(0, \ldots, 0) = 0$, and each neighbourhood U of zero in A, there is a neighbourhood V such that $x_i \in V$, $i = 1, \ldots, n$, implies $p(x_1, \ldots, x_n) \in U$.*

Proof. Let $p = q_1 + \ldots + q_k$, where q_i is a monomial of order n_i with scalar coefficient c_i, $1 \le n_i \le n$. Find a neighbourhood V_1 of the origin so that $V_1 + \ldots + V_1 \subset U$ (with k copies of V_1) and a neighbourhood V which satisfies $V^{n_i} \in c_i^{-1}V_1$ for all i; this can be done by (1). It is now clear that V satisfies the conclusion of the lemma.

The next lemma is due to Grauert and Remmert ([5], Chapter I, Remark 2 in the Appendix to §5); they needed it for proving that a noetherian Banach algebra is necessarily finite-dimensional. They formulated it for Banach algebras, but the proof works in a more general context. For the convenience of the reader we reproduce it here.

LEMMA 4. *Let A be a real or complex commutative unital F-algebra which is also a Q-algebra. Let I be a proper ideal in A whose closure $\overline{I}$ is a finitely generated ideal. Then I is closed.*

Proof. Since A is a Q-algebra, $\overline{I}$ is a proper ideal in A and, by assumption, $\overline{I}$ is of the form (2) with $(x_1, \ldots, x_n) \in \overline{I}$. Then

$$\Phi(u_1, \ldots, u_n) = x_1u_1 + \ldots + x_nu_n$$

is a continuous linear map from A^n to A. Since A^n is also an F-space (with the product topology), and Φ is onto, Banach's theorem ([2] or [8]) says that Φ is open, and so, for every neighbourhood V of zero in A, the set $S(V) = \Phi(V, \ldots, V) = x_1V + \ldots + x_nV$ is a neighbourhood of zero in $\overline{I}$. Since I is dense in $\overline{I}$ we have

(12) $$I + S(V) = \overline{I}$$

for each V. The elements x_i are in $\overline{I}$ and so (12) implies that for any neighbourhood V of zero in A there are $u_{k,i}$ in V and y_i in I, $1 \le i, k \le n$, such that $x_k = y_k + \sum_{i=1}^n x_iu_{k,i}$, or

(13) $$y_k = x_k - \sum_{i=1}^n x_iu_{k,i}, \quad k = 1, \ldots, n.$$

We can treat (13) as a system of linear equations with given y_k and $u_{k,i}$ and find $x_1, \ldots, x_n$ by the Cramer formulas. If C is the matrix $(u_{i,j})$, then (13) can be written as $(y_1, \ldots, y_n) = (\mathrm{Id} - C)(x_1, \ldots, x_n)$, where Id is the identity matrix. Thus we need to know that the A-valued determinant of $\mathrm{Id} - C$ is invertible in A. It is easy to see that this determinant is of the form $e - p$, where p is a polynomial in the variables $u_{i,j}$ satisfying $p(0, \ldots, 0) = 0$. Since A is a Q-algebra, there is a neighbourhood U of zero such that $e + U \subset G(A)$. Using now Lemma 3 for U and p we obtain a neighbourhood V so that the determinant is invertible in A. The Cramer formulas imply $x_k = \sum_{i=1}^n y_iv_{k,i}$

for suitable $v_{k,i}$, and so the elements x_i are in I. Thus, by (2), $\bar{I} \subset I$ so that $I = \bar{I}$.

We can now prove our main result.

THEOREM 5. *Let A be a commutative real or complex unital Fréchet algebra. Then A has all ideals closed if and only if it is noetherian.*

Proof. If A has all ideals closed, then it is noetherian by Lemma 2. If A is noetherian then, by Proposition 1 it is a Q-algebra, and all ideals in A are closed by Lemma 4.

Observe that the above result is also true for a non-unital algebra. This follows from the following, rather obvious, fact: a commutative topological algebra has all ideals closed if and only if its unitization has all ideals closed. Also if I is a proper ideal in A, then it is a proper ideal in the unitization of A after the natural imbedding of A in this unitization (for details cf. [14]). We do not know, however, whether the non-commutative versions of Proposition 1 and Theorem 5 hold true.

Theorem 5 can be used to show that in the algebras constructed in [3] all ideals are closed. This fact also follows from Theorem 2.6 in [4]. These examples, and also other known examples of noetherian F-algebras (e.g. the algebras of all power series in one or several variables, see [4]) are multiplicatively convex. Within the class of locally convex algebras no other examples are possible, since by Proposition 1 commutative noetherian F-algebras are Q-algebras, and commutative locally convex F-algebras which are Q-algebras must be multiplicatively convex (see [10]; the argument given there works only in the complex case, but the result can also be obtained in the case of a real algebra, see [13]). Noetherian F-algebras which are not locally convex can be obtained by considering p-homogeneous seminorms, $0 < p < 1$, instead of homogeneous seminorms in the constructions given in [3]. In this way one can obtain multiplicatively pseudoconvex noetherian F-algebras which are not locally convex. The definition of an m-pseudoconvex algebra is analogous to that of an m-convex algebra: the topology of such an algebra can be given by means of a family of p-homogeneous submultiplicative seminorms, $0 < p \leq 1$, where the value of p depends upon the seminorm. When extending the construction of [3] to the locally pseudoconvex case we consider the same value of p for all seminorms (for more details see [14]).

We now show that the metrizability and completeness assumptions are essential in Proposition 1 and Theorem 5. The following example shows that there can exist a non-noetherian complete m-convex Q-algebra with all ideals closed. We do not know, however, whether a commutative complete noetherian topological algebra must have all ideals closed.

EXAMPLE 6. Let X be an infinite-dimensional vector space equipped with a trivial (zero) multiplication and with the maximal locally convex topology $\tau_{\max}^{\mathrm{LC}}$, given by all seminorms on X. It is known ([9], Example on p. 56) that X is a complete locally convex space with all linear subspaces closed, and since all seminorms are submultiplicative, it is a complete m-convex algebra. Let A be the unitization of this algebra. It is again complete and m-convex, it is also a Q-algebra since all elements of the form $\lambda e + x$, where λ is a non-zero scalar and $x \in X$, have inverses $\lambda^{-2}(\lambda e - x)$. Clearly all ideals of A are closed (it is easy to see that the ideals of A coincide with the linear subspaces of X). Since non-invertible elements in A coincide with elements of X, every proper principal ideal in A is of the form $xA = Kx$, where K is the field of scalars and $x \in X$, so it is one-dimensional. Thus every finitely generated ideal is finite-dimensional and, in particular, X is an ideal which is not finitely generated. Thus A is not noetherian and has all ideals closed.

The following example shows that it is possible to have a commutative complete noetherian topological algebra which is not a Q-algebra and has all ideals closed. Thus Proposition 1 does not extend to the non-metrizable case.

EXAMPLE 7. Let A be the (real or complex) algebra of all polynomials in one variable; it is clearly noetherian. Equipping this algebra with the topology $\tau_{\max}^{\mathrm{LC}}$ we obtain a complete noetherian non-Q-algebra with all ideals closed.

Finally we show that it is possible to have a noetherian (incomplete) normed commutative topological Q-algebra with a non-closed ideal. Thus Theorem 5 does not hold true for an incomplete algebra, even if it is a Q-algebra.

EXAMPLE 8. Let A be one of the algebras constructed in [3]. Its maximal ideal space is the closed unit disc $\Delta \subset \mathbb{C}$ and it consists of fuctions continuous in Δ and holomorphic in its interior, and contains all polynomials. If we equip it with the supremum norm $|\cdot|_\infty$, we obtain an incomplete normed algebra which is noetherian (cf. condition (a) above) and a Q-algebra. The latter follows from the fact that an element x of A is invertible if it does not vanish on Δ, so that $|e - x|_\infty < 1$ implies the invertibility of x. Now it is easy to see that the ideal $(e - z)^2 A$ is dense in $(e - z)A$ in the supremum norm (here $e(\zeta) = 1$ and $z(\zeta) = \zeta$ for $\zeta \in \Delta$); for details cf. [14].

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