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On pointwise estimates for maximal and singular integral operators

by

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Abstract. We prove two pointwise estimates relating some classical maximal and singular integral operators. In particular, these estimates imply well-known rearrangement inequalities, $L^p$ and BLO-norm inequalities.

Introduction. For a locally integrable function $f$ on $\mathbb{R}^n$, define the Hardy-Littlewood and Fefferman-Stein maximal functions by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where $f_Q = \frac{1}{|Q|} \int_Q f$, the supremum is taken over all cubes $Q$ containing $x$, and $|Q|$ denotes the Lebesgue measure of $Q$.

We also define the Calderón-Zygmund maximal singular integral operator by

$$T^*f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} f(y) k(x-y) \, dy \right|,$$

where the kernel $k(x)$ satisfies the standard conditions:

$$|k(x)| \leq \frac{c}{|x|^n}, \quad \int_{R_1 < |x| < R_2} k(x) \, dx = 0 \quad (0 < R_1 < R_2 < \infty),$$

$$|k(x) - k(x-y)| \leq \frac{c|y|^\alpha}{|x|^{n+\alpha}} \quad (|y| \leq |x|/2, \alpha > 0).$$

Let $\omega$ be a non-negative, locally integrable function. Given a measurable set $E$, let $\omega(E) = \int_E \omega(x) \, dx$. We say that $\omega$ satisfies Muckenhoupt's

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condition $A_\infty$ if there exist $c, \delta > 0$ so that for any $Q$ and $E \subseteq Q$, 
$$\omega(E) \leq c(|E|/|Q|)^\delta \omega(Q).$$

For $\omega \in A_\infty$, it is well known (see [7, 9, 10]) that

(2) \quad \|T^*f\|_{p,\omega} \leq c\|Mf\|_{p,\omega},

(3) \quad \|Mf\|_{p,\omega} \leq c\|f^\#\|_{p,\omega}

for all $p > 0$, where $\|f\|_{p,\omega} \equiv (\int\int f(x)^p \omega(x) \, dx)^{1/p}$.

BMO estimates for $T^*f$ go back to [15, 16]:

(4) \quad \|T^*f\|_* \leq c\|f\|_*.

For the Hardy–Littlewood maximal function a BMO estimate was established later in [4]:

(5) \quad \|Mf\|_* \leq c\|f\|_*.

These estimates were strengthened in [13] and [3] respectively:

(6) \quad \|T^*f\|_{\text{BLO}} \leq c\|f\|_{\infty},

(7) \quad \|Mf\|_{\text{BLO}} \leq c\|f\|_*.

The space BLO [8] consists of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BLO}} = \sup_Q (f_Q - \inf_Q f) < \infty.$$}

It is easy to see that $\text{BLO} \subseteq \text{BMO}$, moreover $\|f\|_* \leq 2\|f\|_{\text{BLO}}$.

Note that the estimates (2), (3) were proved in [7, 9, 10] with the help of so-called good $\lambda$ inequalities. Afterwards, rearrangement inequalities for $Mf, f^\#, T^*f$ were obtained (see [1, 2, 5]), which also imply (2), (3).

The non-increasing rearrangement of $f$ with respect to $\omega$ [6, p. 32] is defined by

$$f_*(t) = \sup_{\omega(B) = t} \inf_{x \in B} f(x) \quad (0 < t < \infty).$$

If $\omega \equiv 1$ we use the notation $f_*(t)$.

A key role in our work is played by the maximal function (see [11, 19])

$$m_\lambda f(x) = \sup_{Q \ni x} (f_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1).$$

In terms of this function we establish pointwise estimates for the operators $Mf, f^\#, T^*f$. In particular, these estimates imply all the above mentioned results, namely rearrangement inequalities, $L^p_\omega$ and BLO-norm estimates (2)–(7).

Our main results are the following.

**Theorem 1.** For any function $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and for all $x \in \mathbb{R}^n$, 

$$m_\lambda(T^*f)(x) \leq c_{\lambda,n} Mf(x) + T^*f(x) \quad (0 < \lambda < 1).$$

**Theorem 2.** For any function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$, 

$$m_\lambda(Mf)(x) \leq c_{\lambda,n} f^\#(x) + Mf(x) \quad (0 < \lambda < 1).$$

Inequalities (2)–(7) follow from these theorems in view of the next main lemma.

**Lemma 1.** Let $f$ and $g$ be non-negative functions on $\mathbb{R}^n$. Suppose that for any $\lambda, 0 < \lambda < 1$, there exists a constant $c_\lambda > 0$ so that 

$$m_\lambda f(x) \leq c_\lambda g(x) + f(x)$$

for all $x \in \mathbb{R}^n$, and let $\omega \in A_\infty$. Then

(i) there exists a constant $c' > 0$ so that 

$$f_*(t) \leq c' f_*(2t) + f_*(t)$$

for all $t > 0$;

(ii) if $f_*(+\infty) = 0$, then

$$\|f\|_{L^p_\omega} \leq c_p \|g\|_{L^p_\omega} \quad (0 < p < \infty);$$

(iii) if $g \in L^\infty$, then

$$\|f\|_{\text{BLO}} \leq c\|g\|_{\infty}.$$

The proof of (iii) is essentially based on the inequality

$$\|f\|_* \leq c_n \sup_Q \inf_{c \in E} (f(c\chi_Q^*)^*(\lambda|Q|))$$

which was proved by F. John [12] and J-O. Strömberg [19] in the cases $0 < \lambda < 1/2$ and $\lambda = 1/2$ respectively. For $\lambda > 1/2$ this inequality fails.

First, we prove Theorems 1, 2, and then Lemma 1.

**Proof of Theorem 1.** Here it is convenient to use the maximal function

$$\tilde{m}_\lambda f(x) = \sup_{B \ni x} (f_{\lambda B})^*(\lambda|B|),$$

where the supremum is taken over all balls $B$ centered at $x$. It is easy to see that for any cube $Q$ containing $x$ there is a ball $B$ centered at $x$ which contains $Q$ such that $|B| = c_n |Q|$. From this property, for any $x \in \mathbb{R}^n$ we have

$$m_\lambda f(x) \leq \tilde{m}_\lambda f(x).$$

By (9), it suffices to get the required estimate for $\tilde{m}_\lambda$. Let $B$ be an arbitrary ball with center at $x$. From the definition of $T^*$ it follows that

$$T^*(f_{\lambda B})(x) \leq T^* f(x).$$

Further, by (1), the standard arguments (see, for example, [18, p. 59]) show that for all $y \in B$,

(11) \quad $T^*(f_{\lambda B})(y) \leq cMf(x) + T^*(f_{\lambda B})(x)$. 

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On the other hand, by weak type (1, 1) of $T^* [17, p. 42]$ we have
\[
(T^*(f\chi_Q))^*(\lambda |Q|) \leq c \frac{1}{|Q|} \int_B |f(y)| \, dy \leq c M f(x).
\]
From this and (10), (11) we get
\[
((T^* f)\chi_Q)^*(\lambda |Q|) \leq c M f(x) + T^* f(x).
\]
Taking the upper bound over all balls $B$ centered at $x$ proves the theorem.

**Proof of Theorem 2.** We shall use the following elementary property of cubes: if cubes $Q_1$ and $Q_2$ intersect then either $Q_1 \subset 3Q_2$ or $Q_2 \subset 3Q_1$ (as usual, $kQ$ denotes the cube concentric with $Q$ and having edge length $k$ times as large).

Let $Q$ be an arbitrary cube containing the point $x$. Take an arbitrary point $y \in Q$ and suppose a cube $Q'$ contains $y$. If $Q' \subset 3Q$, then
\[
|f|_{Q'} \leq |f - f|_{3Q} + |f|_{3Q} \leq M((f - f|_{3Q})\chi_{3Q})(y) + M f(x).
\]
Assume now that $Q' \not\subset 3Q$. Then $Q \subset 3Q'$ and in this case
\[
|f|_{Q'} \leq |f - f|_{3Q} + |f|_{3Q'} \leq 3^n f^*(x) + M f(x).
\]
Thus, for all $y \in Q$
\[
M f(y) = \max \left( \sup_{Q' \subset 3Q} |f|_{Q'}, \sup_{Q' \supset Q} |f|_{Q'} \right)
\leq M((f - f|_{3Q})\chi_{3Q})(y) + 3^n f^*(x) + M f(x).
\]
Using the weak type (1, 1) of the operator $M$, we get
\[
((M f)\chi_Q)^*(\lambda |Q|) \leq (M((f - f|_{3Q})\chi_{3Q}))^*(\lambda |Q|) + 3^n f^*(x) + M f(x)
\leq c \frac{1}{|Q|} \int_{3Q} |f - f|_{3Q} + 3^n f^*(x) + M f(x)
\leq c f^*(x) + M f(x).
\]
Taking the upper bound over all $Q \ni x$ yields the theorem.

**Proof of Lemma 1.** Choose $\lambda$ so that $c(2^n \lambda)^c = 1/4$, where $c, \delta$ are the constants from the definition of $A_{\infty}$, and put $c' = c_0$.

Let $E$ be an arbitrary set with $\omega(E) = t$. Applying the Calderón–Zygmund decomposition to the function $\chi_E$ and number $\lambda$, we get pairwise disjoint cubes $Q_i$ such that
\[
\lambda |Q_i| < |E \cap Q_i| \leq 2^n \lambda |Q_i|.
\]
From the definition of $A_{\infty}$ it follows that
\[
\omega(E) = \sum_i \omega(E \cap Q_i) \leq c \sum_i \left( \frac{|E \cap Q_i|}{|Q_i|} \right)^c \omega(Q_i) \leq c(2^n \lambda)^c \omega \left( \bigcup_i Q_i \right).
\]
So, we have $\omega(\bigcup_i Q_i) \geq 4t$. From this and the left-hand inequality of (12) we obtain
\[
\inf_{x \in E} |f(x)| \leq \inf_{x \in 3E \cap Q_i} |f(x)| \leq \inf_{x \in (f\chi_{Q_i})^*} (\lambda |Q_i|)
\leq \inf_{x \in (f\chi_{Q_i})^*} m_{\lambda, f}(x) = \inf_{x \in \bigcup_i Q_i} m_{\lambda, f}(x) \leq (m_{\lambda, f})^*(4t).
\]
Taking the supremum over all sets $E$ with $\omega(E) = t$, we get $f^*_\omega(t) \leq (m_{\lambda, f})^*(4t)$.

From this and simple properties of rearrangement it follows that
\[
f^*_\omega(t) \leq \lambda \omega(2E \cap Q_i) \leq \lambda \omega(2t) + f^*_\omega(2t).
\]
So, we get (i). Iterating this inequality we obtain (ii) in a standard way (see, for example, [14]).

It remains to prove (iii). This follows immediately from the following BLO criterion.

**LEMMA 2.** Let $\lambda \leq 1/2$. Then a non-negative function $f$ belongs to BLO iff $m_{\lambda, f} - f \in L^\infty$. Moreover,
\[
\|f\|_{BLO} = \|m_{\lambda, f} - f\|_{\infty}.
\]

**Proof.** Define $A = \|m_{\lambda, f} - f\|_{\infty}$. It is clear that
\[
\inf_{Q} ((f - c)\chi_Q)^*(\lambda |Q|) \leq ((f - c)\chi_Q)^*(\lambda |Q|) = (f\chi_{Q})^*(\lambda |Q|) - \inf_{Q} f \leq A.
\]
Since $\lambda \leq 1/2$, by John and Strömberg's theorem (see (8)) it follows that $f \in BMO$ and $\|f\|_{\infty} \leq c A$. Further, note that for any cube $Q$, $\omega(Q_i)$.
\[
f_q \leq \inf_{x \in Q} \left( |f(x) - f| + |f(x)| \right) \leq \left( \frac{|f - f_q| + |f(x)|}{|Q_i|} \right)^* (\lambda |Q|)
\leq ((f - f_q)\chi_{Q})^*(\lambda |Q_i|/2) + (f\chi_{Q})^*(\lambda |Q_i|/2) \leq 2\|f\|_{\infty} + (f\chi_{Q})^*(\lambda |Q_i|/2).
\]
From this and (13) we get
\[
\|f\|_{BLO} = \sup_{Q} (f - f_q) \chi_{Q} \leq \sup_{Q} (2\|f\|_{\infty} + (f\chi_{Q})^*(\lambda |Q|) - \inf_{Q} f) \leq (2c + 1) A.
\]
Conversely, let $f \in BLO$. Then
\[
(f\chi_{Q})^*(\lambda |Q|) \leq ((f - f_q)\chi_{Q})^*(\lambda |Q|) + f_q
\leq \frac{1}{\lambda} \|f\|_{\infty} + \|f\|_{BLO} + \inf_{Q} f \leq (2/\lambda + 1) \|f\|_{BLO} + \inf_{Q} f.
\]
Thus,
\[ m_{\lambda}f(x) \leq (2/\lambda + 1)||f||_{\text{BLO}} + f(x). \]
The lemma is proved.

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References