References

[AM] M. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. 320 (1990), 727–735.

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, New York, 1988.

[OK] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Res. Notes Math. Ser. 219, Longman Sci. & Tech., Harlow 1990.

[S] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.

[T] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Pure Appl. Math. 123, Academic Press, New York, 1986.

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On pointwise estimates for maximal and singular integral operators

by

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Abstract. We prove two pointwise estimates relating some classical maximal and singular integral operators. In particular, these estimates imply well-known rearrangement inequalities, L^p_ω and BLO-norm inequalities.

Introduction. For a locally integrable function f on $\mathbb{R}^n$, define the Hardy–Littlewood and Fefferman–Stein maximal functions by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

$$f^{\#}(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy,$$

where $f_Q = |Q|^{-1}\int_Q f$, the supremum is taken over all cubes Q containing x, and $|Q|$ denotes the Lebesgue measure of Q.

We also define the Calderón–Zygmund maximal singular integral operator by

$$T^*f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} f(y)k(x-y)\,dy \Big|,$$

where the kernel $k(x)$ satisfies the standard conditions:

$$(1)\qquad \begin{aligned} &|k(x)| \le \frac{c}{|x|^n}, \qquad \int_{R_1<|x|<R_2} k(x)\,dx = 0 \quad (0<R_1<R_2<\infty),\\ &|k(x)-k(x-y)| \le \frac{c|y|^\alpha}{|x|^{n+\alpha}} \quad (|y|\le |x|/2,\ \alpha>0). \end{aligned}$$

Let ω be a non-negative, locally integrable function. Given a measurable set E, let $\omega(E) = \int_E \omega(x)\,dx$. We say that ω satisfies *Muckenhoupt's*

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condition A_∞ if there exist $c,\delta>0$ so that for any Q and $E\subset Q$,

$$\omega(E)\le c(|E|/|Q|)^\delta\omega(Q).$$

For $\omega\in A_\infty$, it is well known (see [7, 9, 10]) that

$$\|T^*f\|_{p,\omega}\le c\|Mf\|_{p,\omega}, \tag{2}$$

$$\|Mf\|_{p,\omega}\le c\|f^\#\|_{p,\omega} \tag{3}$$

for all $p>0$, where $\|f\|_{p,\omega}\equiv(\int_{\mathbb{R}^n}|f(x)|^p\omega(x)\,dx)^{1/p}$.

BMO estimates for T^*f go back to [15, 16]:

$$\|T^*f\|_*\le c\|f\|_\infty. \tag{4}$$

For the Hardy–Littlewood maximal function a BMO estimate was established later in [4]:

$$\|Mf\|_*\le c\|f\|_*. \tag{5}$$

These estimates were strengthened in [13] and [3] respectively:

$$\|T^*f\|_{\mathrm{BLO}}\le c\|f\|_\infty, \tag{6}$$

$$\|Mf\|_{\mathrm{BLO}}\le c\|f\|_*. \tag{7}$$

The space BLO [8] consists of all functions $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{BLO}}=\sup_Q(f_Q-\inf_Q f)<\infty.$$

It is easy to see that BLO $\subset$ BMO, moreover $\|f\|_*\le 2\|f\|_{\mathrm{BLO}}$.

Note that the estimates (2), (3) were proved in [7, 9, 10] with the help of so-called good λ inequalities. Afterwards, rearrangement inequalities for $Mf, f^\#, T^*f$ were obtained (see [1, 2, 5]), which also imply (2), (3).

The *non-increasing rearrangement* of f with respect to ω [6, p. 32] is defined by

$$f^*_\omega(t)=\sup_{\omega(E)=t}\inf_{x\in E}|f(x)|\quad(0<t<\infty).$$

If $\omega\equiv 1$ we use the notation $f^*(t)$.

A key role in our work is played by the maximal function (see [11, 19])

$$m_\lambda f(x)=\sup_{Q\ni x}(f\chi_Q)^*(\lambda|Q|)\quad(0<\lambda<1).$$

In terms of this function we establish pointwise estimates for the operators $Mf, f^\#, T^*f$. In particular, these estimates imply all the above mentioned results, namely rearrangement inequalities, L^p_ω and BLO-norm estimates (2)–(7).

Our main results are the following.

THEOREM 1. *For any function* $f\in L^p(\mathbb{R}^n)$ $(1\le p<\infty)$ *and for all* $x\in\mathbb{R}^n$,

$$m_\lambda(T^*f)(x)\le c_{\lambda,n}Mf(x)+T^*f(x)\quad(0<\lambda<1).$$

THEOREM 2. *For any function* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and for all* $x\in\mathbb{R}^n$,

$$m_\lambda(Mf)(x)\le c_{\lambda,n}f^\#(x)+Mf(x)\quad(0<\lambda<1).$$

Inequalities (2)–(7) follow from these theorems in view of the next main lemma.

LEMMA 1. *Let f and g be non-negative functions on $\mathbb{R}^n$. Suppose that for any $\lambda, 0<\lambda\le 1/2$, there exists a constant $c_\lambda>0$ so that*

$$m_\lambda f(x)\le c_\lambda g(x)+f(x)$$

for all $x\in\mathbb{R}^n$, and let $\omega\in A_\infty$. Then

(i) *there exists a constant $c'>0$ so that*

$$f^*_\omega(t)\le c'g^*_\omega(2t)+f^*_\omega(2t)$$

for all $t>0$;

(ii) *if $f^*_\omega(+\infty)=0$, then*

$$\|f\|_{L^p_\omega}\le c_p\|g\|_{L^p_\omega}\quad(0<p<\infty);$$

(iii) *if $g\in L^\infty$, then*

$$\|f\|_{\mathrm{BLO}}\le c\|g\|_\infty.$$

The proof of (iii) is essentially based on the inequality

$$\|f\|_*\le c_n\sup_Q\inf_c((f-c)\chi_Q)^*(\lambda|Q|) \tag{8}$$

which was proved by F. John [12] and J.-O. Strömberg [19] in the cases $0<\lambda<1/2$ and $\lambda=1/2$ respectively. For $\lambda>1/2$ this inequality fails.

First, we prove Theorems 1, 2, and then Lemma 1.

Proof of Theorem 1. Here it is convenient to use the maximal function

$$\widetilde{m}_\lambda f(x)=\sup_{B\ni x}(f\chi_B)^*(\lambda|B|),$$

where the supremum is taken over all balls B centered at x. It is easy to see that for any cube Q containing x there is a ball B centered at x which contains Q such that $|B|=c_n|Q|$. From this property, for any $x\in\mathbb{R}^n$ we have

$$m_\lambda f(x)\le\widetilde{m}_{\lambda/c_n}f(x). \tag{9}$$

By (9), it suffices to get the required estimate for $\widetilde{m}_\lambda$. Let B be an arbitrary ball with center at x. From the definition of T^* it follows that

$$T^*(f\chi_{\mathbb{R}^n\setminus 2B})(x)\le T^*f(x). \tag{10}$$

Further, by (1), the standard arguments (see, for example, [18, p. 59]) show that for all $y\in B$,

$$T^*(f\chi_{\mathbb{R}^n\setminus 2B})(y)\le cMf(x)+T^*(f\chi_{\mathbb{R}^n\setminus 2B})(x). \tag{11}$$

On the other hand, by weak type (1,1) of T^* [17, p. 42] we have

$$(T^*(f\chi_{2B}))^*(\lambda|B|) \le \frac{c}{|B|}\int_{2B}|f(y)|\,dy \le cMf(x).$$

From this and (10), (11) we get

$$((T^*f)\chi_B)^*(\lambda|B|) \le cMf(x) + T^*f(x).$$

Taking the upper bound over all balls B centered at x proves the theorem.

Proof of Theorem 2. We shall use the following elementary property of cubes: if cubes Q_1 and Q_2 intersect then either $Q_1 \subset 3Q_2$ or $Q_2 \subset 3Q_1$ (as usual, kQ denotes the cube concentric with Q and having edge length k times as large).

Let Q be an arbitrary cube containing the point x. Take an arbitrary point $y \in Q$ and suppose a cube Q' contains y. If $Q' \subset 3Q$, then

$$|f|_{Q'} \le |f - f_{3Q}|_{Q'} + |f|_{3Q} \le M((f - f_{3Q})\chi_{3Q})(y) + Mf(x).$$

Assume now that $Q' \not\subset 3Q$. Then $Q \subset 3Q'$ and in this case

$$|f|_{Q'} \le |f - f_{3Q'}|_{Q'} + |f|_{3Q'} \le 3^n f^\#(x) + Mf(x).$$

Thus, for all $y \in Q$,

$$Mf(y) = \max\Big(\sup_{\substack{Q'\ni y\\ Q'\subset 3Q}} |f|_{Q'},\ \sup_{\substack{Q'\ni y\\ Q\subset 3Q'}} |f|_{Q'}\Big)$$
$$\le M((f - f_{3Q})\chi_Q)(y) + 3^n f^\#(x) + Mf(x).$$

Using the weak type (1,1) of the operator M, we get

$$((Mf)\chi_Q)^*(\lambda|Q|) \le (M((f - f_{3Q})\chi_{3Q}))^*(\lambda|Q|) + 3^n f^\#(x) + Mf(x)$$
$$\le \frac{c}{|Q|}\int_{3Q}|f - f_{3Q}| + 3^n f^\#(x) + Mf(x)$$
$$\le cf^\#(x) + Mf(x).$$

Taking the upper bound over all $Q \ni x$ yields the theorem.

Proof of Lemma 1. Choose λ so that $c(2^n\lambda)^\delta = 1/4$, where c, δ are the constants from the definition of A_∞, and put $c' = c_\lambda$.

Let E be an arbitrary set with $\omega(E) = t$. Applying the Calderón–Zygmund decomposition to the function χ_E and number λ, we get pairwise disjoint cubes Q_i such that

$$\lambda|Q_i| < |E \cap Q_i| \le 2^n\lambda|Q_i|. \tag{12}$$

From the definition of A_∞ it follows that

$$\omega(E) = \sum_i \omega(E\cap Q_i) \le c\sum_i\Big(\frac{|E\cap Q_i|}{|Q_i|}\Big)^\delta \omega(Q_i) \le c(2^n\lambda)^\delta\omega\Big(\bigcup_i Q_i\Big).$$

So, we have $\omega(\bigcup_i Q_i) \ge 4t$. From this and the left-hand inequality of (12) we obtain

$$\inf_{x\in E}|f(x)| \le \inf_i \inf_{x\in E\cap Q_i}|f(x)| \le \inf_i (f\chi_{Q_i})^*(\lambda|Q_i|)$$
$$\le \inf_i \inf_{x\in Q_i} m_\lambda f(x) = \inf_{x\in\bigcup_i Q_i} m_\lambda f(x) \le (m_\lambda f)^*_\omega(4t).$$

Taking the supremum over all sets E with $\omega(E) = t$, we get

$$f^*_\omega(t) \le (m_\lambda f)^*_\omega(4t).$$

From this and simple properties of rearrangement it follows that

$$f^*_\omega(t) \le (c'g + f)^*_\omega(4t) \le c'g^*_\omega(2t) + f^*_\omega(2t).$$

So, we get (i). Iterating this inequality we obtain (ii) in a standard way (see, for example, [14]).

It remains to prove (iii). This follows immediately from the following BLO criterion.

LEMMA 2. *Let $\lambda \le 1/2$. Then a non-negative function f belongs to BLO iff $m_\lambda f - f \in L^\infty$. Moreover,*

$$\|f\|_{\mathrm{BLO}} \asymp \|m_\lambda f - f\|_\infty.$$

Proof. Define $A = \|m_\lambda f - f\|_\infty$. It is clear that

$$(f\chi_Q)^*(\lambda|Q|) \le A + \inf_Q f \tag{13}$$

for any cube Q. From this it follows that

$$\inf_c((f - c)\chi_Q)^*(\lambda|Q|) \le ((f - \inf_Q f)\chi_Q)^*(\lambda|Q|)$$
$$= (f\chi_Q)^*(\lambda|Q|) - \inf_Q f \le A.$$

Since $\lambda \le 1/2$, by John and Strömberg's theorem (see (8)) it follows that $f \in \mathrm{BMO}$ and $\|f\|_* \le cA$. Further, note that for any cube Q,

$$f_Q \le \inf_{x\in Q}(|f(x) - f_Q| + |f(x)|) \le ((|f - f_Q| + |f|)\chi_Q)^*(|Q|)$$
$$\le ((f - f_Q)\chi_Q)^*(|Q|/2) + (f\chi_Q)^*(|Q|/2) \le 2\|f\|_* + (f\chi_Q)^*(|Q|/2).$$

From this and (13) we get

$$\|f\|_{\mathrm{BLO}} = \sup_Q(f_Q - \inf_Q f) \le \sup_Q(2\|f\|_* + (f\chi_Q)^*(|Q|/2) - \inf_Q f)$$
$$\le \sup_Q(2cA + (f\chi_Q)^*(\lambda|Q|) - \inf_Q f) \le (2c+1)A.$$

Conversely, let $f \in \mathrm{BLO}$. Then

$$(f\chi_Q)^*(\lambda|Q|) \le ((f - f_Q)\chi_Q)^*(\lambda|Q|) + f_Q$$
$$\le \frac{1}{\lambda}\|f\|_* + \|f\|_{\mathrm{BLO}} + \inf_Q f \le (2/\lambda + 1)\|f\|_{\mathrm{BLO}} + \inf_Q f.$$

Thus,

$$m_\lambda f(x) \le (2/\lambda + 1)\|f\|_{\mathrm{BLO}} + f(x).$$

The lemma is proved.

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References

[1] R. J. Bagby and D. S. Kurtz, *Covering lemmas and the sharp function*, Proc. Amer. Math. Soc. 93 (1985), 291–296.

[2] —, —, *A rearranged good λ inequality*, Trans. Amer. Math. Soc. 293 (1986), 71–81.

[3] C. Bennett, *Another characterization of BLO*, Proc. Amer. Math. Soc. 85 (1982), 552–556.

[4] C. Bennett, R. DeVore and R. Sharpley, *Weak-L^∞ and BMO*, Ann. of Math. 113 (1981), 601–611.

[5] C. Bennett and R. Sharpley, *Weak-type inequalities for H^p and BMO*, in: Proc. Sympos. Pure Math. 35, Amer. Math. Soc., 1979, 201–229.

[6] K. M. Chong and N. M. Rice, *Equimeasurable Rearrangements of Functions*, Queen's Papers in Pure and Appl. Math. 28, Queen's University, Kingston, Ont., 1971.

[7] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 15 (1974), 241–250.

[8] R. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. 79 (1980), 249–254.

[9] A. Córdoba and C. Fefferman, *A weighted norm inequality for singular integrals*, Studia Math. 57 (1976), 97–101.

[10] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[11] B. Jawerth and A. Torchinsky, *Local sharp maximal functions*, J. Approx. Theory 43 (1985), 231–270.

[12] F. John, *Quasi-isometric mappings*, in: Seminari 1962–1963 di Analisi, Algebra, Geometria e Topologia (Roma, 1964), Ediz. Cremonese, Roma, 1965, 462–473.

[13] M. A. Leckband, *Structure results on the maximal Hilbert transform and two-weight norm inequalities*, Indiana Univ. Math. J. 34 (1985), 259–275.

[14] A. K. Lerner, *On weighted estimates of non-increasing rearrangements*, East J. Approx. 4 (1998), 277–290.

[15] S. Spanne, *Sur l'interpolation entre les espaces $L_k^{p\Phi}$*, Ann. Scuola Norm. Sup. Pisa 20 (1966), 625–648.

[16] E. M. Stein, *Singular integrals, harmonic functions, and differentiability properties of functions of several variables*, in: Proc. Sympos. Pure Math. 10, Amer. Math. Soc., 1967, 316–335.

[17] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[18] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[19] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), 511–544.

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